SPECTRAL RESULTS FOR THE GRAPH LAPLACIAN

JUSTIN TITTELFITZ

Abstract. In this paper, we consider the Laplacian Operator on graphs, along with its eigenvectors and eigenvalues. After establishing preliminaries, we give eigenvector expansions for solutions of Electrical Network Boundary Value Problems. We then state some results for the nodal domains of our eigenvectors.

Contents

1. Introduction

In this paper, we will explore properties of the Graph Laplacian, with a specific focus on the eigenvalues and eigenvectors. We will mainly concern ourselves with the Weighted Laplacian, which is the appropriate operator for Electrical Conductivity Networks. This belongs to a more general set of objects that can be referred to as Graph Laplacians.

1.1. Graph Laplacians.

Definition 1.1. Consider a connected, undirected graph G, with vertices V and edges σ. Consider also a vector f defined at the vertices of G. The Laplacian Operator Δ, acts on f as

$$\Delta f(x) := \sum_{x \sim y} (f(x) - f(y))$$

This operation can be represented by the adjacency matrix L of this graph

$$l_{ij} = \begin{cases} -1, & \text{if } i \neq j \text{ and there is an edge connecting } v_i \text{ and } v_j \\ -\sum\limits_{i \neq j} l_{ij}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

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Definition 1.2. We can also consider the Laplacian for graphs with an edge function γ, with the requirement that it is symmetric, ie, $\gamma(x,y) = \gamma(y,x)$. The Weighted Laplacian Operator Δ_γ, acts on f as

$$\Delta_\gamma\ f(x) := \sum_{x\sim y} \gamma(x,y)(f(x) - f(y))$$

This operation can be represented by a matrix K given by

$$k_{ij} = \begin{cases} -\gamma_{ij}, & \text{if } i \neq j \text{ and there is an edge connecting } v_i \text{ and } v_j \\ \sum\limits_{i\neq j} \gamma_{ij}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Note that this definition coincides with that of the Kirchoff matrix, K, for Electrical Conductivity networks. We can also view the Laplacian from our last definition as a special case of the Weighted Laplacian; this would correspond to a graph with every edge weight equal to 1. We will refer to these types of graphs and operators as 'Regular'.

Definition 1.3. Consider a sub-graph S of G. The Weighted Sub-Graph Laplacian Operator Δ_γ^S, acts on f as

$$\Delta_\gamma^S\ f(x) := \sum_{x,y\in S} \gamma(x,y)(f(x) - f(y)) + w(x)f(x)$$

where $w(x) = \sum\limits_{y\notin S} \gamma(x,y)$. Note $w(x) \geq 0$ for all $x \in S$ and $w(x) > 0$ for some $x \in S$ as long as $S \neq G$. This operation can be represented by the appropriate submatrix of K; the submatrix C is a prime example.

It is important to note that all of these constructions yield symmetric matrices, a critical property for the analysis which will follow. At this point we should also note that the first two constructions are positive semi-definite, while the third is positive definite. This can be easily verified by examining the quadratic forms of these matrices.

1.2. **Eigenfunctions.** Throughout this paper, we will focus on eigenvalues and eigenvectors of these matrices, and it will be convenient to establish some basic facts and notation conventions. Since the matrices we are considering are symmetric, they are also orthogonally diagonalizable. We will order our eigenvalues in ascending order

$$\lambda_1 \leq \lambda_2 \ldots \leq \lambda_i \leq \ldots \leq \lambda_n$$

We will also need to allow for the possibility of repeated eigenvalues

$$\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+r} = \lambda_{k+r-1} < \lambda_{k+r}$$

Here, the eigenvalue λ_k has a multiplicity of r. When speaking in the context of repeated eigenvalues, we will use k^- to denote the lowest of these indices, and k^+ to indicate the greatest. Finally, we will choose an orthonormal basis of eigenvectors so that ϕ_i is the eigenvector corresponding to λ_i.

Definition 1.4. Consider an Electrical Conductivity Network on a graph G, with Kirchoff matrix K. A Dirichlet eigenfunction for an Electrical Conductivity Network is a function f, such that $f(x) = 0$ for $x \in \partial V$, and that there is a function ϕ so that

$$\phi(x) = f(x) \ for x \in int(V)$$
$$and \ C\phi = \lambda\phi$$

The way we have used boundary data to think about and define a Dirichlet eigenfunction leads to the following interpretation of what it means to be a Neumann eigenfunction. It may, however, be equally (or more) reasonable to think of eigenvectors of the Kirchoff matrix as Neumann eigenvectors, due to their usefulness in solving Neumann problems. This will be discussed further in the next section.

Definition 1.5. A Neumann eigenfunction for an Electrical Conductivity Network is a function f, such that $\nabla_{e\perp} f(x) = 0$ for $x \in \partial V$, and that there is a function ϕ so that

$$\phi(x) = f(x) \ for x \in int(V)$$
$$and \ (C - B^T A^{-1} B)\phi = \lambda\phi$$

Definition 1.6. The Neumann response matrix for an Electrical Conductivity Network is a matrix Λ_N which is given by taking the Schur complement of K with A. In other words, $\Lambda_N = C - B^T A^{-1} B$. Note the difference between this and the usual response matrix for an Electrical Network, given by $\Lambda = A - BC^{-1}B^T$. It should be noted that this matrix effectively switches the role of interior and boundary nodes, and if multiplied by a vector of potentials on the interior, produces the current from the interior of the graph to the boundary.

Lemma 1.7. *For a positive semi-definite, symmetric matrix K, 0 is an eigenvalue of multiplicity one, and the corresponding eigenvector is a scalar multiple of* $[1, 1, \ldots, 1]^T$.

Proof. Because K is symmetric, the algebraic and geometric multiplicities of its eigenvalues are the same. Now consider the quadratic form

$$\langle \phi, K\phi \rangle = \frac{1}{2} \sum_{x,y:x\sim y} \gamma(x,y)(f(x) - f(y))^2 = 0$$

if and only if $f(x) = f(y)$ for all $x, y \in V$. □

1.3. **Comparison to Continuous Problems / Gamma Harmonicity.** At this point, the reader may note our eigenfunctions are not, in general, gamma-harmonic. It is critical to point out that this is neither required, nor detrimental. We will briefly describe some continuous analogs in order to dispel any fears. In the continuous case, the wave equation is given as

$$\partial^2_{tt} u = \Delta u$$

In order to solve Boundary Value/ Initial Value Problems involving this PDE, one generally separates variables, and, in the spatial case, arrives at the eigenvalue problem

$$\Delta\phi = \lambda\phi$$

It is also required that ϕ satisfy the same boundary data as u. While solutions of this equation will not satisfy the original PDE, they are enormously useful in expressing the solutions we are interested in. Another interesting PDE to consider is Poisson's equation

$$\Delta u = \rho$$

Boundary Value Problems involving this PDE can be solved using a Green's Function approach, or an eigenfunction expansion. This second approach once again considers solutions of

$$\Delta \phi = \lambda \phi$$

that satisfy the same boundary data as u. Again, it is clear that these functions do not satisfy the PDE; their use is as a basis for the solutions we are interested in.

2. Solutions of Forward Problems

Throughout this section, we will consider the following (familiar) boundary value problem

$$Ku = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} u_B \\ u_I \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} = g \tag{1}$$

The Dirichlet boundary value problem assumes u_B, while the Neumann assumes g.

Theorem 2.1. *Consider the Dirichlet boundary value problem for an Electrical Conductivity Network Let $\{\lambda_i, \phi_i\}$ be solutions of the eigenvalue problem*

$$C\phi = \lambda \phi$$

The solution of the above BVP can be expressed as

$$u_I = -\sum_{i=1}^{n} (1/\lambda_i)\langle B^T u_B, \phi_i \rangle \phi_i \tag{2}$$

Where $\langle , \rangle$ is the usual inner product.

Proof. Assume that u_I can be expressed in the form $u_I = \sum\limits_{i=1}^{n} a_i \phi_i$. From matrix multiplication, we also know that

$$-B^T u_B = Cu_I = \sum_{i=1}^{n} a_i C\phi_i = \sum_{i=1}^{n} a_i \lambda_i \phi_i$$

Taking an inner product with ϕ_j then yields $-\langle B^T u_B, \phi_j \rangle = a_j \lambda_j$, and finally gives $a_i = -(1/\lambda_i)\langle B^T u_B, \phi_i \rangle$, thus giving the solution as stated in the theorem. Note that Δ_γ^S is a positive definite form, and so this last manipulation is valid. □

Theorem 2.2. *Consider the Neumann boundary value problem for an Electrical Conductivity Network. Let $\{\lambda_i, \phi_i\}$ be solutions of the eigenvalue problem*

$$K\phi = \lambda \phi$$

The solution of the above BVP can be expressed as

$$u = -\sum_{i=2}^{n} (1/\lambda_i)\langle g, \phi_i \rangle \phi_i \; + \; c\phi_1 \tag{3}$$

The last term of this expansion is due to the property that solutions to Neumann problems are unique up to a constant.

Proof. Assume that u can be expressed in the form $u = \sum_{i=1}^{n} a_i\phi_i$. From our boundary data, we know

$$g = Ku = \sum_{i=1}^{n} a_i K\phi_i = \sum_{i=1}^{n} a_i\lambda_i\phi_i$$

At this point, the analysis differs somewhat from that of the Dirichlet case; we now have a positive semi-definite form, and as such, our first eigenvalue is zero. This has the result of leaving a_1 undetermined. We are still able develop an expression for the other coefficients by taking an inner product, yielding $a_i = -(1/\lambda_i)\langle g, \phi_i\rangle$ for $2 \le i \le n$, and finally giving the solution stated in the theorem. $\square$

We can use a similar approach to solve the Neumann problem in terms of the eigenvectors of Λ and Λ_N.

Theorem 2.3. *Consider the Neumann boundary value problem for an Electrical Conductivity Network. Let $\{\lambda_i, \phi_i\}$ and $\{\mu_i, \psi_i\}$ be solutions of the eigenvalue problems*

$$\Lambda\phi = \lambda\phi$$
$$\Lambda_N\psi = \mu\psi$$

The solution of the above BVP can be expressed as

$$u_B = \sum_{i=2}^{n}(1/\lambda_i)\langle f, \phi_i\rangle\phi_i \ + \ c\phi_1 \tag{4}$$

$$u_I = -\sum_{i=2}^{n}(1/\mu_i)\langle B^TA^{-1}f, \psi_i\rangle\psi_i \ + \ c\psi_1 \tag{5}$$

Again, the constant terms in the expansion are from the non-uniqueness of Neumann problems.

Proof. Assume that u_B and u_I can be expressed as linear combinations of their eigenvectors. From matrix multiplication, we know $Au_B + Bu_I = f$ and thus, $u_B = A^{-1}(f - Bu_I)$. Note A is invertible, as any priniciple, proper submatrix of K is positive definite. We also know that $B^Tu_B + Cu_I = 0$. Substituting for u_B gives $B^TA^{-1}f - B^TA^{-1}Bu_I + Cu_I = 0$. Rearranging this expression finally yields $\Lambda_N u_I = -B^TA^{-1}f$. We also have, as a basic result, $\Lambda u_B = f$. From our assumptions, we then know

$$f = \Lambda u_B = \sum_{i=1}^{n} a_i\Lambda\phi_i = \sum_{i=1}^{n} a_i\lambda_i\phi_i$$
$$-B^TA^{-1}f = \Lambda_N u_I = \sum_{i=1}^{n} b_i\Lambda_N\psi_i = \sum_{i=1}^{n} b_i\mu_i\psi_i$$

Once again, we have positive semi-definite forms, and as such, our first eigenvalues are zero, and the first constants in our expansion are undetermined. Still, we take an inner product, solve for the coefficients, and get the solution stated in the theorem. $\square$

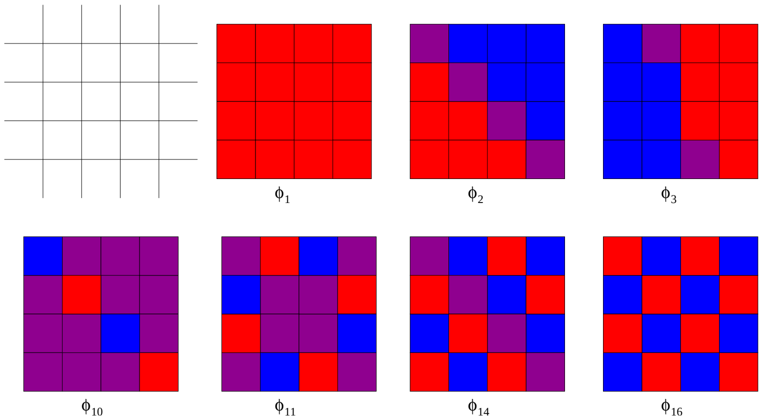

FIGURE 1. Nodal Domains for Dirichlet Eigenfunctions

Corollary 2.4. *Using the eigenvalues and eigenvectors of* Λ*, we can define the Neumann to Dirichlet map* $\tilde{\Lambda}$*, as*

$$\sum_{i=2}^{n}(1/\lambda_i)\phi_i\phi_i^T$$

Then, up to a constant

$$u_B = \tilde{\Lambda} f$$

In this sense, $\tilde{\Lambda}$ acts as a kind of inverse to the response matrix (though neither of these matrices are invertible). Consider the $n-1$ dimensional subspace $W \subset \mathbb{R}^n$ spanned by the eigenvectors of Λ that are orthogonal to the constant vector $C = [1,1,\ldots,1,\ldots,1]^T$. Recall that for our BVP to be well defined, the current on the boundary must sum to zero. This is equivalent to $f \in W$. Then Λ surjectively maps potentials from $\mathbb{R}^n$ to currents in W, and $\tilde{\Lambda}$ injectively maps currents from W to potentials in $\mathbb{R}^n$. If we restrict our potentials to $\mathbb{R}^n/C$, then these maps are bijective.

3. Nodal Domain Theorems

Definition 3.1. A Strong Nodal Domain, D, of a function f on a graph G is a maximal, connected set of vertices such that, for all $x, y \in D$, $f(x)f(y) > 0$. A Weak Nodal Domain, D,of a function f on a graph G is a maximal, connected set of vertices such that, for all $x, y \in D$, $f(x)f(y) \geq 0$.

In particular, we are interested in the nodal domains of our eigenfunctions on the interior of our graph. In figure 1 on page 6, some nodal domains of a regular graph are illustrated. The red areas correspond to vertices where the eigenfunction takes on a positive value, the blue areas are negative, and the purple are zeros. The eleventh eigenfuntion, ϕ_{11} has 8 strong nodal domains, and 2 weak nodal domains. In the continuous case, there is a rather famous theorem due to Courant that states

Theorem 3.2. *Given the self-adjoint, second order differential equation*

$$L[u] + \lambda\rho u = 0 \ (\rho > 0)$$

if the eigenfunctions are ordered according to increasing eigenvalues, then the nodes of the n-th eigenfunction divide the domain into no more than n subdomains.

The reader is directed to [2] for the proof. There is a analog to this theorem in the discrete case.

Theorem 3.3. *If ϕ is an eigenfunction of Δ_γ^S corresponding to λ_k, then ϕ divides S into no more than k^- Weak Nodal Domains, and no more than k^+ Strong Nodal Domains*

A detailed proof is given in [3].

Theorem 3.4. *Consider a Dirichlet eigenfunction f on an Electrical Conductivity Network. If we let v be a restriction of f to one of its Strong Nodal Domains D, then v is a Dirichlet eigenfunction of D.*

Proof. Let

$$v(x) := \begin{cases} \phi(x), & \text{if } x \in D \\ 0, & \text{otherwise} \end{cases}$$

We have $C\phi = \lambda\phi$. We would like to show $C'v = \lambda v$. Before we can begin the proof, we must give a construction for C'. We will begin with the square submatrix of C whose rows and columns are those corresponding to the points in D. Then, we will construct a boundary for D, by inserting a boundary vertex in the middle of any edge that connects a vertex $x \in D$ to another vertex in $y \in int(V) - D$. This will have the effect of turning our original edge, with conductance γ into two new edges, whose conductance we will designate with γ' and γ''. We will require that this pair of edges has the same effective conductance as the original, in other words

$$\frac{\gamma'\gamma''}{\gamma' + \gamma''} = \gamma$$

We will also require that the net current flow is 0 at this new vertex, in other words

$$\gamma'\phi(x) + \gamma''\phi(y) = 0$$

Simple manipulation shows these two conditions are equivalent to

$$\gamma' = \frac{\gamma[\phi(x) - \phi(y)]}{\phi(x)}$$

Then C' is formed by switching entries corresponding to γ with the appropriate value γ'. We will now complete the proof by showing that $C\phi(x) = C'v(x)$ for $x \in D$. This expression can be written as

$$\sum_{y\sim x} \gamma_{xy}[\phi(x) - \phi(y)] + w(x)\phi(x) = \sum_{y\sim x} \gamma'_{xy}[v(x) - v(y)] + w(x)v(x)$$

For any edges whose conductance did not need to be changed, the terms of this sum will cancel, since $\phi(x) = v(x)$ for $x \in D$ by definition. The term involving $w(x)$ is unchanged in our construction, so this cancels as well. We are left to verify

$$\gamma_{xy}[\phi(x) - \phi(y)] = \gamma'_{xy}[v(x) - v(y)]$$

for the edges we changed. By construction, $v(x) = \phi(x)$, $v(y) = 0$, and $\gamma' = \frac{\gamma[\phi(x)-\phi(y)]}{\phi(x)}$, finishing the proof. □

4. Suggestions for Future Work

For those interested in pursuing spectral questions, [4] will be an invaluable reference. In the future, it may be productive to ask

- What useful properties or recoverability questions can be found by examining $\tilde{\Lambda}$?
- The solutions presented in this paper can be thought of as Fourier series on graphs. For infinite graphs, is there a reasonable analog of the Fourier transform?
- What useful properties (if any) does Λ_N have? Specifically, for what kinds of graphs (if any) could this be useful, in terms of recoverability?
- Is there a good way to characterize or utilize the bases of Λ and Λ_N in terms of the basis for K?

References

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University of Oregon
E-mail address: `jtittelfitz@gmail.com`