Eigenvalues for n-Lattices and n-Stars

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August 12, 2005

ABSTRACT. We are concerned with the response matrices of γ-harmonic lattice and star networks. We look at the physical interpretation of the eigenvalues and eigenvectors of these matrices, when possible. Computing the characteristic polynomial of the response matrix is also utilized as a last resort.

Contents

1 Introduction

Let $G = (V, E)$ be a graph where V is the set of vertices and E is the set of edges. Let there be a network of resistors, $\Omega = (G, \gamma)$ (1), where the vertices are the nodes of the network, and γ is a conductivity function on the edges of the network. Ω is partitioned into boundary nodes, $\partial\Omega$, and interior nodes, intΩ. Let u be a vector of voltages in intΩ and let p and q be interior nodes. If $\sum_{q\sim p} \gamma_{pq}(u(p) - u(q)) = 0$, u is called γ-harmonic. We are concerned only with networks that are γ-harmonic.

There exists a Kirchhoff matrix, K, which contains information about an electrical network. The k_{ij}'th entry $({}_i \neq{}_j)$ of the Kirchhoff matrix gives the negative value of the conductivity between nodes i and j. The k_{ii}'th entry of K is the negative sum of all other entries in the i'th row, such that the row sums

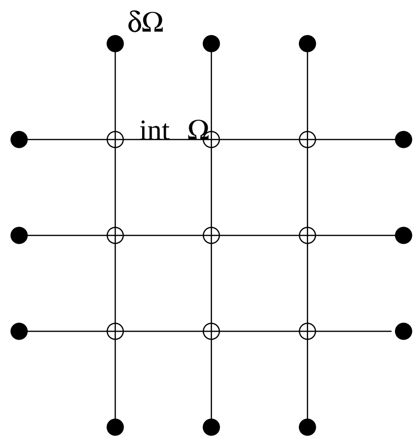

Figure 1: 3-lattice

of K are zero. K is an upper-triangular, positive semi-definite matrix. It can be divided into submatrices, A, B, B^T, and C, where A shows the connections between boundary nodes, B shows the connections from boundary nodes to interior nodes, and C shows the connections between interior nodes.

$$K = \begin{array}{cc} & \begin{array}{cc} \partial\Omega & int\Omega \end{array} \\ \begin{array}{c} \partial\Omega \\ int\Omega \end{array} & \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \end{array}$$

Let ϕ be a voltage function on $\partial\Omega$ and ψ be the current function on $\partial\Omega$. According to Ohm's Law, $I = (\triangle v)\gamma$, where I is the current and $\triangle v$ is the voltage drop between nodes. Keeping this in mind, there is a response matrix, Λ, which is derived by taking the Schur compliment of K. In other words, $\Lambda = A - BC^{-1}B^T$. The response matrix is a map from the boundary voltages to boundary currents, $\Lambda\phi = \psi$.

2 Definitions

Definition 2.1 *An n-lattice* is a square lattice with n boundary nodes on each side, numbered counterclockwise starting at the upper right boundary vertex.

Definition 2.2 If Ω has *constant conductivity* then for each edge, $e \in \Omega$, $\gamma(e) = c$, where c is some constant.

Definition 2.3 An edge is *incident* to a node if that node is an endpoint of that edge.

Definition 2.4 Two boundary nodes are *adjacent* if they are numbered, i and $i + 1$. e.g. boundary nodes 2 and 3 are adjacent.

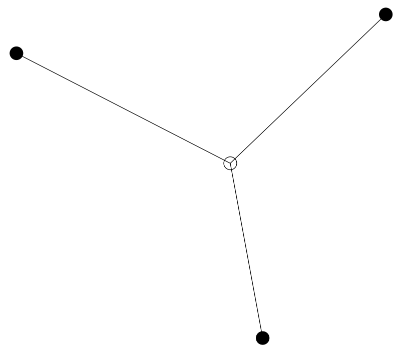

Figure 2: 3-star

Definition 2.5 Two nodes are *neighbors* if they are the endpoints of the same edge.

Definition 2.6 This definition is taken verbatim from Michael Goff's article [2]: A *boundary antenna* is a pair of boundary spikes that share a common vertex.

Definition 2.7 If a graph has n boundary nodes, all of which are neighbors of the same single interior node and no other nodes, call this graph an *n-star (2).*

Definition 2.8 Divide an n-lattice,Ω, diagonally in half, so that on one side of the diagonal Ω has conductivity, a, and on the other side Ω has conductivity, b. Ω has *split conductivity.* (3)

3 Eigenvalues of Λ

For the first theorem, rather than investigating the characteristic polynomial of Λ to find its eigenvalues, we investigate the actual electrical network, keeping in mind the physical interpretation of the eigenvalues and eigenvectors of the response matrix: λ is an eigenvalue of Λ if $\Lambda\phi = \lambda\phi$. Recall that Λ is a map that takes boundary voltages to boundary currents ($\Lambda\phi = \psi$). Thus, if λ is an eigenvalue, $\lambda\phi = \psi$. In other words, the current at a boundary node is a scalar multiple of the voltage at that node. It follows that ϕ is an eigenvector of Λ (4).

4 Results

Lemma 4.1 *For all response matrices, zero is an eigenvalue.*

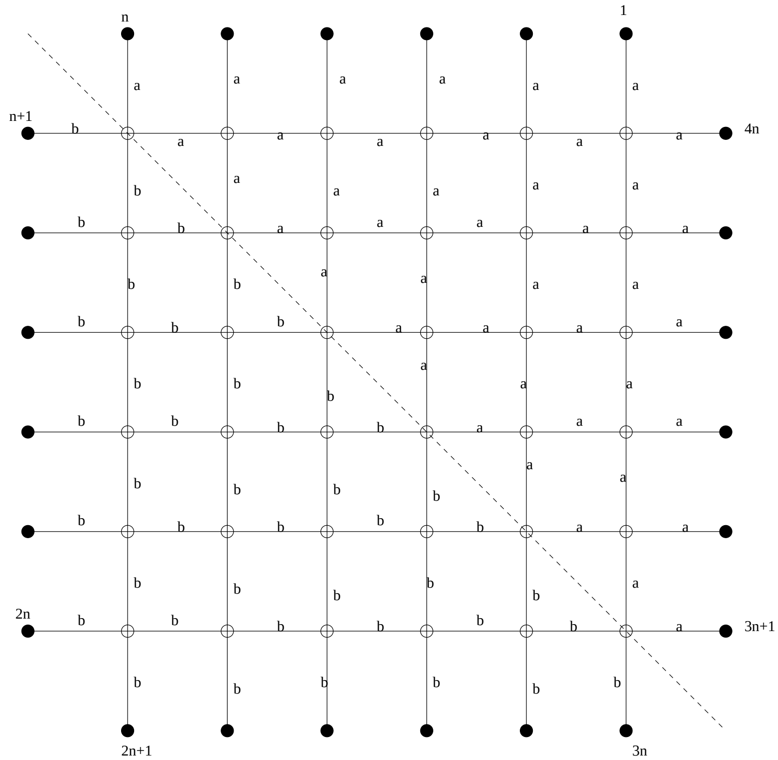

Figure 3: Split conductivity

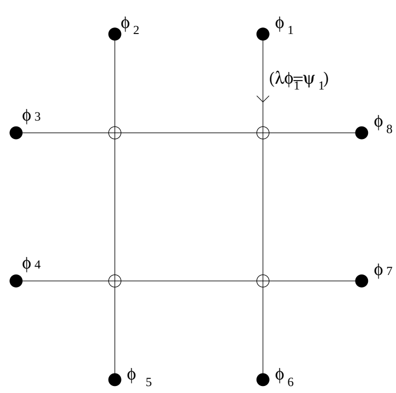

Figure 4: 2-lattice with boundary potentials and current at node 1

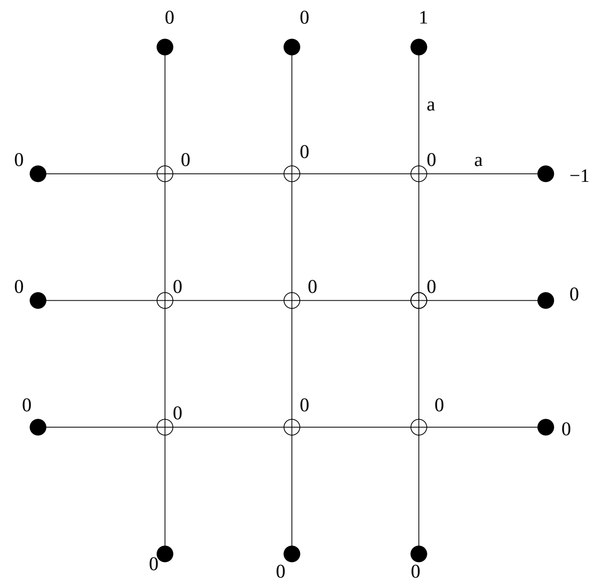

Figure 5: Proof of Lemma 4.2

Proof: If the voltage at every node of a network is equal, then no current flows through the network. When the current equals zero, $\psi = 0$, giving the equation $\lambda\phi = 0$. If ϕ is any constant vector, then $\lambda = 0$. $\Box$

Lemma 4.2 *In a lattice network, if a boundary antenna has constant conductivity, a, then a is an eigenvalue for the response matrix of that network.*

Proof: It suffices to give an eigenvector belonging to $\lambda = a$ to prove that a is an eigenvalue of the response matrix. Let a lattice network satisfy the hypothesis of this lemma. Consider the pair of boundary nodes of the antenna with conductivity, a. Induce a voltage of 1 to one of these nodes, and a voltage of -1 to the other. Let all other voltages in the network equal 0 (5). This gives us the eigenvector for $\lambda = a$, where the eigenvector's entries consist of zeros, except for the two entries equal to 1 and -1, respectively, corresponding to the boundary nodes with those voltages. We know this is an eigenvector because the current out of the boundary node with voltage 1 is a, and the current out of the boundary node with voltage -1 is $-a$ (by Ohm's Law). $\Box$

Theorem 4.3 *Let a lattice-shaped network have constant conductivity, $a > 0$. The response matrix for this network will have eigenvalues 0 and a. Let λ be any other eigenvalue for Λ, then $0 < \lambda < a$.*

Proof: Part A. Lemma 4.1 gives a proof of why 0 is an eigenvalue.

Part B. Assume $\lambda < 0$. Let pq be a boundary edge of Ω such that the boundary node, p, is induced with maximum positive voltage. The voltage at

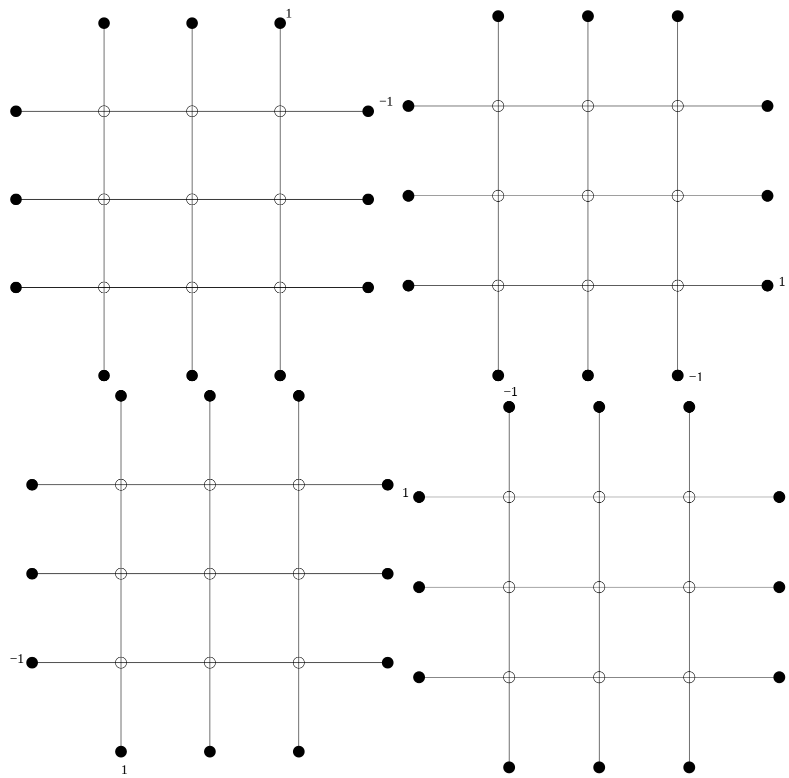

Figure 6: Physical interpretation of eigenvectors belonging to $\lambda = a$

q must be greater than the voltage at p. Let rs be a boundary edge of Ω such that r is the boundary node induced with the minimum negative voltage. The voltage at s must be less than the voltage at r. Either case contradicts the Maximum Principle for Harmonic Functions [1]. Thus $\lambda \leq 0$. An alternative argument for $\lambda \leq 0$ is the fact that positive semidefiniteness is a property of all response matrices, meaning all eigenvalues of a response matrix are nonnegative.

Part C. Lemma 4.2 gives a proof of why a is an eigenvalue of the response matrix. In fact, for all n-lattices except the 1-lattice, a has multiplicity 4. For the 1-lattice, a has multiplicity 3. (6)(7)

Part D. If $s\Lambda$ is a scalar multiple of Λ, then the eigenvalues of $s\Lambda$ will be the scalar, s, multiplied by the eigenvalues of Λ. The response matrix for a lattice network with constant conductivity, a, is the constant, a, multiplied by the response matrix for the same network, but with constant conductivity, 1. Thus, given any lattice network with constant conductivity, 1, a proof that $\lambda \leq 1$ will suffice to prove that $\lambda \leq a$ for lattice networks with constant conductivity, a.

Case 1 Let a lattice network have conductivity equal to 1 on all edges. We normalize ϕ so that $|\phi_i| \leq 1$. Assume $\lambda > 1$. Let ϕ have a maximum value

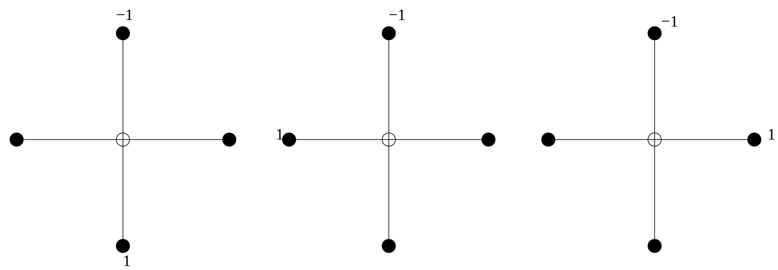

Figure 7: λ=a for the 1-lattice

of 1 at one of the boundary nodes on a boundary antenna of a lattice network (8). By Ohm's Law, the interior node, which we will call p, neighboring that boundary node has voltage $1-\lambda$, and the other boundary node on that antenna must also have voltage 1. The current flowing into p is at least 2λ. Thus, a current of at least λ must flow out of one of the remaining edges incident with p. Call this edge pq. By Ohm's Law, the voltage at node q must be less than or equal to $1-2\lambda$. By our assumption that $\lambda > 1$, the voltage at node q is less than -1. This is a contradiction because we normalized ϕ, and there cannot be an extreme voltage value at an interior node. Therefore, $\lambda \leq 1$, if the maximum voltage is imposed at a corner boundary node.

Case 2. Let a lattice network have conductivity equal to 1 on all edges. Again, we normalize ϕ so that $|\phi_i| \leq 1$, and we assume $\lambda > 1$. Now let ϕ have a maximum value of 1 at a boundary node, l, that is not on a bounary antenna (9). Let the boundary nodes adjacent to this node have voltages x and y such that $x, y \leq 1$. The current flowing out of node l is λ. By Ohm's law, the voltage at the interior node neighboring l, called p has a voltage of $1-\lambda$. The interior nodes neighboring x and y, called q and r, have voltages of $x(1-\lambda)$ and $y(1-\lambda)$, respectively. The voltage drop from q to p is $(1-\lambda)(x-1)$, which, by our assumptions, must be greater than or equal to zero. Likewise, the voltage drop from r to p is $(1-\lambda)(y-1)$, which also must be greater than or equal to zero. We already know the current flowing from l to p is $\lambda > 1$. Thus, current is flowing into p at three edges, which means current must flow out of p at the fourth edge incident with p. Call the fourth neighboring node of p, w. The current flowing from p to w must be greater than or equal to λ. By Ohm's law, the voltage at w is less than or equal to $1-2\lambda$, which means that the voltage at w is less than or equal to -1. Again, this is a contradiction because it would cause an extreme value to lie on an interior node. Thus, $\lambda \leq 1$, if the maximum voltage is not imposed on a corner boundary node. Therefore $\lambda \leq 1$ in all cases, implying that if constant conductivity is equal to a in a lattice network, then $\lambda \leq a$. $\square$

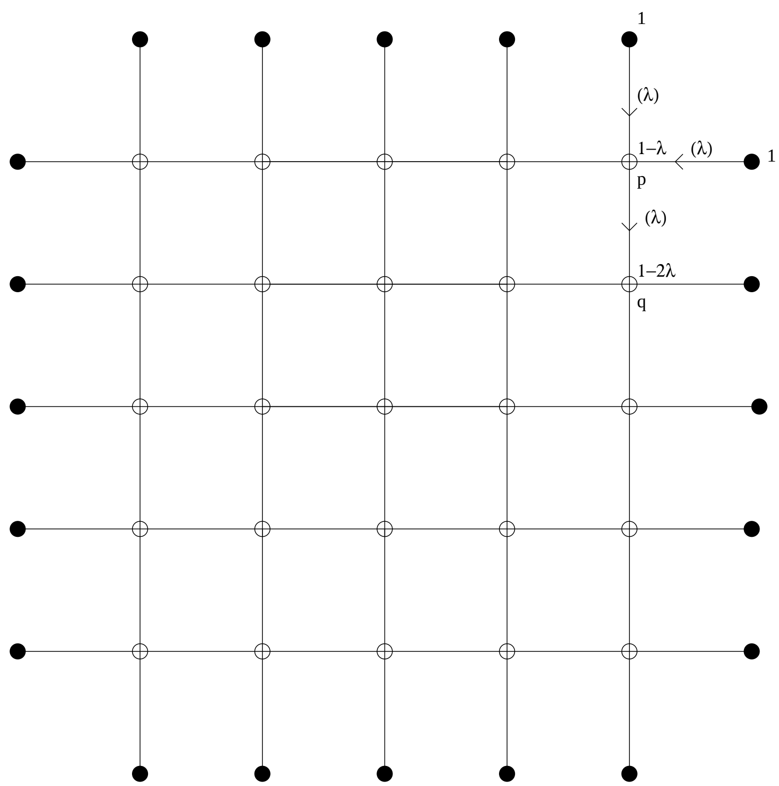

Figure 8: Part D, Case 1 of proof for Theorem 4.3

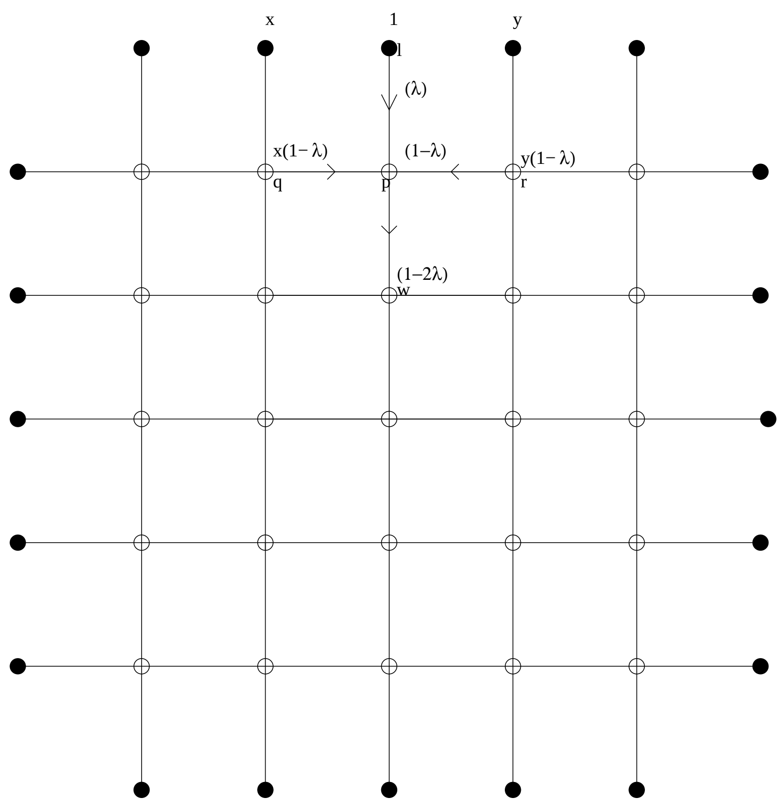

Figure 9: Part D, Case 2 of proof for Theorem 4.3

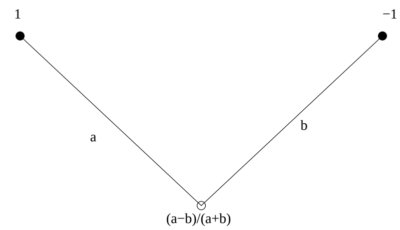

Figure 10: 2-star with Eigenvector

4.1 Conjectures

Conjecture 4.4 *Let a lattice network have split conductivity, a and b. The response matrix for this network has eigenvalues $\lambda = 0, a$, and b, and eigenvalues $0 \leq \lambda \leq max(a, b)$.*

The part of this conjecture that states $\lambda = 0, a, b$ is true by the lemmas preceding the Theorem 4.3. However, I am yet unable to prove that $max\lambda = max(a, b)$. To begin the proof, one must note that the maximum voltage on a network with split conductivity must be induced on a boundary node of an antenna with split conductivity, and on no other type of boundary node. I challenge the reader to complete the proof, or to find a counterexample.

4.2 N-stars and Eigenvalues

The initial aim of this research was to ultimately find results about n-lattice networks with arbitrary conductivities. However, the simple issue of split conductivity on an n-lattice was a significant obstacle. Thus, a new approach to the problem was taken, by exploring much simpler networks: n-stars. If the conductivities on an n-star are arbitrary, what, if anything, can we say about the eigenvalues of the response matrix?

We begin with the 2-star. Recall the the response matrix of an n-star is $n \times n$ and therefore has n eigenvalues. Recall also that zero is an eigenvalue for all response matrices, with corresponding eigenvector **1.** Thus there is only one unknown eigenvalue of the response matrix for the 2-star. This eigenvalue was discovered using similar methods to what the proof of Theorem 4.3 used. It was hypothesized that an eigenvector might have entries 1 and -1, and hence these voltages were induced to the boundary nodes of a 2-star with conductivities a and b. By Ohm's Law, the voltage at the interior node was $(a - b)/(a + b)$ (10). Then, if λ is an eigenvalue, we are given two equations: $\lambda = (1-(a-b)/(a+b))a$ and $-\lambda = (-1 - (a - b)/(a + b))b$. In both cases, $\lambda = 2ab/(a + b)$. Therefore, by guess and check, the eigenvalues of the response matrix for a 2-star with conductivities a and b are zero and $2ab/(a + b)$.

We now look at the eigenvalues of a 3-star with conductivities a, b, and c. Note that the methods of research change at this point. Rather than using figures to aid our researh, we must now utilize matrix algebra in order to find eigenvalues. Again, recall that zero is an eigenvalue. We will use a process called "matrix deflation" [3] to take the 3×3 response matrix of the 3-star and change it to a 2×2 matrix with the same eigenvalues, excluding zero. The characteristic polynomial gives us the remaining eigenvalues:

$$\lambda = \frac{ab+bc+ac \pm \sqrt{(ab+bc+ac)^2 - 3abc(a+b+c)}}{a+b+c}$$

There appears to be some symmetry in the eigenvalues of the 2-star and 3-star. But, is there a formula that gives the eigenvalues for any n-star with arbitrary conductivities? We inspect the eigenvalues of the 4-star to aid in the search of a pattern. Let the conductivities on a 4-star be a, b, c, and d. With the knowledge that zero is an eigenvalue, we again perform matrix deflation on the response matrix, resulting in a 3×3 matrix with the remaining nonzero eigenvalues. (For more details about the process of deflation, see [3]). Although matrix deflation was meant to simplify the computation of eigenvalues, we are given a result that is very difficult to work with:

$$\begin{aligned}
&\lambda^3(cd^2-c-ad^2+a)+\lambda^2[2(c^2d+ac^2+bc^2+bcd-a^2d-abd-a^2c-a^2b)+ad^2-cd^2]+\\
&\lambda[3(-ac^2d^2-ac^3d-bc^2d-b^2c^2d+a^2cd-abcd^2)+abc(-c^2-a^2+ab-bc)+bcd(-bd+\\
&bd^3)+abcd(c+b+ab-d^2-a^2-bd^2)+abd(-a^2b+ad^2+a^3+bd)+2(-5abc^2d+\\
&a^3cd+a^2bd^2+a^2bc^2+a^2b^2d+2a^3bd+4a^2bcd+a^2bcd^2+a^2b^2d^2-a^3bd^2-ab^2cd^2)+\\
&a^3cd]+[abcd(-a^3+ad-a^4+a^2c^2+a^2cd+a^2d^2+a^3d-acd^2-ad^3+a^2bc+a^2b^2-\\
&a^2bd+bcd^2+b^2d^2+bd^3-acd^2+c^2d^2+cd^3-abc^2-ab^2c+bcd^2-7acd-3abcd-3abc-\\
&ac^2-bcd-b^2d-bd^2-ab^2+7c^2d+3cd^2+5bc^2+b^2c)+2abcd(-a^2b-ab^2d+a^2bd-\\
&abd^2-ac^2d+bc^2d+b^2cd-2ad^2+2c^3-2a^2c)+b^2c^2d^2(b+d-cd-bd-d^2+c]=0
\end{aligned}$$

This computation has not been checked for errors and henceforth may be slightly incorrect. However there is no obvious pattern between the eigenvalues of the response matrices for the 2-, 3-, and 4-stars.

Searching for more intuition about eigenvalues of n-stars, we let two conductivities on the 4-star be equal. Let the conductivities now be a, a, c, and d. Note that we now have a boundary antenna with constant conductivity. Therefore, zero and a are eigenvalues of this 4-star. Replacing b with a in the deflated matrix will give us the remaining two eigenvalues:

$$\lambda = \frac{3ac+2cd+3ad \pm \sqrt{(3ac+2cd+3ad)^2 - 16acd(2a+c+d)}}{2(2a+c+d)}$$

Again, this computation has not been checked for accuracy. Nonetheless, the only major similarity between this and the eigenvalues of the 3-star is the quadratic formula. Sadly, as response matrices grow in size, eigenvalues will not be given by this formula. Hence, it is impossible to declare any pattern in the eigenvalues of n-stars thus far.

5 Continued Research and Conclusions

Much work has been left unfinished in this research. Many aspects of this paper can be corrected, added on to, or simplified. What I am most interested in finding is an upper bound to response matrix eigenvalues, yet I believe that a new approach to the problem is in order. Computing the actual eigenvalues of response matrices is an extremely difficult (and messy) way to look for results. Perhaps there is a less complex way to do this research. The reader might continue this research by:

- proving the split-conductivity conjecture (or giving a counterexample)
- checking eigenvalue computations for accuracy and possible simplifaction and/or clarification
- finding an alternate way to search for information about eigenvalues of response matrices of n-lattices and/or n-stars.

Perhaps when significant and plentiful results have been discovered about the eigenvalue problem, it can be turned into an inverse problem. Given the eigenvalues of a response matrix, what can we say about the actual electrical network?

6 Acknowledgement

I would like to thank Dr. Jim Morrow, Sam Cosky, Ernie Esser, and Jenny French for helping me formulate and work on this research.

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