THREE 3-TO-1 GRAPHS WITH POSITIVE CONDUCTIVITIES

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1. Introduction

I have looked at variations of the "threehands" graph (Fig 1) and tried to figure out how we can look for response matricies that correspond to all-positive Kirchoff matrix. So far, I was unable to make this into an exact science, so some arguments deal with each graph messily and individually. However, I have found that:

(1) The graph with no inversions has only one positive conductance corresponding to a response matrix. (This is old news)
(2) The graph with one inversion can have three different positive Kirchoff matrix that correspond to the same response matrix.
(3) The graph with two inversions can have also have three.
(4) That same graph is labeled "recoverable" by Nick Addington's program. So, either the program is buggy (which is unlikely. I may verify it by hand, but I tend to trust the program), or this is a counterexample to Nick Addington's method.
(5) However, Nick Addington's method is still meaningful for that graph. While I don't yet understand the way it works for the graph, it is unlikely that it is a coincidence that the graph Addington's method labels "recoverable" can't have all of its Kirchoff matrices positive. All of my methods of proving that are very ugly.

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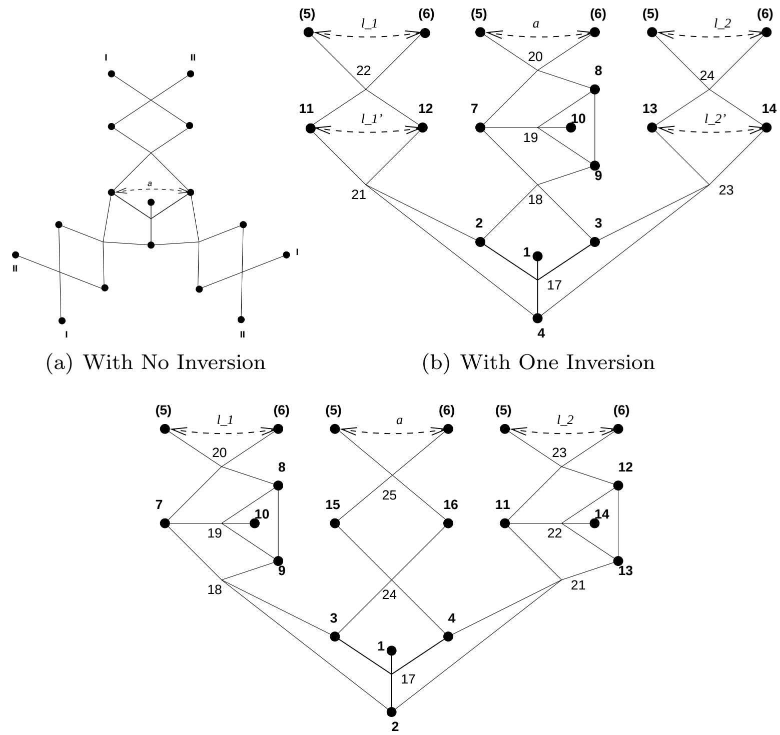


(a) With No Inversion (b) With One Inversion

(c) With Two Inversions

Figure 1. “Threehands” Graphs. Note the subtle differences in the numberins of the vertices.

This turned out to be a lie. Both the graph with one or two inversions can have three real positive Kirchoff matrices correspond to the same response matrix.

2. Parametrizing Response Matrices

First of all, let us figure out the conditions for a matrix Λ to be a response matrix that corresponds to any (signed, or even complex) Kirchoff matrix. This is possible because the graphs we deal with are composed from four-stars that are connected in a very simple way (as “multiplexers”). In fact, we could say that we are dealing with this kind of graphs exactly because it is easy to parametrize their response matrices.

Four every four-star (or even an n-star), the response matrices are parametrized easily. As in my other paper, consider the four-star with boudary vertices 1, 2, 3, 4 and an interior vertex 5. If a response matrix satisfies the equation $\lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{kl}$ for all i, j, k, l, we simply use the following formulas to get the conductances in the Kirchoff matrix:

$$\gamma_i = -\lambda_{ii} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} = \sum_{i \neq j} \lambda_{ij} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} \tag{1}$$

If the response matrix does not satisfy $\lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{kl}$, we say that no Kirchoff matrix corresponds to it.

Now, as Jeff Russel discusses in [5], if all of the λ's are positive, this corresponds to our usual notion of response matrices as Schur complements of response matrices. However, we do not want to limit ourself to these cases. Instead, we allow λ's to have any complex or infinite values (as long as the equations (1) make sense, but this is always the case for our multiplexers) and then use the equations (1) as axioms.

So, from now on,

Definition 2.1. For a graph G with n vertices and b boundary vertices,

- By a *response matrix,* we mean any $b \times b$ matrix Λ with possibly complex or infinite entries that is symmetric and has row sums 0.
- By a *parametrization of response matrices* we mean some correspondence that assigns a finite (possibly empty) set of Kirchoff matrices (any $n \times n$ matrices with row sums 0) to every response matrix.
- We say that G is *recoverable* if for every response matrix, there is at most one Kirchoff matrix that corresponds to it.

We will never use the exact nature of this correspondence (Schur complements, etc.) in what follows. We will only use equations (1), some rules for parametrizations of larger graphs in relation to their parts, and the fact that:

Theorem 2.2. *For real, positive response matrices Λ for a graph G made of n-stars, their notion of parametrization induced by* (1) *in the sense of definition 2.1 and our usual notion of parametrization from the Schur complement is the same.*

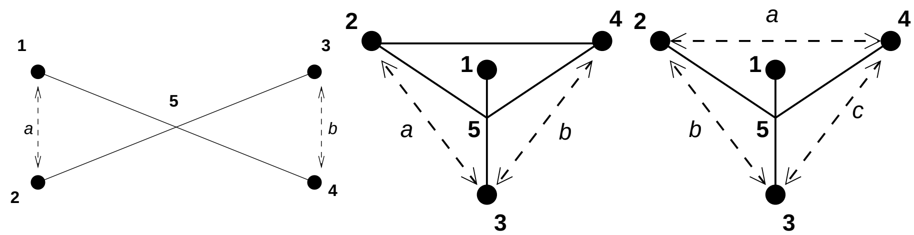
Proof. Omitted. It should be clear; what is lacking for now is a precise definition of how to build parametrizations of larger matrices from smaller ones in full generality. For the way we are going to do it here, see below. □

2.1. **... abstractly.** In our case, we consider the four-stars as "multiplexers" and the graph as a series of multiplexers. The properties of the multiplexers are designed to ease the process of building a large graph for which we know how to parametrize the response matrices from small ones. If we try to parametrize the response matrix of a mutliplexer, we find that we can consider some of the entries in this matrix (the "outputs") separately from the rest. More precisely,

Definition 2.3. A graph $M = (V; E)$ together with a set $B \subset V$ of boudary edges and a set $O \subset B \times B$ is a mutiplexer (or an n-multiplexer if $n = |O|$) if:

1. The set $B \times B$ is partitioned into disjoint subsets as $B \times B = D \sqcup O \sqcup I$, where D is the diagonal $\{v \times v \mid v \in V\}$, O are the *outputs* of the multiplexer, and I are the *interior responses* of the mutliplexer. We consider all of these as addresses of entries in the response matrix.
2. The response matrices of M are parametrized as follows: there are fixed relations between the entries addressed by I. That is, there is a fixed set $A(I) \subset \mathbb{R}^{|I|}$ of "allowed" vectors of values we can assign to these entries.

   For every allowed choice of values for entries in I, we can pick any output entry $o \in O$. There should be a single Kirchoff matrix corresponding to a choice of *any complex number or* ∞ for the entry of Λ addressed by o.



(a) 2-Multiplexer (b) 2-Multiplexer that creates the "inversion" (c) 3-Multiplexer

FIGURE 2. Multiplexers we use.

That is, a choice of a value for o imposes a value on all the other entries of O. Any complex value for o is valid, and the graph is recoverable - no more that one (complex) Kirchoff matrix is ever associated to a response matrix.

(3) All the other response matrices do not correspond to any Kirchoff matrices.

We usually want to consider a mutliplexer as a black box with a number of outputs. The outputs have some relations between them. While these relations are dependent on the interior responses, we are only concerned about the relations themselves, and largely try to ignore the interior responses.

The three mutlipexers we use are shown in Fig. 2. Their properties are summarized below:

Explain what outputs are.

(1) For the first multiplexer, $b = \frac{C}{a}$, where $C = \lambda_{13}\lambda_{24} = \lambda_{13}\lambda_{23}$. Note that we can obtain a multiplexer with any desired real value of $C > 0$ by setting $\lambda_{13} = \lambda_{24} = \lambda_{13} = \lambda_{24} = \sqrt{C}$.
(2) For the second multiplexer, $b = Ca$, where $C = \frac{\lambda_{14}}{\lambda_{12}}$. Note that the properties of this graph as a multiplexer do not depend on λ_{24} at all as changing λ_{24} only affects the conductance of the one edge between vertices 2 and 4. So, from now on, we will ignore λ_{24}, and simply assume that it is large enough for the conductance between 2 and 4 to remain positive whenever necessary.
(3) For the third multiplexer, $b = C_1 a$ and $c = C_2 a$, where $C_1 = \frac{\lambda_{13}}{\lambda_{14}}$ and $C_2 = \frac{\lambda_{13}}{\lambda_{12}}$.
(4) If all the interior responses are positive, then if a is real and positive, so are b and c, and furthermore all the conductances we get in the multiplexer are real and positive.

These properties, as well as the fact that these things are actually multiplexers, should be in the prevous paper.

<partial conductances> <ignore int. resp., equations between the outputs>

Now, all the graphs G we are considering satisfy the following properties:

Definition 2.4. A graph G is "good" if every pair of distinct boundary vertices (i, j) for G satisfies exactly one of the following:

(1) There are no connections between i and j. That is, $\lambda_{i,j} = 0$.
(2) There is a single multiplexer M such that every connection between i and j involves only the vertices and edges of M. (i, j) is an interior response of M.
(3) Every connection between the vertices i and j involves the vertices and edges of only one multiplexer. For every such multiplixer, the pair (i, j) is one of its "outputs".

In such a graph, we can consider *partial responses* between two boundary vertices along specific multiplexers. These satisfy the property that the sum of all the partial responses between i and j is $\lambda_{i,j}$. WayMoreExplanationHere

Now, suppose we have our graph G and we want to find the exact conditions that a matrix Λ has to satisfy for it to correspond to at least one (possible complex) Kirchoff matrix. We find:

(1) All of its entries of the first type have to be 0.
(2) Every entry (i, j) of the second type corresponds to a single mutliplexer M. Let I_M be the set of the interior responses of that mutliplexer and O_M be the set of all the outputs of that multiplexer. Since every connection between i and j involves only the vertices and edges of M, the value of $\lambda_{i,j}$ is the same as the partial response between i and j along M. Since this is true for all the entries of I_m, we know that the whole set of allowed values for I_m is the same as if the multiplexer was not a part of a larger graph.

    It is clear that if we pick non-allowed values for entries in I_M for any M, we will not have any corresponding Kirchoff matrices no matter what the other entries are.
(3) For every fixed value of the interior responses in the graph, we obtain relations between the outputs of every single multiplexer.

    For entries (i, j) of the third type, however, it is no longer true that partial responses are the same as actual responses. Instead, $\lambda_{i,j}$ is the sum of the partial responses between i and j over all the mutliplexers.

    So, we can fix all the $\lambda_{i,j}$'s of the third sort to arbitrary values. If a set of partial responses does not satisfy either the relations between the outputs of any mutliplexer or the condition that the sum of partial responses between i and j should be $\lambda_{i,j}$, it clearly does not correspond to a Kirchoff matrix. If it does, then by properties of the multiplexers, we can find a unique set of conductances for every multiplexer. So, we have a unique conductance for every edge of the graph.

In other words, we have:

Lemma 2.5. *For a "good" graph, fix a response matrix Λ that has 0's for all the responses of the first type, allowed values for interior responses of every multiplexer that is a part of the graph, and arbitrary values for responses of the third type.*

Then, we for every pair of vertices (i, j) of the third type, we consider partial responses along each multiplexer M, $\lambda^M_{i,j}$ as a variable (and $\lambda^M_{i,j} = 0$ if there is no connection between i and j that involves M). There are as many Kirchoff matrices that correspond to Λ as there are solutions to the following equations:

(1) *For fixed (i, j) of the third type, $\sum_{M \text{ a multiplexer}} \lambda^M_{i,j} = \lambda_{i,j}$.*
(2) *For a fixed multiplexer M, all of the partial responses for its outputs satisfy the appropriate relations*

(a) *explain*

.

All the other response matricies have no corresponding Kirchoff matrices.

Note that, for the multiplexers we use, if all of the partial responses in the lemma are real and positive, the corresponding Kirchoff matrix will also be all-positive

2.2. **...concretely.** In all the "threehands" graphs, pick a single partial conductance in the graph, call it a, and call the three partial conductances between the edges (5,6) l_1, l_2, and l_3. If we apply lemma 2.5, we find that the two conditions in the lemma simplify and are equivalent to the following two:

(1) All the partial conductances, including $l_1(a)$, $l_2(a)$, and $l_3(a)$, are specific linear functionals of a. The coefficients of the linear fractionals depend on which of the three graphs we are looking at, and on all the responses except for one, $\lambda_0 = \lambda_{5,6}$. They are obtained from continued fractions.
(2) $l_1(a) + l_2(a) + l_3(a) = \lambda_0$. This equation is at most a cubic.

I prefer to pick $a = l_3(a)$, which allows us to write the equation as:

$$a + l_1(a) + l_2(a) = \lambda_0 \tag{2}$$

The exact partial response that I picked to be a is shown in Fig. 1.

We allow a to vary and consider λ as a function of a: $\lambda(a) = a + l_1(a) + l_2(a)$. It should be clear from the discussion of the previous section that the response matrix $\Lambda(a)$ that has λ_0 replaced with $\lambda(a)$ and the restriction that the appropriate partial response is a corresponds to a unique Kirchoff matrix.

Furthermore, if we now assume that the response matrices are both real and positive, the functions $l_1(a)$ and $l_2(a)$ are either strictly increasing or strictly decreasing (proof in the other paper). Due to our choice of a,

(1) In the graph with no inversions, both l_1 and l_2 are strictly increasing.
(2) In both of the graphs with inversions, both l_1 and l_2 are strictly decreasing.

Need More Stuff Here

3. Finding the Possibilities for the Number of Real Roots

We know that $l_1(a)$ and $l_2(a)$ are both linear fractionals of a. So, both l_1 and l_2 are continuous as functions from projectve space $\mathbb{R} \cup \infty$ to itself, and there is a single value a_1 such that $l_1(a_1) = \infty$ and a single value a_2 such that $l_2(a_2) = \infty$. (WLOG, we assume that $a_1 < a_2$.) The function $\lambda(a)$ is continuous and real-valued everywhere except for three points: a_1, a_2, and $a_3 = \infty$.

The domain of a's is then naturally divided into three regions: the leftmost region I $(-\infty, a_1)$, the middle region II (a_1, a_2) and the rightmost region III (a_2, ∞).

There may actually be fewer regions in singular cases (if $a_1, a_2, a_3 = \infty$ aren't all distinct), but we will ignore these in this paper.

3.1. **Graph with no inversion.** For the graph with no inversions, $\lambda(a)$ goes strictly increasingly form $-\infty$ to ∞ in every one of the regions. So, for every value of λ_0, there will be one value of a in every region corresponding to it.

<Missing proof that only one of these values corresponds to an all-positive matrix>

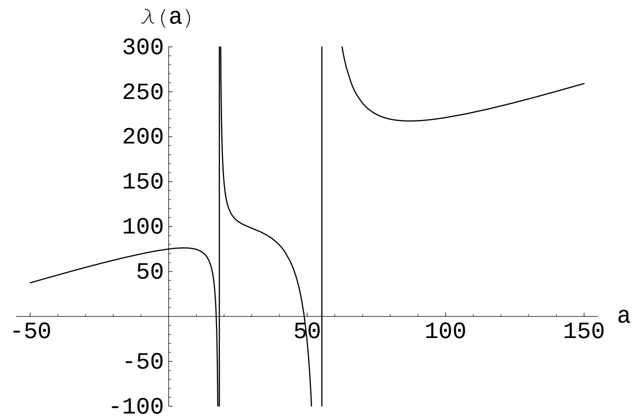


FIGURE 3. Sample plot of $\lambda(a)$ for graphs with inversions.

3.2. **Graphs with one or two inversions.** A sample plot of $\lambda(a)$ as a function of a for a graph with inversions is shown on Fig. 3. It has to have the following features:

(1) In region I, the functions comes from $-\infty$, approaches a maximum, and then goes to $-\infty$.
(2) In region II, the function comes from $+\infty$ and goes to $-\infty$. Depending on the parameters, it may be strictly decreasing, or it may asume a single maximum and a single minimum (there's a "wrinkle").
(3) In region II, the function comes from $+\infty$ and goes to $-\infty$.

Now, let us attempt to solve the equation $\lambda(a) = \lambda_0$ for various λ_0.

- If λ_0 is large, there will be one solution of a in region II, and two solutions in region III. There cannot be more than two solutions in region III because the equation $\lambda(a) = \lambda_0$ is a cubic. For a single value of λ_0, the two solutions in region III are the same (at the minimum).
- As λ_0 decreases further, there is a real solution in region II and two complex solutions.
- If the function is not strictly decreasing in region II, there is a range of values of λ_0 for which there are three real solutions for a, all in region II.
- As λ_0 decreases further, there is again a real solution in region II and two complex solutions.
- Finally, for smaller values of λ_0,there is a single real solution in region II, and two solutions in region I

So, we can suspect a likely possibility of three real, positive conductances corresponding to the same response matrices in case there is this "wrinkle".

Note that this rough method does not distinguish between the graph with a single inversion and the graph with two inversions.

4. Region of All-Positiveness

Recall that for an all-positive response matrix of one of our graphs to correspond to an all-positive Kirchoff matrix, it is necessary and sufficient for all the partial conductances in the graph to be positive.

<not quite true - non-important partial conductances>

I haven't yet proved the following conjecture:

Conjecture 4.1. *If we fix all the responses in a reponse matrix except for $\lambda(a)$, the set $P \subset \mathbb{R}$ of values of a that correspond to all-positive real Kirchoff matrices is an open interval.*

Since the set where a single linear fractional is positive consists of at most two rays, it is clear that P can be at most a union of two open intervals. However, if the conjecture is proven true, it will be clear that P has to lie completely inside one of the regions I, II, or III of the graph.

However, it turns out that we do not really need to know anything about all of P to do useful things, for two reasons:

Lemma 4.2. *If the set P contains a single point p, it also contains a small closed interval around p.*

Proof. P is defined by some finite number of strict inequalities. If a strict inequality is true for $p \in P$, it must also be true on a neighborhood of p.

For our convenience, we would like to deal with closed intervals. Of course, every neighborhood must contain a closed interval for us to use. □

Lemma 4.3. *In all the three graphs we are dealing with, we can express $l_1(a) = \frac{C_1}{l_1'(a)}$ and $l_2(a) = \frac{C_2}{l_2'(a)}$, where $l_1'$and l_2' are some partial conductances. We can create response matrices that change both C_1 and C_2 to any positive values without changing the set P in any way.*

Proof. The partial l_1 and l_2 are of the appropriate form because the mutliplexer between them and the responses l_1' and l_2' is the multiplexet on Fig. 2a).

Example

(Why we can create the mutliplexer with the right value of C_1. Note that all of these responses are all positive)

The set P is defined by the fact that all partial responses in the graph are positive. Now, $l_1(a) > 0$ if and only if $l_1'(a) > 0$, for any positive value of C_1. Clearly P is not dependent on C_1 and, by the same argument, on C_2 (as long as both are positive). □

So, we always have a closed interval $I \subset P$ and we want to make the behavior of $\lambda(a) = a + l_1(a) + l_2(a)$ to suit our fancy. I can't contain any discontinuities of λ. We know that the *signs* of the derivatives of l_1 and l_2 are imposed on us. Furthermore, the derivative of a is $+1$ everywhere. However, we can now modify the magnitude of the derivatives of l_1 and l_2 however we want.

From now on, let us only deal with graphs whose l_1 and l_2 are *decreasing* functions. The function $\frac{d\lambda}{da}$ varies continuously with C_1 and C_2 of I. Now, if we set both C_1 and C_2 to be sufficiently small, we will have, for $a \in I$:

$$\frac{d\lambda}{da} = 1 - C_1 \cdot \frac{d}{da}\left(\frac{1}{l_1'(a)}\right) - C_2 \cdot \frac{d}{da}\left(\frac{1}{l_2'(a)}\right) > 1 - \epsilon$$

On the other hand, if we set both C_1 and C_2 to be large, $\frac{d\lambda}{da}$ will be large and negative.

By the Intermediate Value Theorem, for every point $a_0 \in I$, there are such values of C_1 and C_2 such that $\frac{d\lambda}{da}\Big|_{a_0} = 0$. Then, for a value of λ_0 sufficiently close to $\lambda(a_0)$, there will be two values a_1, $a_2 \in I$ s.t. $\lambda(a_1) = \lambda(a_2) = \lambda_0$.

(Pictures)

(Lemma. Why this matters. What to do with the two-minimae's case)

In fact, the following holds true:

(Which partial responses have to be positive)(Interval. Small interval is sufficient)(Does not generalize)

Lemma 4.4. *For our graphs, if there is an all-positive Kirchoff matrix whose response matrix corresponds to a point in one of the regions (either I, II, or II) of the function $\lambda(a)$, then there is a response matrix that corresponds to the maximum number of Kirchoff matrices possible for that region, and they are all positive.*

Proof. The above proves this for regions I and III. The proof for region II is forthcoming - so far I got there by tweaking C_1 and C_2 enough manually. □

So, for the graphs with one or two inversions, if we find a single Kirchoff matrix that corresponds to a point in region II, we will be able to produce a response matrix that corresponds to three real, positive Kirchoff matrices. Furthermore, neither of these graphs is "recoverable" in the sense that all of its response matrices correspond to at most one *real, positive* Kirchoff matrix. Indeed, if we find a single Kirchoff matrix in either region I or II, the lemma will allow us to find a response matrix that corressponds to two real, positive Kirchoff matrices. Since there exist real, positive non-singular Kirchoff matrices for both graphs, they must lie in at least one of the regions I, II, or III.

5. Graph with Three Real Positive Conductances

By the previous section, all we need to find three real, positive Kirchoff matrices corresponding to the same response matrix for either one of the graphs with inversions, is to find a response matrix whose a lies in the second region of $\lambda(a)$. For the graph with one inversion, this is possible. We vary C_1 and C_2 to crete three values a_1, a_2, a_3 such that they all correspond to all-positive matrices and $\lambda(a_1) = \lambda(a_2) = \lambda(a_3)$. Then, we change some of the entries of the response matrix to $\sqrt{C_1}$ and $\sqrt{C_2}$ to obtain the new response matrix. We can recover the Kirchoff matrices from that however we want. My method was to use Nick Addington's program to parametrize the Kirchoff matrices by one conductance related to a. I then plugged in a_1, a_2, and a_3 to obtain the three response matrices (see Fig. 4).

6. Graph with Only Two

For the graph with one inversion, it turns out that it is impossible to obtain a response matrix that corresponds to an a in the second region of $\lambda(a)$. I suspect that a proof similar to Nick Addington's method can be used to show this. The proof that I have, however, is ugly.

Proof. <Oops, it seems I forgot it. Watch this space.> □

In fact, in the porcess of remembering the proof, I found a counterexample (see Fig. 5). I'll try to put the detailed description of how that "remembering" process went here.

Appendix A. The Output of Nick Addington's Program

This is what Nick Addington's program outputs for the graph with two inversions. It takes about 40 minutes and 2G of RAM on William Stein's "SAGE" computer for the first line to come up, the rest is quick.

```
R^18_21: recovered 10,10 from 9 10 11 x 3 6 10
R^17_19: recovered 9,8 from 9 10 x 8 10
R^17_19: recovered 8,8 from 8 9 x 8 10
R^17_19: recovered 8,7 from 8 10 x 7 8
R^17_19: recovered 9,9 from 3 8 9 x 7 9 10
R^17_19: recovered 7,7 from 7 9 10 x 7 9 10
R^17_19: recovered 4,3 from 2 4 7 10 x 1 3 7 8
R^17_17: recovered 1,1 from 1 4 x 1 3
R^17_17: recovered 2,2 from 1 2 x 1 2
R^17_17: recovered 2,3 from 1 2 x 2 3
R^17_17: recovered 2,4 from 1 2 x 3 4
R^17_17: recovered 3,3 from 2 3 x 3 4
R^17_17: recovered 4,4 from 3 4 x 3 4
R^18_18: recovered 3,3 from 2 3 x 3 7
R^18_18: recovered 2,2 from 2 3 x 2 3
R^18_18: recovered 7,7 from 2 7 x 2 7
R^18_18: recovered 7,9 from 2 7 x 7 9
R^18_18: recovered 9,9 from 7 9 x 2 9
R^20_20: recovered 5,5 from 5 8 x 5 7
R^20_20: recovered 6,5 from 5 6 x 5 7
R^20_20: recovered 6,6 from 6 7 x 5 6
R^20_20: recovered 7,7 from 6 7 x 6 7
R^20_20: recovered 8,8 from 7 8 x 7 8
R^21_21: recovered 4,4 from 2 4 x 4 11
R^21_21: recovered 2,2 from 2 4 x 2 4
R^21_21: recovered 11,11 from 2 11 x 2 11
R^21_21: recovered 11,13 from 2 11 x 11 13
R^21_21: recovered 13,13 from 11 13 x 2 13
R^22_22: recovered 12,13 from 11 12 x 13 14
R^22_22: recovered 13,13 from 12 13 x 13 14
R^22_22: recovered 11,11 from 11 13 x 11 13
R^22_22: recovered 12,11 from 11 12 x 11 13
R^22_22: recovered 12,12 from 12 13 x 11 12
R^22_22: recovered 14,14 from 12 14 x 12 14
R^23_23: recovered 5,5 from 5 12 x 5 11
R^23_23: recovered 6,5 from 5 6 x 5 11
R^23_23: recovered 6,6 from 6 11 x 5 6
R^23_23: recovered 11,11 from 6 11 x 6 11
R^23_23: recovered 12,12 from 11 12 x 11 12
R^24_24: recovered 16,15 from 4 16 x 3 15
R^24_24: recovered 3,3 from 3 16 x 3 15
R^24_24: recovered 4,4 from 3 4 x 3 4
R^24_24: recovered 15,15 from 4 15 x 4 15
R^24_24: recovered 16,16 from 15 16 x 15 16
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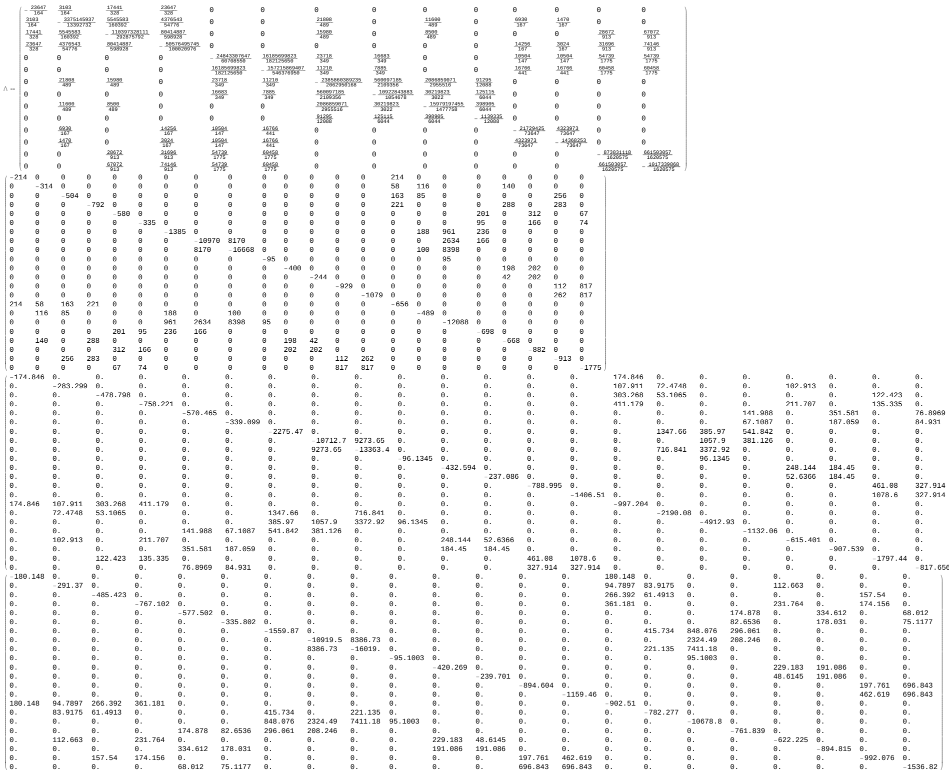

These picture are actually only 22x22, not 24x24. (I entered K22 instead of K24 somewhere). Fix that.

FIGURE 4. Three real positive Kirchoff matrices for the graph with one inversion that correspond to the same response matrix

```
R^25_25: recovered 5,5 from 5 16 x 5 15
R^25_25: recovered 6,6 from 5 6 x 5 6
R^25_25: recovered 15,15 from 6 15 x 6 15
R^25_25: recovered 16,16 from 15 16 x 15 16
Graph is recoverable.
```

References

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[5] Russel, Jeffrey. "Star and K Solve the Inverse Problem." 2003.

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$$\Lambda = \begin{pmatrix}
-12000 & 4000 & 4000 & 4000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4000 & -4070-20\sqrt{5}-40\sqrt{10} & 20 & 50 & 0 & 0 & 10\sqrt{5} & 0 & 10\sqrt{5} & 0 & 20\sqrt{10} & 0 & 20\sqrt{10} & 0 & 0 & 0 \\
4000 & 20 & -4110-20\sqrt{5} & 50 & 0 & 0 & 10\sqrt{5} & 0 & 10\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 20 & 20 \\
4000 & 50 & 50 & -4140-40\sqrt{10} & 0 & 0 & 0 & 0 & 0 & 0 & 20\sqrt{10} & 0 & 20\sqrt{10} & 0 & 20 & 20 \\
0 & 0 & 0 & 0 & -\frac{2942}{5}-40\sqrt{2}-20\sqrt{39} & \frac{1942}{5} & 10\sqrt{39} & 10\sqrt{39} & 0 & 0 & 20\sqrt{2} & 20\sqrt{2} & 0 & 0 & 100 & 100 \\
0 & 0 & 0 & 0 & \frac{1942}{5} & -\frac{2942}{5}-40\sqrt{2}-20\sqrt{39} & 10\sqrt{39} & 10\sqrt{39} & 0 & 0 & 20\sqrt{2} & 20\sqrt{2} & 0 & 0 & 100 & 100 \\
0 & 10\sqrt{5} & 10\sqrt{5} & 0 & 10\sqrt{39} & 10\sqrt{39} & -4350-20\sqrt{5}-20\sqrt{39} & 200 & 150 & 4000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 10\sqrt{39} & 10\sqrt{39} & 200 & -404200-20\sqrt{39} & 400000 & 4000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10\sqrt{5} & 10\sqrt{5} & 0 & 0 & 0 & 150 & 400000 & -404150-20\sqrt{5} & 4000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4000 & 4000 & 4000 & -12000 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 20\sqrt{10} & 0 & 20\sqrt{10} & 20\sqrt{2} & 20\sqrt{2} & 0 & 0 & 0 & 0 & -4350-40\sqrt{2}-40\sqrt{10} & 150 & 200 & 4000 & 0 & 0 \\
0 & 0 & 0 & 0 & 20\sqrt{2} & 20\sqrt{2} & 0 & 0 & 0 & 0 & 150 & -404150-40\sqrt{2} & 400000 & 4000 & 0 & 0 \\
0 & 20\sqrt{10} & 0 & 20\sqrt{10} & 0 & 0 & 0 & 0 & 0 & 0 & 200 & 400000 & -404200-40\sqrt{10} & 4000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4000 & 4000 & 4000 & -12000 & 0 & 0 \\
0 & 0 & 20 & 20 & 100 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -280 & 40 \\
0 & 0 & 20 & 20 & 100 & 100 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 40 & -280
\end{pmatrix}$$

$$K_1 = \begin{pmatrix}
-2.17921\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 2.17921\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -4303.83 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4022.15 & 69.9559 & 0. & 0. & 211.726 & 0. & 0. & 0. & 0. \\
0. & 0. & -4217.34 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4022.15 & 69.9559 & 0. & 0. & 0. & 0. & 0. & 125.234 & 0. \\
0. & 0. & 0. & -4359.11 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4022.15 & 0. & 0. & 0. & 211.726 & 0. & 0. & 125.234 & 0. \\
0. & 0. & 0. & 0. & -1158.27 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 211.926 & 0. & 0. & 93.0493 & 0. & 853.293 \\
0. & 0. & 0. & 0. & 0. & -1158.27 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 211.926 & 0. & 0. & 93.0493 & 0. & 853.293 \\
0. & 0. & 0. & 0. & 0. & 0. & -4759.25 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 123.978 & 4331.12 & 304.157 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -404525. & 399890. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4331.12 & 304.157 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 399890. & -404345. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 123.978 & 4331.12 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -156965. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 156965. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -4776.92 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 314.209 & 4318.42 & 144.286 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -404357. & 399894. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4318.42 & 144.286 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 399894. & -404526. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 314.209 & 4318.42 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -162742. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 162742. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -320. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 58.7717 & 261.228 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -320. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 58.7717 & 261.228 \\
2.17921\times10^{6} & 4022.15 & 4022.15 & 4022.15 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -2.19128\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 69.9559 & 69.9559 & 0. & 0. & 0. & 123.978 & 0. & 123.978 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -387.868 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 4331.12 & 4331.12 & 4331.12 & 156965. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -169958. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 211.926 & 211.926 & 304.157 & 304.157 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1032.17 & 0. & 0. & 0. & 0. & 0. \\
0. & 211.726 & 0. & 211.726 & 0. & 0. & 0. & 0. & 0. & 0. & 314.209 & 0. & 314.209 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1051.87 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4318.42 & 4318.42 & 4318.42 & 162742. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -175698. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 93.0493 & 93.0493 & 0. & 0. & 0. & 0. & 144.286 & 144.286 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -474.671 & 0. & 0. \\
0. & 0. & 125.234 & 125.234 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 58.7717 & 58.7717 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -368.012 & 0. \\
0. & 0. & 0. & 0. & 853.293 & 853.293 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 261.228 & 261.228 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -2229.04
\end{pmatrix}$$

$$K_2 = \begin{pmatrix}
-1.40441\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1.40441\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -4299.72 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4034.47 & 61.7397 & 0. & 0. & 203.509 & 0. & 0. & 0. & 0. \\
0. & 0. & -4213.23 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4034.47 & 61.7397 & 0. & 0. & 0. & 0. & 0. & 117.018 & 0. \\
0. & 0. & 0. & -4355. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4034.47 & 0. & 0. & 0. & 203.509 & 0. & 0. & 117.018 & 0. \\
0. & 0. & 0. & 0. & -1158.27 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 196.617 & 0. & 0. & 86.2689 & 0. & 875.382 \\
0. & 0. & 0. & 0. & 0. & -1158.27 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 196.617 & 0. & 0. & 86.2689 & 0. & 875.382 \\
0. & 0. & 0. & 0. & 0. & 0. & -4778.38 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 162.242 & 4273.72 & 342.42 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -404525. & 399909. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4273.72 & 342.42 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 399909. & -404345. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 162.242 & 4273.72 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -187362. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 187362. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -4786.93 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 334.234 & 4288.39 & 164.311 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -404357. & 399904. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4288.39 & 164.311 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 399904. & -404526. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 334.234 & 4288.39 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -178444. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 178444. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -320. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 60.7743 & 259.226 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -320. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 60.7743 & 259.226 \\
1.40441\times10^{6} & 4034.47 & 4034.47 & 4034.47 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.41652\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 61.7397 & 61.7397 & 0. & 0. & 0. & 162.242 & 0. & 162.242 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -447.963 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 4273.72 & 4273.72 & 4273.72 & 187362. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -200183. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 196.617 & 196.617 & 342.42 & 342.42 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1078.08 & 0. & 0. & 0. & 0. & 0. \\
0. & 203.509 & 0. & 203.509 & 0. & 0. & 0. & 0. & 0. & 0. & 334.234 & 0. & 334.234 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1075.49 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4288.39 & 4288.39 & 4288.39 & 178444. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -191309. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 86.2689 & 86.2689 & 0. & 0. & 0. & 0. & 164.311 & 164.311 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -501.16 & 0. & 0. \\
0. & 0. & 117.018 & 117.018 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 60.7743 & 60.7743 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -355.585 & 0. \\
0. & 0. & 0. & 0. & 875.382 & 875.382 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 259.226 & 259.226 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -2269.22
\end{pmatrix}$$

$$K_3 = \begin{pmatrix}
-1.05317\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1.05317\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -4295.85 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4046.1 & 53.9869 & 0. & 0. & 195.757 & 0. & 0. & 0. & 0. \\
0. & 0. & -4209.35 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4046.1 & 53.9869 & 0. & 0. & 0. & 0. & 0. & 109.265 & 0. \\
0. & 0. & 0. & -4351.12 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4046.1 & 0. & 0. & 0. & 195.757 & 0. & 0. & 109.265 & 0. \\
0. & 0. & 0. & 0. & -1158.27 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 174.29 & 0. & 0. & 80.9969 & 0. & 902.982 \\
0. & 0. & 0. & 0. & 0. & -1158.27 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 174.29 & 0. & 0. & 80.9969 & 0. & 902.982 \\
0. & 0. & 0. & 0. & 0. & 0. & -4827.55 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 260.576 & 4126.22 & 440.755 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -404525. & 399958. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4126.22 & 440.755 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 399958. & -404345. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 260.576 & 4126.22 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -392294. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 392294. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -4798.56 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 357.486 & 4253.51 & 187.564 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -404357. & 399915. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4253.51 & 187.564 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 399915. & -404526. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 357.486 & 4253.51 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -201344. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 201344. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -320. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 63.0995 & 256.9 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -320. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 63.0995 & 256.9 \\
1.05317\times10^{6} & 4046.1 & 4046.1 & 4046.1 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.06531\times10^{6} & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 53.9869 & 53.9869 & 0. & 0. & 0. & 260.576 & 0. & 260.576 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -629.126 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 4126.22 & 4126.22 & 4126.22 & 392294. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -404673. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 174.29 & 174.29 & 440.755 & 440.755 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1230.09 & 0. & 0. & 0. & 0. & 0. \\
0. & 195.757 & 0. & 195.757 & 0. & 0. & 0. & 0. & 0. & 0. & 357.486 & 0. & 357.486 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1106.49 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 4253.51 & 4253.51 & 4253.51 & 201344. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -214104. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 80.9969 & 80.9969 & 0. & 0. & 0. & 0. & 187.564 & 187.564 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -537.121 & 0. & 0. \\
0. & 0. & 109.265 & 109.265 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 63.0995 & 63.0995 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -344.73 & 0. \\
0. & 0. & 0. & 0. & 902.982 & 902.982 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 256.9 & 256.9 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -2319.76
\end{pmatrix}$$

FIGURE 5. Three real positive Kirchoff matrices that correspond to the same response matrix in the graph with two inversions.