The Discrete Laplacian and the Hotspot Conjecture

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Abstract

An exploration into the discrete analog of the Laplacian and the hot spot conjecture for graphs.

Contents

1 An Introduction to the Hotspot Conjecture

1.1 Preliminaries

The spatially discrete heat equation with continuous time is,

$$u_t = -Ku \tag{1}$$

Using separation of variables we can reduce the spatial part of the heat equation to the simple eigenvalue problem:

$$K\phi_i = \lambda_i \phi_i \tag{2}$$

Where we let, $0 = \lambda_1 \leq \cdots \leq \lambda_i \leq \cdots \leq \lambda_n$ be the eigenvalues of K, and $\{\phi_1, \cdots, \phi_i\}$ be the corresponding eigenvectors. If K is a symmetric, $n \times n$ matrix, then the eigenvectors are orthogonal (Therefore $\phi_i \cdot \phi_j = 0$ for all $i \neq j$) and span R^n. With that in mind we can see than any vector $u(t)$ can be written as such:

$$u(t) = \sum_{i=1}^{n} \beta_i(t)\ \phi_i$$

We can simplify this equation by substituting this form of u into the initial heat equation.

$$\begin{aligned} u_t(t) &= -Ku(t) \\ \sum_{i=1}^{n} \frac{d\beta_i(t)}{dt}\ \phi_i\ &= -K\sum_{i=1}^{n} \beta_i(t)\ \phi_i \\ &= -\sum_{i=1}^{n} \lambda_i\ \beta_i(t)\ \phi_i \end{aligned}$$

$$\Longrightarrow \beta_i(t) = \alpha_i\ e^{-\lambda_i t} \ \text{ and } u(t) \text{ can now be written: } u(t) = \sum_{i=1}^{n} \alpha_i\ e^{-\lambda_i t}\ \phi_i$$

The equation can be rewritten as:

$$u(t) = \alpha_1\ + \alpha_2\phi_2\ e^{-\lambda_2 t} + R(t)$$

Where $R(t)$ is a remainder function that approaches zero faster than the second term.

1.2 The Conjecture

The hot spot conjecture was first introduced in 1974 by Jeff Rauch.

Conjecture 1.1. *The Hotspot Conjecture: Any second eigenfunction for the Laplacian with Neumann boundary conditions in a bounded Euclidean domain attains its maximum at the boundary.*

An equivalent way of stating the conjecture is...

Conjecture 1.2. *The Hotspot Conjecture (rephrased): The hot spots or maximal temperature values of bounded Euclidean domain move to the boundary as t goes to ∞.*

This conjecture has been shown to be generally false in the continuous case. However for 2-d simply-connected domains it holds.

2 A Discrete Example of the Hotspot Conjecture

2.1 The 5×5 Grid

The first example of the hotspot conjecture on graphs will be the 5×5 square grid. The eigenfunction is the set of numbers in the second eigenvector of the Laplacian. The second eigenvalue of the 5×5 grid has multiplicity 2 thus there are two independent eigenfunctions corresponding to this eigenvalue. Figure 1 displays the two 'temperature curves'. The machinery for forming these graphs will be explained later.

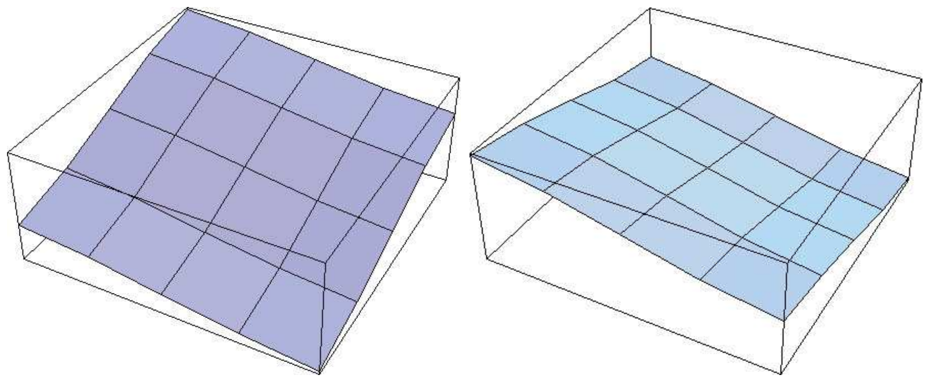

Figure 1: The Two Second Eigenfunctions of the 5×5 Grid

3 The Rectangular Grid

3.1 Forming Notation - The Rectangular Grid Laplacian

We start creating our Kirchhoff matrix for a $m \times n$ rectangular grid where $m < n$ by numbering the nodes left to right starting at the top of the gird and moving down similar to the way one would read. It's important to note that the edge nodes are not all contained in a upper left hand submatrix. The edge is not arbitrary but defined geometrically. For a typical rectangular grid (Figure 2) the edge set is the set of nodes (intersection points) that have 2 or 3 connections.

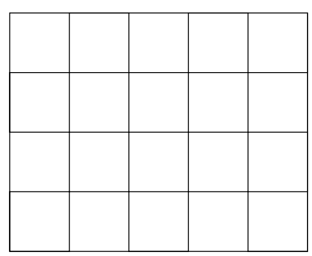

Figure 2: A Typical Grid

With this construction we can see that the Kirchhoff matrix has this form:

$$K = \begin{array}{cc} & m \times n \\ m \times n & \begin{bmatrix} D_1 & -I & & & 0 \\ -I & D_2 & \ddots & & \\ & -I & \ddots & -I & \\ & & \ddots & D_2 & -I \\ 0 & & & -I & D_1 \end{bmatrix} \end{array}$$

where,

$$D_1 = \begin{array}{cc} & n \\ n & \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 3 & \ddots & & \\ & -1 & & & \\ & & \ddots & -1 & \\ & & \ddots & 3 & -1 \\ 0 & & & -1 & 2 \end{bmatrix} \end{array} \quad D_2 = \begin{array}{cc} & n \\ n & \begin{bmatrix} 3 & -1 & & & 0 \\ -1 & 4 & \ddots & & \\ & -1 & & & \\ & & \ddots & -1 & \\ & & \ddots & 4 & -1 \\ 0 & & & -1 & 3 \end{bmatrix} \end{array}$$

The K matrix is composed of identity matrices on the immediate off diagonals, and 2 D_1 matrices and $m-2$ D_2 matrices on the diagonal.

3.2 An Example - The 3×4 Grid

Using the above notation the Kirchhoff matrix for the 3×4 grid is,

$$\begin{bmatrix}
2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2
\end{bmatrix}$$

The first non-zero eigenvalue and corresponding eigenvector for the Laplacian of the 3×4 grid are $\lambda_2 = 0.585786$ and
$\phi_2 = \begin{bmatrix} -1, & -0.414, & 0.414, & 1, & -1, & -0.414, & 0.414, & 1, & -1, & -0.414, & 0.414, & 1 \end{bmatrix}$

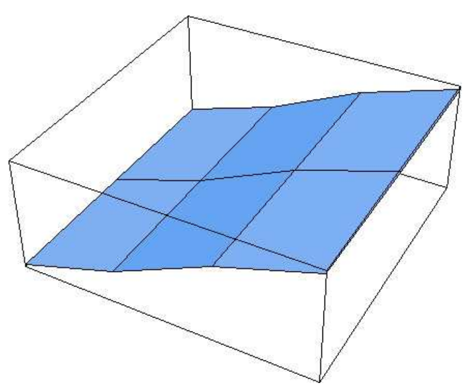

Figure 3: The Second Eigenfunction of the 3×4 Grid

3.3 Proof of the Hotspot Conjecture for Rectangular Graphs

Theorem 3.1. *For an $m\times$ n rectangular graph with uniform conductivity the second eigenfunction of the Laplacian attains its highest values at the edge nodes. In other words for a vector Ψ_2 that satisfies $K\Psi_2 = \lambda_2\Psi_2$, Ψ_2 attains its maximal values on the edge nodes.*

Conjecture 3.1. *The following is a eigenvector of the Laplacian:*

$$\Phi_2 = \begin{bmatrix} \begin{pmatrix} \phi_2(1) \\ \vdots \\ \phi_2(n) \end{pmatrix} \\ \begin{pmatrix} \phi_2(1) \\ \vdots \\ \phi_2(n) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \phi_2(1) \\ \vdots \\ \phi_2(n) \end{pmatrix} \end{bmatrix} = \begin{array}{c} 1 \\ \\ 2 \\ \\ \vdots \\ \\ m \end{array} \begin{bmatrix} \vec{\phi_2} \\ \\ \vec{\phi_2} \\ \\ \vdots \\ \\ \vec{\phi_2} \end{bmatrix}, \ \textit{where} \ \ \phi_2(x) = \cos[\frac{\pi}{n}(x - \frac{1}{2})]$$

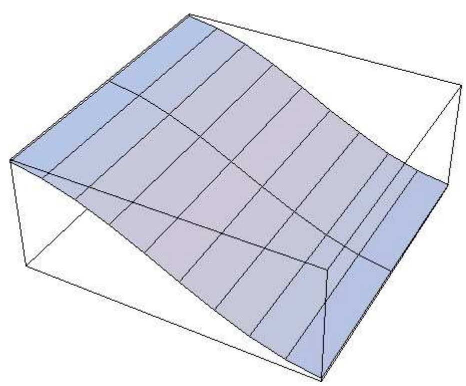

Figure 4: The Second Eigenfunction of the 3×10 Grid

Proof.

$$K\Phi_2 = \lambda_2\Phi_2 \iff \begin{bmatrix} D_1 & -I & & & 0 \\ -I & D_2 & \ddots & & \\ & -I & & & \\ & & \ddots & -I & \\ & & \ddots & D_2 & -I \\ 0 & & & -I & D_1 \end{bmatrix} \begin{bmatrix} \vec{\phi_2} \\ \\ \vec{\phi_2} \\ \\ \vdots \\ \\ \vec{\phi_2} \end{bmatrix} = \begin{bmatrix} D_1\vec{\phi_2} - I\vec{\phi_2} \\ \\ D_2\vec{\phi_2} - 2I\vec{\phi_2} \\ \\ \vdots \\ \\ D_1\vec{\phi_2} - I\vec{\phi_2} \end{bmatrix} = \lambda \begin{bmatrix} \vec{\phi_2} \\ \\ \vec{\phi_2} \\ \\ \vdots \\ \\ \vec{\phi_2} \end{bmatrix}$$

Looking at the is $D_1\vec{\phi_2} + I\vec{\phi_2} = \lambda\vec{\phi_2}$. This matrix can be written as a system of equations:

$$
\begin{aligned}
\phi_2(1) - \phi_2(2) &= \lambda\phi_2(1) \\
-\phi_2(1) + 2\phi_2(2) - \phi_2(3) &= \lambda\phi_2(2) \\
-\phi_2(2) + 2\phi_2(3) - \phi_2(4) &= \lambda\phi_2(3) \\
&\vdots \\
-\phi_2(n-2) + 2\phi_2(n-1) - \phi_2(n) &= \lambda\phi_2(n-1) \\
-\phi_2(n-1) + \phi_2(n) &= \lambda\phi_2(n)
\end{aligned}
$$

If we can show that the following two expressions are equivalent then we have shown that Φ as defined above is an eigenfunction of K.

$$
\begin{aligned}
-\phi_2(i) + 2\phi_2(i+1) - \phi_2(i+2) &= \lambda\phi_2(i+1) \\
-\phi_2(i+1) + 2\phi_2(i+2) - \phi_2(i+3) &= \lambda\phi_2(i+2)
\end{aligned}
$$

Solving for λ, we get,

$$
\lambda = \frac{-\phi_2(i) + 2\phi_2(i+1) - \phi_2(i+2)}{\phi_2(i+1)} = \frac{-\phi_2(i+1) + 2\phi_2(i+2) - \phi_2(i+3)}{\phi_2(i+2)}
$$

Cross-multiplying...

$$
\begin{aligned}
[\phi_2(i)\phi_2(i+2)]\,\phi_2(i+2) &= [\phi_2(i+1) + \phi_2(i+3)]\,\phi_2(i+1) \\
\longrightarrow \left[\cos(\frac{\pi}{n}(i-\frac{1}{2})) + \cos(\frac{\pi}{n}(i+\frac{3}{2}))\right]\cos(\frac{\pi}{n}(i+\frac{3}{2})) &= \left[\cos(\frac{\pi}{n}(i+\frac{1}{2})) + \cos(\frac{\pi}{n}(i+\frac{5}{2}))\right]\cos(\frac{\pi}{n}(i+\frac{1}{2})) \\
\longrightarrow \left[2\cos(\frac{\pi}{n}(\frac{2i+1}{2}))\cos(\frac{\pi}{n}(\frac{2}{2}))\right]\cos(\frac{\pi}{n}(i+\frac{3}{2})) &= \left[2\cos(\frac{\pi}{n}(\frac{2i+3}{2}))\cos(\frac{\pi}{n}(\frac{2}{2}))\right]\cos(\frac{\pi}{n}(i+\frac{1}{2})) \\
\longrightarrow \left[2\cos(\frac{\pi}{n}(i+\frac{1}{2}))\cos(\frac{\pi}{n}(1))\right]\cos(\frac{\pi}{n}(i+\frac{3}{2})) &= \left[2\cos(\frac{\pi}{n}(i+\frac{3}{2}))\cos(\frac{\pi}{n}(1)))\right]\cos(\frac{\pi}{n}(i+\frac{1}{2}))
\end{aligned}
$$

We can see now that the equality is true and therefore we have shown that Φ_2 (as defined earlier) is in fact an eigenvector of our Kirchhoff matrix. Out of this calculation we also get a closed expression for λ.

$$
\lambda = \frac{\cos(\frac{\pi}{2n}) - \cos(\frac{3\pi}{2n})}{\cos(\frac{\pi}{2n})} \tag{3}
$$

□

Conjecture 3.2. *The Φ_2 is the second eigenvector of the Laplacian:*

If this conjecture can be proved it's trivial to show that our Φ_2 maintains it's maximum at the edge nodes and thus Theorem 3.1 is true.

Conjecture 3.3. *All the eigenvector has the form:*

$$\Phi_{lk} = \left[\begin{array}{c} \begin{pmatrix} \phi(1,1) \\ \vdots \\ \phi(n,1) \end{pmatrix} \\ \begin{pmatrix} \phi(1,2) \\ \vdots \\ \phi(n,2) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \phi(1,m) \\ \vdots \\ \phi(n,m) \end{pmatrix} \end{array} \right] \text{ where } \phi(x,y) = \cos\left(\frac{\pi(2k-1)}{n}\left(x-\frac{1}{2}\right)\right)\cos\left(\frac{\pi(2l-1)}{m}\left(y-\frac{1}{2}\right)\right)$$

4 Counterexample to the Hotspot Conjecture for Graphs

It could be conjectured that the hotspot conjecture works for all connected graphs. Here's a counterexample...

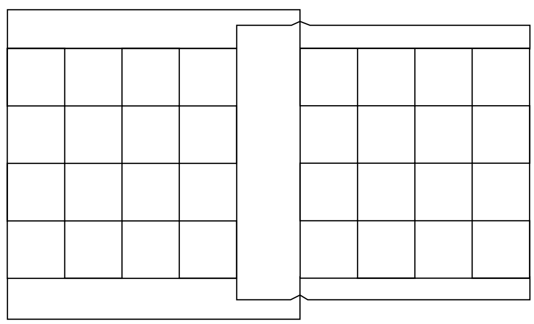

Figure 5: Counterexample Graph

The second eigenfunction has the following form:

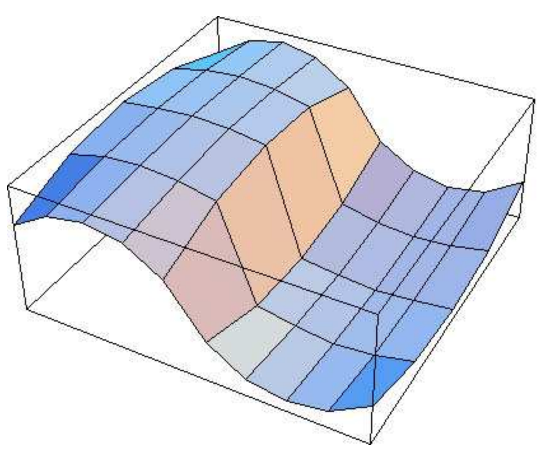

Figure 6: Counterexample Graph

The highest and lowest values of this eigenfunction are contained in the centers of the two square grids.

This result doesn't tell us much. In fact our definition of boundary is somewhat unspecified. This seems to erode the very foundation of this paper, for the hotspot conjecture wouldn't make much sense if we didn't have a predetermined definition the boundary. It may be better to think of this conjecture as only working for graphs that are analogous to simple geometrical objects in the continuous realm. Another possible solution would be to reverse the conjecture and state the boundary nodes are those nodes that attain the highest values of the second eigenfunction. This may be an interesting way of forming a notion of boundary for graphs.

5 Further Work

1. Prove Conjectures 3.2 and 3.3
2. Look at the Hotspot Conjecture for graphs other than the rectangular grid (Figure 7).
3. Explore the other eigenfunctions of the Kirchhoff Matrix
4. See if the Hotspot Conjecture holds for non-constant 'conductivities'.
5. Form or refine a definition of boundary for graphs that is analogous to continuous domains.

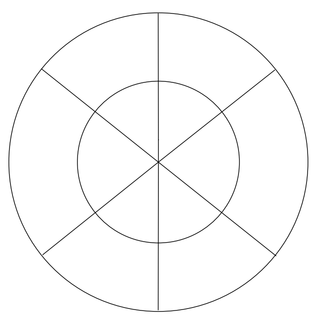

Figure 7: Another Interesting Graph

References

[1] Sam, Jeff, Ernie. "A LaTeX Starter File." 2003.

[2] Curtis, B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.

[3] Devries, Timothy. "Recoverability of Random Walk Networks." 2003.

[4] Tittelfitz, Justin. "The Heat Equation on Discrete Domains ." 2006.

[5] F. R. K. Chung. "Spectral Graph Theory" CBMS – Volume 92. American Mathematical Society, ©1997.