ANGLE-SYSTEMS AND SELF-DUALITY

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Abstract. This paper serves as a broad introduction to the theory and application of Angle-Systems. Angle-Systems are defined as an algebraic structure, and their basic properties are introduced. The correspondence between Angle-Systems and embeddings of graphs upon an orientable surface is discussed, and various graph theoretic results are formalized as properties of algebraic structures. Symmetry patterns in Angle-Systems are explored, particularly the problem of characterizing self-dual graphs.

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1. Introduction

Angle-Systems arise as an effort to express all relevant information about a graph and its embedding in a concise structure. In particular, topologically equivalent embeddings must be represented by the same Angle-System. Thus only two properties are critical.

- Vertices, edges, and faces must be described.
- The ordering of edges around vertices and faces must be specified.

Angle-Systems account only for graphs embedded on orientable surfaces, allowing a meaningful distinction between clockwise and counterclockwise in the orderings mentioned.

All theory regarding Angle-Systems presupposes only basic group theory (e.g. [3]). Numerous analogies are made to graph theoretic notions discussed in [2] and related papers, though these references serve only as motivation and examples, not prerequisites.

2. Basic Structures of Angle-Systems

We first describe the algebraic machinery that will be used throughout this paper. Graph theoretic interpretation of these notions follows in the subsequent section. It is recommended that these sections be read in parallel so that the motivation behind the many definitions is understood. They are presented separately to clearly distinguish between the intuitive and rigorous parts of this paper.

Definition 2.1. An *Angle-System of order n* is a pair (p, q) of permutations of $2n$ elements such that $(qp)^2 = e$, where e denotes the identity permutation. If qp consists only of transpositions (i.e. it has no fixed points), then (p, q) is called a *Proper Angle-System.*

Proposition 2.2. *If (p, q) is an Angle-System, then so is (q, p).*

Proof: It suffices to show that if $qpqp = e$, then $pqpq = e$. This follows since $pqpq = q^{-1}(qpqp)q = q^{-1}eq = q^{-1} = e$.
□

Definition 2.3. The Angle-System (q, p) is the *dual* of the Angle-System (p, q).

Definition 2.4. The *Angles* of an angle system are the $2n$ elements permutated by p and q. A denotes the set of angles, and all permutations mentioned may be regarded as bijections of A. The elements of A will be denoted simply as the positive integers from 1 to $2n$.

Composition of permutations will be denoted simply by juxtaposition, as above. When a permutation is applied to a particular angle $a \in A$, parentheses are used, e.g. $pqp(a)$.

Definition 2.5. If $p_1, \ldots, p_n$ are permutations, then $\ll p_1, \ldots, p_n \gg$ denotes the subgroup generated by $p_1, \ldots, p_n$, i.e. all permutations expressible as a composition of some sequence of the permutations. In particular, $\ll p \gg$ is the cyclic group generated by p.

When working with Angle-Systems, we are only interested in permutations consisting of compositions of p and q.

Definition 2.6. If S is a subgroup of permutations of A, and $a \in A$, then the *equivalence class* of a under S is the set $S(a) = \{b \in A \mid \exists s \in S : s(a) = b\}$. If $Sa = Sb$, we say that a and b are *congruent*, modulo S.

Permutations will often be written in *cycle notation*, the unique decomposition of the permutation into disjoint cycles. For example, $p = (1\ 2\ 3)(4\ 5)(6)$ indicated that p maps 1 to 2 to 3 to 1, interchanges 4 and 5, and fixes 6. Fixed elements will sometimes be omitted from the notation. Note that the elements of the cycles of a permutation p correspond to equivalence classes under $\ll p \gg$. Several cases of interest are given in the next definition.

Definition 2.7. We define the following subgroups of permutations, and their correspondent equivalence classes. Where the number of such equivalence classes is of interest, a label is associated.

- $\ll p, q \gg$ is the *Walk Group*, denoted W. All of the below groups are subgroups of W. The equivalence classes under W are called the *components* of the Angle-System, and N denotes the number of components $|W|$.

- $\ll p \gg$ is the *Vertex Group*, denoted P. The equivalence classes under P are the *vertices* of the Angle-System, and V denotes the number of vertices $|P|$.

- $\ll q \gg$ is the *Face Group*, denoted Q. The equivalence classes under Q are the *faces* of the Angle-System, and F denotes the number of faces $|Q|$.

- $\ll qp \gg$ is the *Edge Group*, denoted D. The equivalence classes under this group are the *edges* of the Angle-System. All edges consist of one or two angles; those consisting of only one angle are *unary* edges, while the rest are *proper* edges. E denotes the number of proper edges. Note that $E = 2n - |D|$ (recall that $2n$ is the number of angles).

- $\ll pq, qp \gg$ is the *Geodesic Group*, denoted G. The equivalence classes under G are the *geodesics* of the Angle-System.

The reason for the somewhat irregular definition for E, the number of edges, is that fixed points of qp are of very little interest, and tend to obscure many theorems we would like to prove. Typically, we deal only with Angle-Systems where qp has no fixed points, but the convention of counting only 2-cycles of D in E allows most results to generalize.

Definition 2.8. An Angle-System is *connected* if it has only one component. Alternatively, we state this definition as $\forall a, b \in A,\ \exists w \in W : w(a) = b$.

Typically, we are only interested in considering connected Angle-Systems. Sometimes, however, a system becomes disconnected in the process of a transformation (to be discussed in §4). In this case, it is often advantageous to speak only of a single component and disregard the rest.

Proposition 2.9. *Each component of an Angle-System can be treated as a separate connected Angle-System.*

Proof: Simply consider the restrictions of p, q to the set Wa, for any angle a in the component. Since $p, q \in W$, their restrictions to Wa have image Wa, and clearly the restriction on pq remains true.
□

Definition 2.10. The *Eulernumber* of an Angle-System, denoted χ, is given by $\chi = F + V - E$.

3. Graph Theoretic Interpretation

3.1. **The Fundamental Conjecture of Angle-Systems.** The critical connection between Angle-Systems and graph theory is stated in the following conjecture, which the author titles, in brazen confidence of its truth, the Fundamental Theorem of Angle-Systems.

Conjecture 3.1. There is a one-to-one correspondence between topologically equivalent embedded graphs on orientable surfaces and those Angle-Systems which have no unary edges.

One portion of this conjecture, namely that any embedding of a graph on an orientable surface, is intuitively clear once the proper interpretations of p, q and A are given. The conjecture in full generality may be proven by several steps by existing embedding theorems. However, the purpose of this paper is to explore the properties of Angle-Systems as algebraic objects independent of graph theory, thus rigorous proof of the relation to graph theory is left to other efforts.

All this papers' results, however, are motivated by an intuitive graph theoretic interpretation, which shall now be outlined.

3.2. **A, p, q and pq.** The angles A correspond to ordered pairs of edges in a graph. Namely, they represent two vertices immediately adjacent in the embedding. The pairs are ordered because one edge is distinguished as the *leading edge*, the other the *following edge*, so that the following edge must be rotating counter-clockwise to coincide with the leading edge, if the orientation of the surface is taken to be out of the page.

Notice that each edge in the graph is thus the leading edge of two angles, and the following edge of two others (in certain special cases these four angles need not be

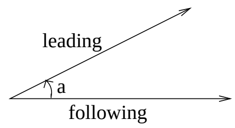

FIGURE 1. Interpretation of an angle.

distinct; this will be discussed later). Using this fact, we interpret the permutations p, q as follows.

- Both p and q map each angle a to one of the two angles whose following edge is the leading edge of a.
- p maps a to the angle around the same vertex, while q maps to the angle around the vertex at the other end.[1]

Observe that the reasoning behind the restriction imposed on qp by Definition 2.1 is now apparent. qp maps from an angle to the unique angle with the same leading edge. Thus this permutation must simply interchange pairs of angles.

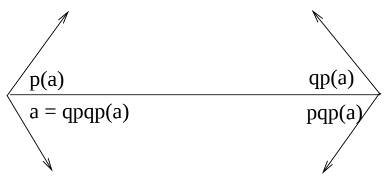

FIGURE 2. The relationship between qp and edges.

3.3. **Primal, Dual, and Medial Graphs.** One reason this interpretation is so powerful and represents so many useful symmetries lies in the evenhandedness with which it applies to both the primal graph and the dual graph. In particular, angles belong not only to one vertex, but also to one face, and it is apparent that q could be interpreted as mapping each a to the unique angle whose following edge is the leading edge of q and which is part of the same *face* as a. Faces and Vertices being duals of each other, q plays the exact same role in the dual graph as p plays in the primal graph, and vice versa. Thus our definition of (q, p) as the dual of (p, q) is logical. See figure 3 to see how angles in the primal graph correspond to angles in the dual graph.

It should be observed that the notions of dual and medial graph discussed here are different from those discussed in [2] and in other REU papers, since no boundary is specified. In most cases, there is a way to map between those two notions, though sometimes it is more difficult. This problem is discussed briefly at the end of the paper.

[1] This may be the same vertex in the case of a loop.

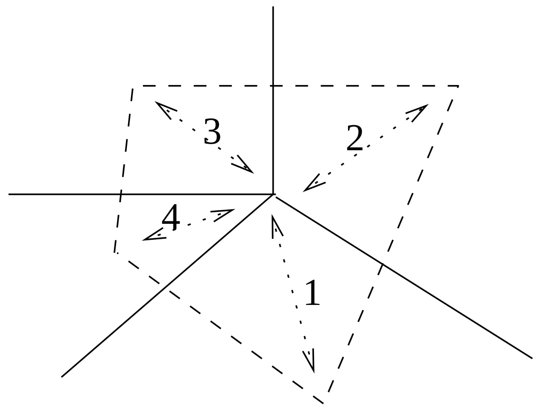

FIGURE 3. Illustration of the correspondence between angles in the primal and dual graph, and the fact that the vertex permutation of the primal is identical to the face permutation of the dual.

Another aesthetically pleasing and conceptually useful characteristic of Angle-Systems is how they can be represented by the medial graph of the correspondent embedded graph. In this setting, angles of the System correspond to edges of the medial graph, with p and d permuting these edges counterclockwise about faces. p rotates about the face of one shading, q about faces of the other shading. pd and dp then interchange adjacent elements of geodesics.

3.4. **The Equivalence Classes.** The equivalence classes described in Definition 2.6 all correspond to the notions for which they are named. The vertices and faces are self-explanatory., while some notes are useful regarding the others.

The walk group W is so-named because any "walk" by repeatedly moving in the manner described by p and q can map any angle to any other angle in a connected part. The equivalence classes thus become connected components.

The edge group defines a pairing of angles which share a leading edge. Since each edge is the leading edge of a pair of angles, the equivalence classes are the edges.

The geodesic group G similarly makes sense since both qp and pq map from an edge in the medial graph to another edge on the same geodesic, and it is easy to see that a series of these two can map from any edge to any other edge on the same geodesic.

3.5. **Example: The Tetrahedron Graph.** We now illustrate the notions described with a simple but interesting graph: the tetrahedron. We display diagrams of the primal, dual, and medial graphs, as well as an overlay of the primal and dual graphs to convey the correspondence between them.

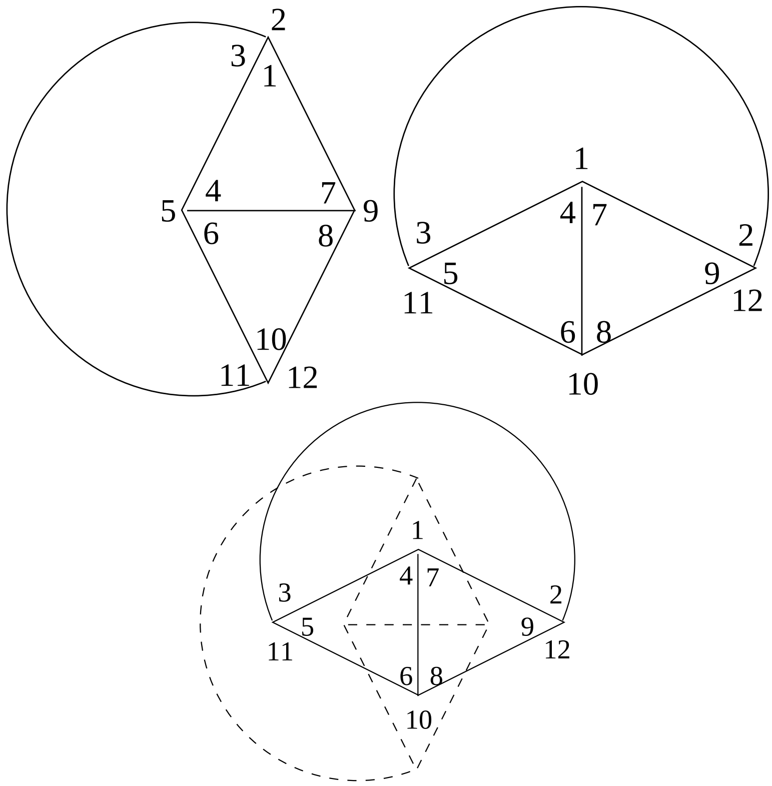

FIGURE 4. The tetrahedron graph, it's dual, and the two overlayed.

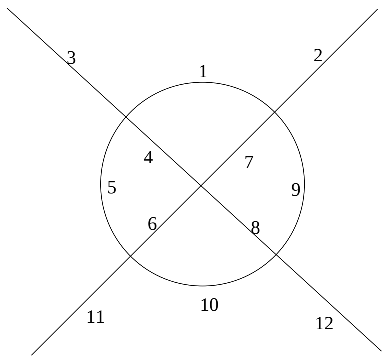

FIGURE 5. The medial graph of the tetrahedron. Note that the four ends are meant to intersect at infinity; the intersection in not drawn so as to avoid clutter.

As can be observed from the diagrams, the Angle-System for the tetrahedron graph is as follows.

$p = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)$
$q = (1\ 4\ 7)(2\ 9\ 12)(3\ 11\ 5)(6\ 10\ 8)$

We also consider the following two compositions, whence we can derive all the equivalence classes we mention.

$qp = (1\ 9)(2\ 11)(3\ 4)(5\ 10)(6\ 7)(8\ 12)$
$pq = (1\ 5)(2\ 7)(3\ 12)(4\ 8)(6\ 11)(9\ 10)$

It is easy enough to observe that interchanging p and q indeed gives the correct Angle-System for the dual graph. Examining how the patterns of the above permutations manifest themselves in the primal, dual, and medial graphs. Further, we enumerate the equivalence classes of the tetrahedron graph, which can all be observed in the diagrams above.

1 Component:	{1,2,3,4,5,6,7,8,9,10,11,12}
4 Vertices:	$\{1,2,3\}\{4,5,6\}\{7,8,9\}\{10,11,12\}$
4 Faces:	$\{1,4,7\}\{2,9,12\}\{3,11,5\}\{6,10,8\}$
6 Edges:	$\{1,9\}\{2,11\}\{3,4\}\{5,10\}\{6,7\}\{8,12\}$
3 Geodesics:	$\{1,9,10,5\}\{2,11,6,7\}\{3,4,8,12\}$

3.6. **Constructing an embedding from an Angle-System.** As a partial justification of why Conjecture 3.1 should be true, we present an intuitive argument for how an embedding can be reconstructed from an Angle-System. The fact that each embedded graph produces a unique Angle-System, and that the Angle-System encapsulates all necessary information about the embedding, should be obvious. It is the converse that presents more technical difficulty.

The most obvious action is to use the equivalence classes mentioned above. For example, examine the cycle notation of p. The cycles represent vertices of the embedding, we can simply draw every vertex as a star, with all angles labeled, and then draw in the edges one by one. It should be noted that if the graph is nonplanar, eventually some of the edges will have to pass under existing edges, and some care will be necessary in determining the exact embedding.

We illustrate this method by example. If the reader is still unsure about the graphical interpretation of Angle-Systems, the author encourages him or her to work several more of these examples. We begin by simply picking *any* p permutation, and any qp permutation composed of 2-cycles, such as the following. Note that we might as well number the angles in p in order, since it makes no difference.

$p = (1\ 2)(3\ 4\ 5)(6\ 7\ 8)$
$qp = (1\ 6)(2\ 4)(3\ 8)(5\ 7)$
$q = qpp^{-1} = (1\ 4\ 8\ 5\ 2\ 6\ 3\ 7)$

The construction and embedding of this graph, which was conceived simply as a pair of permutations obeying our axioms, is displayed in Figure 7.

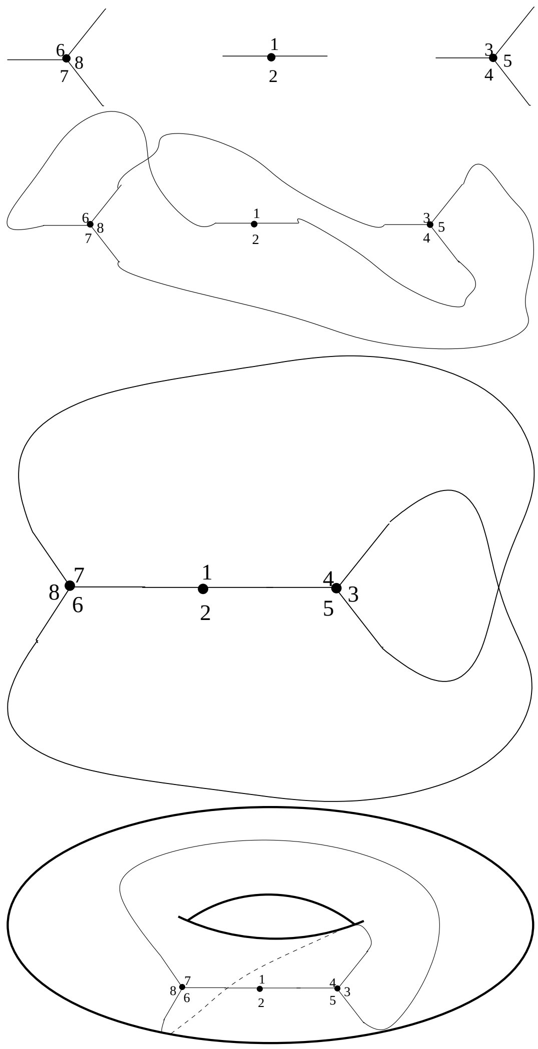

FIGURE 6. First, all vertices are drawn and labeled as stars. Then we identify the leading edge of each angle and connect it to the leading edge of the angle to which the edge permutations pairs it. Next, we tidy up the graph by rotating vertices and redrawing to make it more legible. Finally we observe that indeed this graph can be embedded on a surface of genus 1, as the Euler number would suggest.

3.7. **Unusual Cases.** Another strength of Angle-Systems lies in how they deal with unusual cases, such as edges connecting a vertex to itself. When dealing with the Angle-System as a formal algebraic object, these distinctions, which so often lead to painful casework when performing graph theory, are given formal and distinct representation in Angle-Systems, which can be dealt with in full rigor.

Figure 8 illustrates two of these unusual cases, corresponding to points that are fixed in either p or q. Note how easy these are to spot. Other unusual cases might include an edge connecting back to the same vertex; this can be identified as an angle which is mapped by q back to the same vertex, though perhaps not the same angle, as in the loop shown in Figure 8.

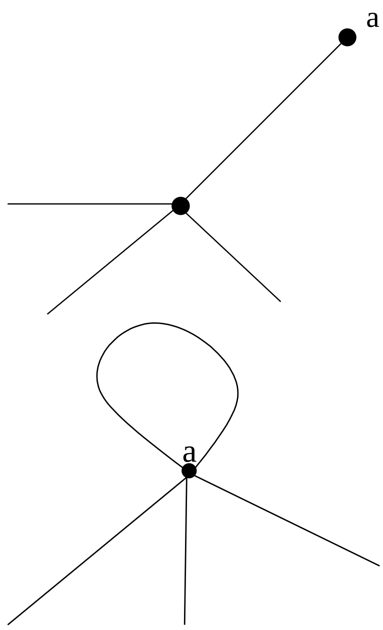

FIGURE 7. If $p(a) = a$, then an isolated vertex is formed, as in the top picture. Dually, if $q(a) = a$, a loop is formed, as in the lower picture. Both of these cases, unusual in typical graph theoretic analysis, are very easy to spot and deal with rigorously in Angle-Systems.

One other strange case that cannot occur in an Angle-System under definition 2.1 is still of interest. If it is allowed that qp have fixed points in addition to two-cycles, then degenerate edges can occur. These are edges which come off of a vertex but do not go to another vertex. They are structurally analogous to loops with no interior face, and are thus drawn as in Figure 9. Their manifestation in the interior graph is also of interest: they correspond to a vertex of degree two (ordinary medial graphs must have only vertices of degree four).

Degenerate edges, prohibited from ordinary Angle-Systems, are usually of no interest. However, they occur inevitably in Quotient Systems to be discussed later. The primary reason for their prohibition is the complications they cause for the computation of χ.

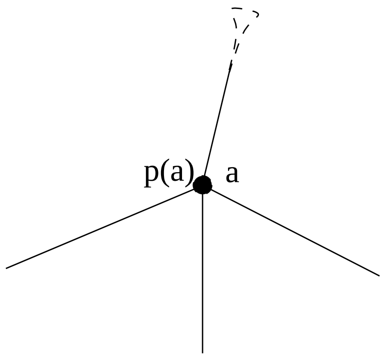

FIGURE 8. Notation for a degenerate edge, which is an edge consisting of only one angle, and correspondent to fixed points in qp, if their prohibition is relaxed.

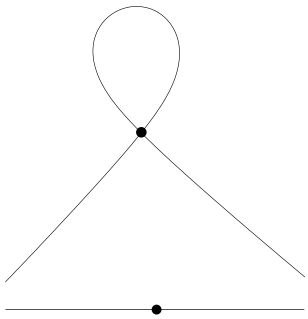

FIGURE 9. The appearance of loops, isolated vertices, and degenerate edges in the medial graph. Loops and isolated vertices, being dual to each other, appear the same in the medial graph: a loop in a geodesic. Degenerate edges correspond to degree two vertices.

4. TRANSFORMATIONS

Substantial information regarding Angle-Systems can be gleaned by considering transformations of existing systems. We consider transformations of the following form.

Definition 4.1. A *standard transform* $T_{m,n}$, where m and n are permutations, from permutation pairs to permutation pairs, is defined by $T_{m,n}(p,q) = (mpn^{-1}, nqm^{-1})$.

Proposition 4.2. *Standard transforms map Angle-Systems to Angle-Systems.*

Proof: Suppose that $T_{m,n}(p,q) = (p_1, q_1)$. Then $q_1p_1 = nqm^{-1}mpn^{-1}nqpn^{-1}$, and q_1p_1. For any angles a, b such that $q_1p_1(a) = b$, premultiplying yields $qpn^{-1}(a) = n^{-1}(b)$. Since qp fixes no points, $n^{-1}(a) \neq n^{-1}(b)$ whence $a \neq b$. Thus (p_1, q_1) is an Angle-System.
□

Observe that the simple transform of relabeling the angles of the system is a standard transform, where $m = n$.

The following observation regarding the composition of permutations will be useful in many of the arguments to follow. All permutations can be decomposed the product of (not generally disjoint) transpositions, which can then be multiplied one at a time.

Proposition 4.3. *If p is a permutation, than composing p with the transposition $(a\ b)$ will increase the number of cycles of p by one if $\ll p \gg a = \ll p \gg b$ (i.e. a and b lie in the same cycle, or $a = p^n(b)$ for some n), and increase the number of cycles by one otherwise.*

Proof: It is not difficult to verify the following identities:

$$(a\ b)(a \dots b \dots) = (a \dots)(b \dots) \tag{1}$$

$$(a\ b)(a \dots)(b \dots) = (a \dots b \dots) \tag{2}$$

$$(\dots a \dots b)(a\ b) = (\dots a)(\dots b) \tag{3}$$

$$(\dots a)(\dots b)(a\ b) = (\dots a \dots b) \tag{4}$$

□

4.1. **Disconnection and Euler's Formula.** We wish to be able to represent the fundamental graph transformations *deletion* and *contraction* on an angle-system. Unfortunately, since both these actions effectively reduce the number of angles by two, they cannot be perfectly represented. The actions can be represented, but always leave an extra component, not connected to the rest of the Angle-System.

Both deletion and contraction hinge on using equations (1) through (4) to break off the edge which is destroyed. All are built up from a basic transformation called *disconnection*.

Definition 4.4. The disconnection of edge e at one of its incident vertices v is $T_{(a,p(a)),e}$, where $a = e \cap v$ (the angle incident at v with leading edge e).

Proposition 4.5. *Upon performing a disconnection, the Euler number χ either remains unchanged or increases by two, remaining unchanged if and only if a and $p(a)$ lie in the same face.*

Proof: Let (p, q) be the original Angle-System, and (p_1, q_1) the result upon applying some disconnection. Then $p_1 = (a, p(a))p$, and since a and $p(a)$ lie in the same cycle of p, $V_1 = V + 1$. $F_1 = F \pm 1$, with the sign dependent on whether a lies in the same face as $p(a)$. $q_1 p_1 = q(a, p(a))(a, p(a))p = qp$, so $E_1 = E$. Thus $\chi_1 = V_1 + F_1 - E_1 = \chi + 1 \pm 1$, and the result follows.
□

We now have the machinery to prove the following crucial result, the analogue of Euler's Theorem in graph theory.

Theorem 4.6. *If (p, q) is a connected Angle-System, then the Euler number χ is an even number less than or equal to* 2*.*

The proof follows directly from the following two lemmas.

Lemma 4.7. *It is possible to apply a series of disconnections to any connected Angle-System, resulting in a system (p_t, q_t) such that $F_t = 1$ and $\chi_t = \chi$.*

Proof: Suppose that the system (p_m, q_m) is a system acquired by applying a series of χ-preserving disconnections so that F_m is minimized. Then for every $a \in A$, $p(a) \in Qa$, for otherwise contraction at a would reduce F and preserve χ. Trivially, $q(a) \in Qa$. Since we can apply any series of p and q and the angle remains in Qa, it follows that $Wa \in Qa$. The system is connected, so $Wa = A$, thus $A \in Qa$. But $Qa \in A$, so $A = Qa$, implying there is only one face on which all angles lie, as desired. □

Lemma 4.8. *For any Angle-System with only one face, χ is even and less than or equal to two.*

Proof: By definition, $F = 1$, so q is a single cycle, and likewise q^{-1}. Now $p = q^{-1}qp$, and qp is expressible as a composition of E transpositions. Each transposition changes the sign of the number of cycles in the resulting permutation, thus carrying out the composition one transposition at a time results in a number of cycles whose sign is congruent to $1 + E \mod 2$. Thus $V \equiv 1 + E \mod 2$, and $\chi = V + F - E \equiv 1 + E + 1 - E \equiv 0 \mod 2$, so χ is even. Also, $V \leq 1 + E$ since V is maximized when each transposition causes an increase in the number of cycles. Thus $\chi = V + F - E \leq 1 + E + 1 - E = 2$. □

With these constraints established on χ, the following definition can be made.

Definition 4.9. For any connected Angle-System, the integer $\frac{1}{2}(\chi - 2)$ is called the *genus* of an Angle-System. When the genus is 0, i.e. $\chi = 2$, the Angle-System is called *planar*.

4.2. **Deletion and Contraction.** As mentioned in the previous section, it is very easy to define deletion and contraction as standard transforms.

Definition 4.10. The *deletion* D_e of edge e is the transform $T_{(a,p(a))(b,p(b)),e}$, where $e = \{a, b\}$ (e in the subscript refers to the identity transform, not the edge, which would be nonsensical). Similarly, the *contraction* C_e of e is the transform $T_{e,(a,q^{-1}(a))(b,q^{-1}(b))}$.

Though somewhat hard on the eyes, this definition follows common sense: to delete an edge, disconnect both its ends, and to contract, perform the dual action of deletion. The inverses must be present to perform the dual action because pq in the dual Angle-System plays the role of qp in the primal, leading to different pairings of angles in edges. Inspection of the example below should make it clear why this is the most convenient definition.

Observe that Both deletion and contraction create disconnected Angle-Systems. In particular, each performs the proper graph-theoretic action on the rest of the angle system and forms an extra component consisting in the two angles a and b. As an example, see Figure 10, where a deletion and contraction are applied to

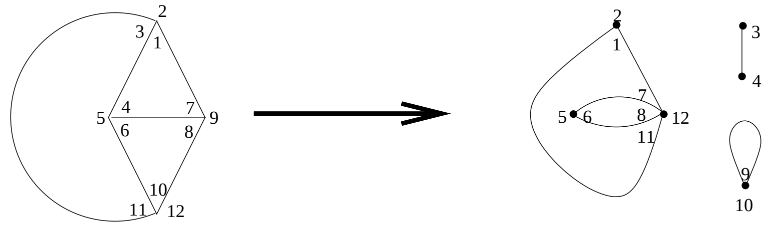

FIGURE 10. The result of performing $D_{\{3,4\}}$ and $C_{\{8,12\}}$ on the tetrahedron graph.

the tetrahedron graph, on edges $\{3, 4\}$ and $\{9, 10\}$, respectively, creating a three-component Angle-System.

In most cases, it is most useful to disregard these extra components, for example in the calculation of χ.

5. Symmetries and Quotient Systems

Another variety of transformation that is of particular interest is a *symmetry*: a permutation on A that preserves the relations p and q. Just like symmetries of plane figures, symmetries of Angle-Systems come in two varieties: proper and reflective. These are defined as follows.

Definition 5.1. A *proper symmetry* of a connected Angle-System is a permutation f such that $fp = pf$ and $fq = qf$. A *reflective symmetry* is a permutation f such that $fp = p^{-1}f$ and $fq = q^{-1}f$.

Proposition 5.2. *The symmetries of an Angle-System are a subgroup of the permutations. Furthermore, the product of two proper or reflective symmetries is a proper symmetry, while the product of a reflective and proper symmetry is a reflective symmetry. The inverse of a proper symmetry is also proper, and likewise for reflective.*

Proof: If f_1, f_2 are two symmetries, the first proper, the second reflective, then $f_1 f_2 p = f_1 p^{-1} f_2 = p^{-1} f_1 f_2$ and $f_1 f_2 q = f_1 q^{-1} f_2 = q^{-1} f_1 f_2$, thus $f_1 f_2$ is a reflective symmetry. The other cases are similar. To see existence of inverses, simply premultiply and postmultiply by f^{-1} in the equations given in the definition. □

One important characteristic of a proper symmetry is that it partitions all the angles into equivalence classes of equal size. This is stated in the following theorem.

Theorem 5.3. *All cycles of a proper symmetry f on a connected Angle-System have equal length.*

Proof: Let a be the angle with minimal order n. Then $f^n w(a) = w f^n(a) = w(a)$, for any $w \in W$. Since the system is connected, any angle can be written as $w(a)$ for some $w \in W$, thus $f^n = e$, and thus n must be the order of all elements. □

Reflective symmetries do not have this property, but they can be simplified in a very pleasing way.

Theorem 5.4. *If a connected Angle-System has a reflective symmetry, then there exists some reflective symmetry with order 2 and at least one fixed point; all reflective symmetries are obtained by composing this point-fixing symmetry with a proper symmetry.*

Proof: Since the square of any reflective symmetry is a proper symmetry, this shall follow as a corollary of the following, more general theorem. □

Theorem 5.5. *If f is a permutation such that f^2 is a proper symmetry, then*

- *Either all elements have the same order, or some have order n, where n is odd, and the rest have order $2n$.*
- *There exist permutations g, h and integer $n \geq 1$ such that h is a proper symmetry, $g^{2^n} = e$, g^2 is a proper symmetry, and $f = gh$. In the case when some element has odd order under f, $n = 1$ and g fixes at least one point.*

Proof: If an element a has order n under f, then if n is odd, a has order n under f^2, otherwise it has order $n/2$ under f^2. Since all elements have the same order under f^2, it follows that if any element has odd order n under f, then all elements have order n or $2n$. If no element has odd order, then all elements have half the order under f^2 as under f, and thus they must all have equal order under f. This establishes the first part of the theorem.

For the second part, there are two cases. If f has elements of odd order n, then $1-n$ is even, thus letting $g = f^n$ and $h = f^{1-n}$ satisfies the stated conditions. If f has only elements of even order $k2^n$, where k is odd, then $g = f^k$ and $h = f^{1-k}$ satisfy the stated conditions. □

Proper symmetries allow for a rather remarkable simplification to take place: the *quotient Angle-System*. All angles which are congruent modulo F (where F denotes $\ll f \gg$) can be identified with each other, and only the relations between these sets need be considered. This notion is made precise by the following theorem.

Theorem 5.6. *If (p, q) is an Angle-System and f is a proper symmetry of the system, there exist permutations p_1, q_1 on the equivalence classes of A under F such that $p_1 Fa = Fpa$, $q_1 Fa = Fqa$, and $(q_1 p_1)^2 = e$, the identity.*

Proof: To see that p_1 and q_1 are well-defined, suppose that $Fa = Fb$. Then $a = f^n(b)$ for some n, so $Fp(a) = Fpf^n(b) = Ff^np(b) = Fp(b)$. The same argument is valid for d. Thus $p_1 Fa$ and $q_1 Fa$ are well-defined. Now note that $(q_1 p_1)^2 Fa = F(qp)^2(a) = Fa$, so $(q_1 p_1)^2 = e$. □

Note that the reason this proof is not valid for reflective symmetries is that n might be odd, in which case $pf^n \neq f^n p$ generally. A more sophisticated definition is required to take quotients by reflective symmetries, which may be developed in later editions of this paper.

Definition 5.7. The pair (p_1, q_1) is called the *quotient Angle-System* of (p, q) by the symmetry f, and is denoted $(p, q)/F$.

One substantial difficulty is that quotient angle systems often degenerate Angle-Systems; it is possible that q_1p_1 has fixed points. It is easy enough to acquire a proper Angle-System from a quotient Angle-System, however: simply remove fixed angles from all cycles. Alternatively, degenerate Angle-Systems can be treated as legitimate structures, with the understanding that they do not have an analogue in graph theory, and if the stipulation is made that only edges containing two angles are counted when calculating χ.

Proposition 5.8. *For any reflective or proper symmetry f, and H being any of F, Q, or G, $fH(a) = Hf(a)$, thus f permutes vertices, faces, and geodesics.*

Proof: If f is a proper symmetry, it commutes with any element of W, and the result is immediate (any holds for any subgroup of W). If f is a reflective symmetry, observe that $fh(a) = h'f(a)$, where h' consists of replacing each instance of p and q in h with its inverse. If $h \in P$ or $h \in Q$, it is clear that h' is in the same subgroup. If $h \in G$, then observe $p^{-1}q^{-1} = qp$ and $q^{-1}p^{-1} = pq$, so $h' \in G$, establishing the result. $\square$

The following theorem may or may not be a discrete version of the Riemann-Hurwitz formula.

Theorem 5.9. *If χ' is the Euler number of the quotient system of an Angle-System with Euler number χ under a symmetry of order n, then*
$$\chi' = \tfrac{1}{n}\chi + \sum_{k|n} \frac{k-1}{n}(V_{n/k} + F_{n/k}) + \frac{1}{n}E_1,$$
where V_k, F_k, E_k denote the number of vertices, faces or edges, respectively, with order k under f.

Proof: Observe $V' = \sum_{k|n} \frac{1}{k}V_k = \sum_{k|n} \frac{k}{n}V_{k/n}$, since all k-cycles of vertices under f are reduced to a single vertex in the quotient system. The second equality results from a change of index and the fact that $k|n$ if and only if $(n/k)|n$. Similarly $F' = \sum_{k|n} \frac{1}{k}F_k = \sum_{k|n} \frac{k}{n}F_{k/n}$. However, we use a different method to calculate E', taking into account that fixed points of q_1p_1 are not to be counted. There are $E - E_1$ edges that are not mapped to themselves. These are constituted from $2(E - E_1)$ angles, which thus become $2(E - E_1)/n$ angles in the quotient system, therefore forming $(E - E_1)/n$ edges. Thus $E' = \frac{1}{n}(E - E_1)$. Using these formulas, we obtain the following.
$$\chi' = \sum_{k|n} \frac{k}{n}V_{k/n} + \sum_{k|n} \frac{k}{n}F_{k/n} - \frac{1}{n}(E - E_1)$$
$$\chi' = \sum_{k|n} \frac{k}{n}(V_{n/k} + F_{n/k}) - \frac{1}{n}(E - E_1)$$
$$\chi' = \frac{1}{n}\sum_{k|n}(V_{k/n} + F_{k/n}) + \sum_{k|n} \frac{k-1}{n}(V_{n/k} + F_{n/k}) - \frac{1}{n}E + \frac{1}{n}E_1$$

$$\chi' = \frac{1}{n}(V + F - E) + \sum_{k|n} \frac{k-1}{n}(V_{n/k} + F_{n/k}) + \frac{1}{n}E_1$$
$$\chi' = \frac{1}{n}\chi + \sum_{k|n} \frac{k-1}{n}(V_{n/k} + F_{n/k}) + \frac{1}{n}E_1. \ \square$$

Corollary 5.10. *A quotient system has lesser or equal genus than the initial Angle-System, with strict inequality when the genus is greater than* 1.

Proof. Notice that $\chi' \geq \frac{1}{n}\chi$. If $\chi = 2$, it follows that $\chi' > 0$ and thus $\chi' = 2$ as well. If $\chi \leq 0$, then $\frac{1}{n}\chi \geq \chi$, with strict inequality if $\chi < 0$, whence $\chi' \geq \chi$, strict if $\chi < 0$. The result follows. $\square$

As an example, the quotient systems created by two different symmetries on the tetrahedron Angle-System shall be presented.

The first symmetry is $f = (1, 10)(2, 11)(3, 12)(4, 8)(5, 9)(6, 7)$. Taking the quotient system, and representing each of the six equivalence classes under F by the angle of lesser index, the following quotient system is obtained, depicted in Figure 11. Observe that two degenerate edges are created.

$p_1 = (1, 2, 3)(4, 5, 6)$
$q_1 = (1, 4, 6)(2, 5, 3)$
$q_1p_1 = (1, 5)(2)(3, 4)(6)$

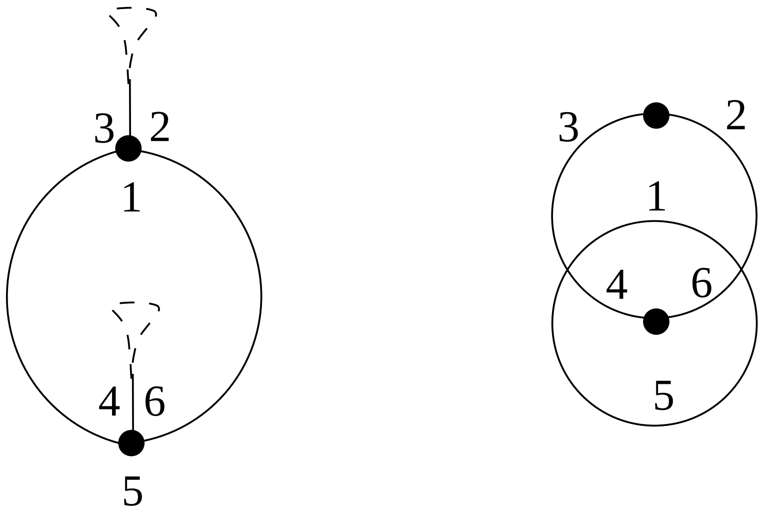

FIGURE 11. The primal and medial graphs of the first example of a quotient system of the tetrahedron.

The second symmetry is $f = (1, 2, 3)(4, 11, 9)(5, 12, 7)(6, 8, 10)$. This produced the quotient system of 4 angles given below and depicted in Figure 11.

$p_1 = (1)(4,5,6)$
$q_1 = (1,4,5)(6)$
$q_1p_1 = (1,4)(5,6)$

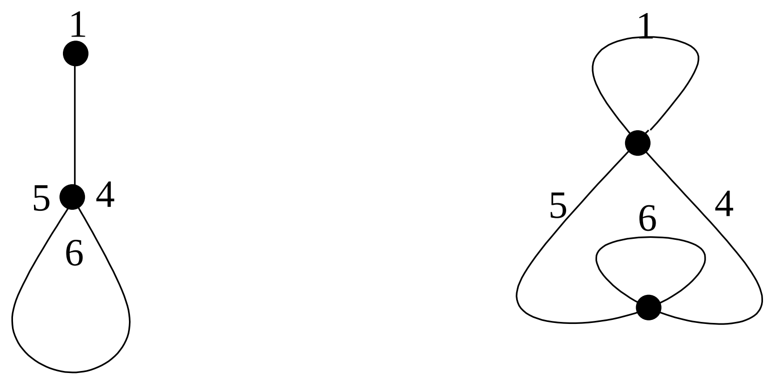

FIGURE 12. The primal and medial graphs of the second example of a quotient system of the tetrahedron.

It is somewhat difficult to gain an intuitive grasp on the geometric effects of quotients on the primal, dual, and medial graphs, but careful study should reveal the underlying patterns.

5.1. **Product Systems.** The natural follow-up question is when it is possible to perform a quotient in reverse, creating a product system. We must understand what must be specified to uniquely determine a product system, and also how to distinguish when several differently specified product systems are isomorphic. This paper answers the first question, but the second is not yet fully explored.

Suppose an Angle-System (p, q) with angles A is given, which is meant to be the quotient system of the product Angle-System (p_P, q_P) with angles A_P, under the proper symmetry f of order n (recall that f must be composed of n-cycles only). Thus each $a \in A$ represents a coset of A_P under f, so we suppose that there is some representative a' such that a corresponds to Fa'. Then for every $a \in A$, there are n angles $f^k(a') \in A_P$. A different notation, however, is slightly more convenient.

Definition 5.11. The angles of a product system are written $\langle a, k\rangle$, $k \in \mathbb{Z}_n$, indicating the kth member of the coset a. Therefore $\langle a, k\rangle = f^k(a')$.

In the above notation, it follows immediately that $f\langle a, k\rangle = \langle a, k+1\rangle$. The representation of p' and q', and what must be specified to uniquely construct them, is given by the following theorem.

Theorem 5.12. $(p, q) = (p_P, q_P)/F$ *if and only if there exist two functions* $u, v : A \to \mathbb{Z}_n$*, such that* p_P *and* q_P *are given by*
$p_P\langle a, k\rangle = \langle p(a), k + u(a)\rangle$
$q_P\langle a, k\rangle = \langle q(a), k + v(a)\rangle$,
and furthermore that $(u + vp + uqp + vpqp)(a) = 0$ *for all* $a \in A$.

Proof: First, suppose that indeed $(p,q) = (p_P, q_P)/F$. Define the function $u' : A \times \mathbb{Z}_n \to \mathbb{Z}_n$ such that for all $\langle a,k\rangle \in A_P$, $p_P\langle a,k\rangle = \langle p(a), u'(a,k)\rangle$. Similarly, define $v' : A \times \mathbb{Z}_n \to \mathbb{Z}_n$ such that $q\langle a,k\rangle = \langle q(a), v'(a,k)\rangle$. These are both well-defined functions since each angle of A_P is uniquely mapped to another by each of p and q. We shall establish that $u'(a,k) = u'(a,0) + k$ and $v'(a,k) = v'(a,0) + k$. The base case $k = 0$ is trivial, and we proceed by induction. Suppose that the result holds for $0 \le k \le h$. Since f is a proper symmetry, $fp_P = p_P f$, thus:
$fp_P\langle a,h\rangle = p_P f\langle a,h\rangle$
$f\langle p(a), u'(a,0) + h\rangle = p_P\langle a, h+1\rangle$
$\langle p(a), u'(a,0) + h + 1\rangle = \langle p(a), u'(a, h+1)\rangle$,
Completing the induction. The proof is identical for v'.
Thus we define $u(a) = u'(a,0)$ and $v(a) = v'(a,0)$, and see that the first two conditions must indeed be true.
To confirm the final inequality, notice that $q_P p_P q_P p_P = e$ implies that for all angles:
$q_P p_P q_P p_P\langle a,k\rangle == \langle a,k\rangle$
$q_P p_P q_P\langle p(a), k + u(a)\rangle == \langle a,k\rangle$
$q_P p_P\langle qp(a), k + u(a) + vp(a)\rangle == \langle a,k\rangle$
$q_P\langle pqp(a), k + u(a) + vp(a) + uqp(a)\rangle == \langle a,k\rangle$
$\langle qpqp(a), k + u(a) + vp(a) + uqp(a) + vpqp(a)\rangle == \langle a,k\rangle$
$\langle a, k + u(a) + vp(a) + uqp(a) + vpqp(a)\rangle == \langle a,k\rangle$,
So indeed we must have $(u + vp + uqp + vpqp)(a) = 0$.
Conversely, suppose that an Angle-System (p,q), and a pair of functions $u, v : A \to \mathbb{Z}_n$, for any n are given. It can be directly verified that if we define a set of angles $A_p == \{\langle a,k\rangle | a \in A, k \in \mathbb{Z}_n$, and permutations $p_P, q_P : A_P \to A_P$ as stated in the theorem, as well as stipulating that $f : A_P \to A_P$ increments the index k of an angle in A_P, then (p_1, q_1) constitutes an angle system, f a proper symmetry of that angle system, and (p,q) corresponds to the quotient system $(p_1,q_1)/F$, as desired.
□

This result allows the construction of all possible product systems from a given Angle-System. What is not clear, however, is how exactly these should be specified, and which product systems are isomorphic to each other. The following theorem, however, gives a means of parameterizing all possible product systems in a fairly efficient manner. First we introduce some terminology.

Definition 5.13. The coefficient k of an angle $\langle a,k\rangle$ in a product system shall be called the *sector* of the angle, and the angle a from the initial system is the *position* of the angle. The functions u and v for each angle of the initial system shall be called, respectively, the *vertex transition* and *face transition* of the angle. The value $(u + vp)(a)$ is called the *edge transition*, which will be denoted $w(a)$.

Thus we interpret the product system as follows: the product creates n sectors of angles, and in each sector is one angle for each position, which correspond to angles of the original system. When any permutation (vertex, face, or edge) is applied to the product system, the position is permuted as in the original system, and then the sector is adjusted by the transition function for that angle and permutation.

Proposition 5.14. *The assignments of vertex and face transitions determine a valid product system if and only if* $w(a) + wqp(a) = 0$ *for all angles. Thus the two angles of every edge have complementary edge transitions.*

Proof: This follows directly from the definition of $w(a)$ and the above theorem. □

We now make the following important observation: we can parameterize all possible product systems of an Angle-System by first assigning not vertex or face transitions, but edge transitions. Simply identify a primary angle for each edge, and assign an edge transition to each primary angle, which uniquely determines the other edge transition. Then either the vertex or edge transitions can be arbitrarily assigned, and the other will be fully determined. For example, if all edge and vertex transitions are knows, we compute the face transition of any angle as $v(a) = (up^{-1} + vpp^{-1} - up^{-1})(a) = (wp^{-1} - up^{-1}(a) = (w - u)p^{-1}(a)$.

The logical next step in the development of a good theory of product systems is to attempt to classify equivalent product systems. For example, of the sector numbers of a given angle of the product system are permuted, the system's structure is unchanged, but the transitions are adjusted. One logical effort might be to attempt to "localize" sectors to the greatest extent possible, so that as many transitions as possible are equal to 0, except at "sector boundaries." All this may be developed in a later edition of this paper, or by some future student.

6. Planarity

It is possible to prove an analogue of the Jordon curve theorem.

Theorem 6.1. *If an Angle-System is planar, then all polygonal paths divide the angles of the Angle-System into two sides.*

The most obvious proof, which is not presented here due to its messy nature and the author's hope that a more elegant solution is yet to be found, is to use standard genus-preserving transforms to shrink the polgonal path down to a single edge, and then showing that at that point a disconnection would increase the Euler number of the graph without creating a new component, showing the Euler number was previously less than 2.

An analogous result for polygonal paths in the medial graph must also hold, and can probably be proved similarly.

This topic is highly incomplete, and future students are highly encouraged to complete it in a satisfying way, and also to extend it to results for higher genuses as well.

7. Self-Duality

We now approach a version of the self-dual problem originally posed by Owen Biesel and Jeff Eaton in [1]: how can one classify those graphs that are isomorphic to their dual? Specifically, [1] deals with planar graphs, and Angle-Systems allow for more headway here than in other cases. The tetrahedron graph mentioned earlier is an example of a self-dual graph.

Definition 7.1. A permutation f of the angles of an Angle-System (p, q) is a proper self-duality if $pf = fq$ and $qf = fp$. It is called a reflective self-duality if $pf = fq^{-1}$ and $qf = fp^{-1}$.

Proposition 7.2. *If f is a self-duality, then f^2 is a proper symmetry.*

Proof: This follows almost directly from the definition.
□

Thus we have the results from a previous section that all the cycles of f either have the same order, or orders n and $2n$, where n is odd, and $f = gh$, where h is a proper symmetry, and g is composed either entirely of 2^{2^n} cycles, for some n, or has order 2. Clearly $g = fh^{-1}$, so we have the following.

Proposition 7.3. *If an Angle-System is self-dual, there exists a self-dual permutation g with all cycles of length 2^{2^n} for some $n \geq 1$, or all cycles of length 1 and 2.*

We will thus typically only concern ourselves with self-dualities of the form above, since they are sufficient to characterize all self-dual graphs.

A neat result that is useful in some casework is the following.

Proposition 7.4. *For any angle a, $Gf(a) = f(Ga)$. Thus f can be regarded as specifying a permutation of the geodesics.*

Proof. It is easy to verify that, in both the reflective and proper cases, $f\{pq, qp\} = \{pq, qp\}f$. Thus $\forall g \in G$, $fg(a) \in Gf(a)$ and $gf(a) \in f(Ga)$, thus $Gf(a) = f(Ga)$. □

7.1. **Constructing Self-Duality.** We now prove a result regarding the uniqueness of the self-dual mapping f. We first must define the dual of a walk, as follows.

Definition 7.5. If we are considering proper self-duality, the dual $\bar{w}$ of any walk $w \in W$ is the walk in W obtained by replacing every p with a q and vice versa. If we are considering reflective self-duality, $\bar{w}$ is obtained by replacing every p with q^{-1} and vice versa. Thus for all $w \in W$, $fw = \bar{w}f$.

Proposition 7.6. *Every valid self-dual on a connected Angle-System is completely determined by its effect on a single angle.*

Proof. Suppose it is given that $f(a) = b$. Connectedness implies that any other angle can be written as $w(a)$, and $fw(a) = \bar{w}f(a) = \bar{w}(b)$. Thus the value of f at every angle is determined. □

This naturally raises the question: for which assignments of $f(a)$ will a valid self-duality f be produced this way? There are two problems that could arise: the definition procedure in the above proposition may be ill-defined, or the function may not be a self-duality. The conditions where these problems do not arise are expressed in the following theorem.

Theorem 7.7. *Let $W_a = \{w \in W | w(a) = a\}$. Then there exists a self-duality on the Angle-System if and only if there exists another angle b such that $\bar{W}_a$ fixes b.*

Proof: To see that this is a necessary condition, note that for all $w \in W_a$, $\bar{w}f(a) = fw(a) = f(a)$, thus $\bar{W}_a$ must fix the point $w(a)$. For sufficiency, note that if such a, b exist, we can define $fw(a) = \bar{w}(b)$. To see that this is well-defined, Suppose that $w, v \in W$ and $w(a) = v(a)$. Then $v^{-1}w$ fixes a, therefore $\bar{v}^{-1}\bar{w}$ fixes b, or $\bar{w}(b) = \bar{v}(b)$, and thus f is well-defined. To see that f is a self-duality is fairly straightforward. □

This theorem provides a closed-form criterion for a graph being self-dual, properly or improperly, but it would be very difficult to verify in practice. It's strength, however, lies in its elimination of the "there exists an f" quantifier.

7.2. **Self-Duality Preserving Transformations and Quotients.** It is sometimes possible to simplify a self-dual angle system by applying a self-dual preserving transformation. It is the author's hope that this will, in time, yield a method that can be used to generate all self-dual graphs. These transformations take a form specified by the following theorem.

Theorem 7.8. *A transformation $T_{m,n}$ preserves self-duality, with the same map f, if the permutations m, n satisfy the following conditions.*

- *For proper self-duals: $mf = fn, nf = fm$.*
- *For reflective self-duals: $mf = fm, nf = fn$.*

Proof: For proper self-duals, if m, n obey the given conditions, then $fmpn^{-1} = nfpn^{-1} = nqfn^{-1} = nqm^{-1}f$, and similarly $fnqm^{-1} = mpn^{-1}f$, so indeed f is still a self-duality. For reflective self-duals, we similarly verify that $fmpn^{-1} = mq^{-1}n^{-1}f = (nqm^{-1})^{-1}f$ and $fnqm^{-1} = np^{-1}m^{-1}f = (mpn^{-1})^{-1}f$, thus f is still a reflective self-dual. □

We also have the following result, which allows substantial simplification of self-dual graphs.

Theorem 7.9. *If f is a self-duality of an Angle-System (p, q), then the permutation f_1 over the equivalence classes F^2A given by $f_1F^2a = F^2f(a)$ is well defined and also a self-duality for the quotient system $(p, q)/F^2$, where F^2 denotes $\ll f^2 \gg$.*

Proof: First, to see that f is a well-defined permutation of the equivalence classes under F^2, note that if $F^2a = F^2b$, then $F^2f(a) = fF^2a = fF^2b = F^2f(b)$. In the proper self-dual case, $fp_1F^2a = fF^2p(a) = F^2fp(a) = F^2qf(a) = q_1F^2f(a) = q_1fF^2a$, and similarly $fq_1F^2a = p_1fF^2a$. The reflective case is analogous. □

Thus any self-dual Angle-System can be reduced by a quotient to a self-dual (possibly degenerate) Angle-System with f of order 2. Alternatively, almost as significant a simplification can be made as in the following proposition.

Proposition 7.10. *Any self-dual graph possesses a self-duality permutation of order 2^n, for some nonnegative integer n.*

Proof: Let k be the highest odd factor of the order of f. Then f^k is the desired permutation. This follows because either all cycles of f have the same order $k2^n$, or some have order k, some order $2k$. □

7.3. **1-2 Self-Duals.** We define those self-dualities where the cycles of f have orders 1 and 2 $1-2$ *Self-Duals*. Any Angle-System with a self-duality having one element of odd order possesses a $1-2$ self-duality. They turn out to be fairly easy to classify, by considering only a single fixed angle.

Suppose f is a self-duality of a connected Angle-System, and a is a fixed point of f. All other angles can be written as $w(a)$ for some $w \in W$. Then f must obey $fw(a) = \bar{w}(a)$. Geometrically, this means that the Angle-System is that of a medial graph which is reflectively symmetric in the ordinary sense, and thus any self-dual embedded graph is topologically equivalent to a graph symmetric in this way. Specifically, if f is proper, the medial graph is symmetric by a reflection through the midpoint of the edge correspondent to a, and if f is reflective, the medial graph is symmetric by a reflection across the geodesic of a.

Some rigor is necessary in this argument, as in all the arguments herein that refer to graphs rather than pure angle-systems. Nevertheless, once this is provided, this provides some vindication for the conjecture at the end of [1].

8. Circular Planar Graphs

One problem in attempting to use Angle-Systems to find results relevant to REU research is that there is no clear definition of boundary nodes, and the notion of the dual graph is notably different in the two cases. Several solutions come to mind, but the author has found none to be totally satisfactory yet. One is simply to take a circular planar graph, and create one more vertex connected to all boundary nodes, but this is not preserved in creating the dual graph. Another idea is to place something identifiable on the boundary nodes, such as a degenerate edge, but again this does not behave well in the dual graph.

The most promising option so far is to revise the definition of Angle-Systems to have *incomplete permutations*, which is to say objects that are like permutations, except that some angles are permuted to nothing. Thus the angles on the boundary simple are the "ends" of cycles. Consider the example in figure 13, which is related to the spherically embedded tetrahedron graph, and is also self-dual in the sense discussed in [1] and other REU papers.

We use square braces (e.g. [ ]) to indicate incomplete permutations, i.e. the final element is not mapped to any element, and no element is mapped to the first. Thus the graph is represented by the following Angle-System of incomplete permutations:

$p = [1\ 2\ 3][4\ 5\ 6](7\ 8\ 9)[10\ 11]$
$q = [6\ 1](2\ 5\ 7)[3\ 9\ 10][11\ 8\ 4]$
$qp = (1\ 5)(2\ 9)[3](4\ 7)[6](8\ 10)[11]$

This representation possessing many aesthetic qualities, and if nothing else is an efficient way to represent a graph with a boundary, in the style used by the REU. It is also easy to convert between the graph and its dual. However, it is not clear how to perform algebra on incomplete permutations. In particular, though they form a monoid (we can compose them), they are not a group, since it is not obvious how to create inverses. It is conceivable that with some effort this model can be revised

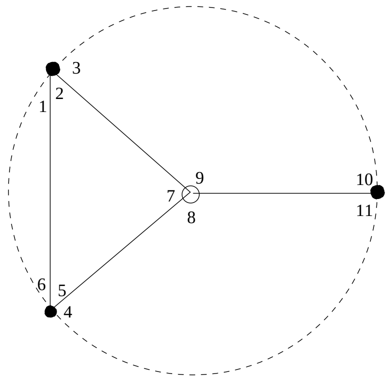

Figure 13. A circular planar graph, in the style used by the REU.

into a robust algebraic structure, lending itself to analysis in the same capacity as Angle-Systems.

9. Further Research

This paper constitutes only a compilation of ideas the author feels could be very productive in an algebraic approach to various graph theoretic problems; it remains to actually solve these problems. There are countless unanswered questions which lend themselves to further research in the vein of the ideas proposed in this paper. Some that the author particular wishes he had the time to explore (and may explore yet) are the following:

- First and foremost, someone with a deeper background in Algebra should carefully examine this theory, and see what marvelous simplifications and extensions may result from less elementary methods.
- The self-dual problem is still significantly incomplete. A little effort might resolve the entire issue with a simple paradigm.
- The more general self-dual problem, where ordering of angles around a vertex is no longer pertinent, might be attacked, though there is strong evidence that Angle-Systems would be unideal in this effort (though their strength lies in the easy creation of dual graphs).
- It should be a fairly routine matter to prove a version of the Jordon curve theorem as a planarity criterion. Further criteria might be explored, as well as a deeper theory of genus in general, and what indicators might exist upon considering only p and qp, without explicitly working out q.
- There is no reason the set of angles must be finite; a great deal of theory might be developed for bijections of the integers, uncountable sets, or even general groups. Applications of such a theory might extend far beyond simple graph theory.
- The topic of geodesics and medial graph has not been explored in great depth. Many of the informal ideas expressed in [1] might find excellent expression if the theory of Angle-System geodesics is developed and understood further.

- The topic of simplifying systems by various standard transforms may provide further headway in the self-dual problem, perhaps classifying a set of "primitive" self dual systems with lead, in turn, to the others. Such primitive systems might have pleasing descriptions in terms of geodesics.
- The problem of applying the methods of Angle-Systems to the graphs with boundaries considered in the REU is very open, as discussed in the final section.
- The abstract structures knows as *Matroids* have provided significant headway in some graph-theoretic problems similar in nature to the problems that inspired Angle-Systems. There may by connections to be found between the two abstractions.
- Angle-Systems provide a good stage for the consideration of *coloring problems*, such as the four-color theorem and its generalizations to higher genus surfaces. Perhaps some interesting results could be discovered in this realm.
- It would be easy enough to program an applet to work with Angle-Systems and perform various actions on them. The creation of a robust software environment might be a good first step in all of the above areas of study.
- This paper should be thoroughly read over and corrected, as the author may have left many technical or more significant errors.

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