ANOTHER LOOK AT CONNECTIONS AND DETERMINANTS

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Abstract. This paper starts by describing a way of visualizing determinants and subdeterminants of an arbitrary square matrix M as connections on a related graph $\mathcal{G}_M$. Several applications of this connection-determinant relationship to the recovery of electrical networks are then given.

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1. Introduction

One of the best tools we have for visualizing the (essentially algebraic) process of recovering electrical networks is the persistent relationship between the connections of the network and subdeterminants of the Kirchhoff matrix. The connection-determinant formula proved on p. 50 of [1] shows part of this connection, but a close study of the Addington recovery method (see [2]) shows that this relationship must run deeper than the formula in [1]. In fact, as we will see, there is a close relationship between the subdeterminants of any square matrix and the connections of an associated directed graph.

The exact numerical form of the relationship is not as important as the broad statement it allows us to make: using the connection-determinant formula, we can determine exactly when a submatrix of Λ (or equivalently, a submatrix of K) is going to be non-zero or zero just by looking at the geometry of the graph. We can then use this visualization of determinants to better understand several aspects of the recovery process.

2. A Geometric View of Determinants

Consider any $k \times k$ matrix M. Its determinant can be found by the formula

$$\det M = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} a_{i,\sigma(i)} \tag{1}$$

where S_k is the set of all permutations on k elements. Each of these permutations σ can be written as a product of disjoint cycles, where a *cycle* is a set of elements in S_k resulting from repeated applications of a permutation σ. For example, the cycle (1 3 2) corresponds to the mapping $1 \mapsto 3$, $3 \mapsto 2$, $2 \mapsto 1$, or $\left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right)$. As another example, the permutation on 5 elements 31254 can be written (1 3 2)(4 5). The sign of the permutation σ is equal to the product of the signs of its component cycles, where the sign of the component cycle is positive if the length of the cycle is odd, and negative if the length of the cycle is even. Put more succinctly, given a cycle $\mathcal{C}$, $\operatorname{sgn}(\mathcal{C}) = (\text{-}1)^{|\mathcal{C}|-1}$, where $|\mathcal{C}|$ is the length of the cycle. If we multiply the signs of the component cycles of σ, we find that $\operatorname{sgn}(\sigma) = (-1)^{k-|\sigma|_c}$, where n is the length of the permutation and $|\sigma|_c$ is the number of cycles in σ. Putting this all together, then, we find that

$$\det M = \sum_{\sigma \in S_k} (-1)^{k-|\sigma|_c} \omega(\mathcal{C}_1)\omega(\mathcal{C}_2)...\omega(\mathcal{C}_n) \tag{2}$$

where $\omega(\mathcal{C}_n)$ denotes the product of all entries $m_{\sigma^i(j),\sigma^{i+1}(j)}$ in the cycle $\mathcal{C}_n$ of σ (σ^i denotes i repeated applications of the permutation σ).

There is a nice geometric interpretation of the determinant of M in terms of graph theory. We create a graph $\mathcal{G}_M$ with k vertices, labelled $1...k$, with an edge set as follows: for each non-zero entry m_{ij} in M, a weighted directed edge is drawn from vertex i to vertex j, with weight m_{ij}. A diagonal entry is interpreted as a loop from a vertex to itself. For example:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & 0 & m_{34} \\ m_{41} & 0 & m_{43} & m_{44} \end{bmatrix}$$

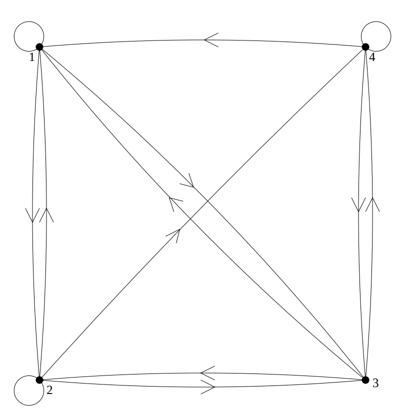

Figure 1. A matrix M and its associated graph $\mathcal{G}_M$

This graph $\mathcal{G}_M$ will be called the *associated graph* of M. Now each of the disjoint cycles in a permutation σ corresponds to a directed loop in the associated graph. The permutation itself corresponds to a loop partition of $\mathcal{G}_M$, where a *loop partition* is defined as a set of loops in $\mathcal{G}_M$ such that each vertex in $\mathcal{G}_M$ is traversed exactly once. The set of all permutations on k elements is then in one-to-one correspondence with $\mathcal{L}(\mathcal{G}_M)$, the set of all loop partitions on $\mathcal{G}_M$.

We can now express the determinant of a $k \times k$ matrix M using these new terms:

$$\det M = \sum_{L \in \mathcal{G}_M} (-1)^{k-|L|_c} \omega(L) \tag{3}$$

where $|L|_c$ is the number of disjoint loops in L and $\omega(L)$ is the product of the weights of all directed edges used in L.

$$(132)(45) \rightarrow \begin{bmatrix} a_{11} & a_{12} & \mathbf{a_{13}} & a_{14} & a_{15} \\ \mathbf{a_{21}} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & \mathbf{a_{32}} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & \mathbf{a_{45}} \\ a_{51} & a_{52} & a_{53} & \mathbf{a_{54}} & a_{55} \end{bmatrix}$$

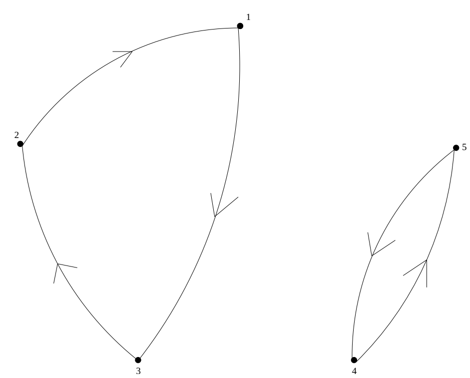

FIGURE 2. A permutation and its corresponding loop partition

The determinant of an arbitrary square submatrix of M can also be expressed in graph-theoretical terms. However, it would suit our purposes better to look at a particular type of square submatrix of M. We divide M into a block form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A and D are square, and define all row indices from 1 to the last row of A to be *boundary indices*, and the rest of the row indices to be *interior indices*. Similar definitions are made for column indices. We now form a submatrix of M as follows: all interior indices (both row and column) shall be in this submatrix, as well as some boundary row indices P and boundary column indices Q ($|P| = |Q|$). The resulting submatrix will be denoted $M(P + D; Q + D)$, in direct accord with the traditional notation of [1].

Now that we have a submatrix, we can find a good expression for its determinant. Following the same procedure as before, we first write the determinant of this matrix as

$$\det M(P + D; Q + D) = \sum_{\sigma \in S_\nu} (-1)^{\nu - |\sigma|} \omega(\mathcal{C}_1)\omega(\mathcal{C}_2)...\omega(\mathcal{C}_n) \tag{4}$$

where $\nu = |P| + |D|$ is the size of $M(P + D; Q + D)$. Our problem is to find an appropriate interpretation of the cycles in a permutation σ in terms of paths on the graph $\mathcal{G}_M$ associated to M. Before we do so, we will find it convenient to denote the vertices in $\mathcal{G}_M$ corresponding to boundary row indices P to be *starting vertices*, and vertices corresponding to Q to be *ending vertices*. Now consider any permutation σ in S_d, and look at a cycle which includes a starting vertex (i.e., a row of P). Eventually this cycle will reach a column corresponding to an ending vertex (a column of Q). Then this part of the cycle from row p_1 to column q_1 corresponds to a directed path in the associated graph $\mathcal{G}_M$ from vertex p_1 to vertex q_1. Since the column index q_1 corresponds to row index (or starting vertex) p_2, we can continue along the cycle to find the next ending vertex q_2. Repeating this process, we eventually find that the cycle ends on a final ending vertex q_n, which corresponds to starting vertex p_1 by definition. We now repeat the process for every cycle which includes a starting vertex p_k, finding its corresponding ending vertex in Q under this permutation.

$$(15)(24)(3) \rightarrow \begin{array}{cc} & \begin{array}{ccccc} 1 & 3 & 7 & 8 & 9 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \\ 9 \end{array} & \begin{bmatrix} a_{11} & 0 & a_{17} & 0 & \mathbf{a_{19}} \\ 0 & 0 & a_{27} & \mathbf{a_{28}} & 0 \\ a_{71} & 0 & \mathbf{a_{77}} & 0 & 0 \\ 0 & \mathbf{a_{83}} & 0 & a_{88} & 0 \\ \mathbf{a_{91}} & a_{93} & 0 & 0 & a_{99} \end{bmatrix} \end{array}$$

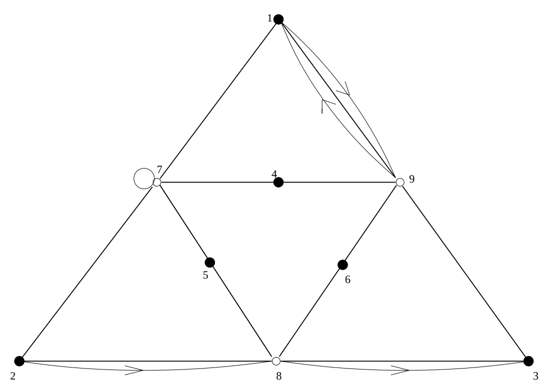

FIGURE 3. A permutation and its corresponding (1,2;1,3) connection on the triangle-in-triangle graph

After finding the map from starting vertices to ending vertices, we write σ as the product of two permutations ϕ and μ, where ϕ is the set of cycles which includes indices of P and Q, and μ consists of the rest of cycles in σ (if there are no more cycles in σ, μ is the identity permutation). ϕ represents the set of disjoint, directed paths from P to Q under the permutation σ, and μ represents the interior-interior connections in σ of all interior nodes not used in ϕ. Just as before, $\text{sgn}(\sigma) = \text{sgn}(\phi)\text{sgn}(\mu)$. Now consider the permutation τ on S_k which takes starting vertices P to ending vertices Q under σ. Since σ and τ map starting to ending vertices in the same way, $|\phi|_c = |\tau|_c$. Then

$$\begin{aligned}\operatorname{sgn}(\phi) &= (-1)^{t-|\phi|_c} \\ &= (-1)^{t+k-k-|\tau|_c} \\ &= (-1)^{t-k}\operatorname{sgn}(\tau) = (-1)^{t+k}\operatorname{sgn}(\tau)\end{aligned}$$

where t is the number of elements in the permutation ϕ.

We are interested in finding a formula for det $M(P+D;Q+D)$ in terms of permutations of τ rather than σ. To that end, we first sum over $S(\phi)$, the set of all σ's which have the same ϕ. This will suppress the part of σ which corresponds to μ into a single term U_ϕ. Denote the set of entries of M in ϕ by E_ϕ and the set of rows (or columns) of M in μ by J_μ. More intuitively, E_ϕ is the set of directed edges in $\mathcal{G}_M$ used in ϕ, and J_μ is the set of vertices in D not used in ϕ. Then

$$\begin{aligned}&\sum_{\sigma\in S(\phi)} \operatorname{sgn}(\sigma)\prod_{i=1}^{\nu} m_{i,\sigma(i)} \\ &\quad= \sum_{\sigma\in S(\phi)} (-1)^{k+t}\operatorname{sgn}(\tau) \prod_{(i,j)\in \mathrm{E}_\phi} m_{i,j}\cdot \operatorname{sgn}(\mu)\cdot \prod_{i\in J_\mu} m_{i,\mu(i)} \\ &\quad= (-1)^k \operatorname{sgn}(\tau)\cdot \Big(\prod_{(i,j)\in\mathrm{E}_\phi} -m_{i,j}\Big)\cdot U_\phi\end{aligned}$$

where U_ϕ is the determinant of the square submatrix of M corresponding to the rows and columns J_μ.

We now sum over all ϕ which induce a particular τ. Then we sum over all $\tau \in S_k$, to obtain the follwing:

Theorem 1. Connection-Determinant Formula. *Suppose M is a square matrix decomposed into $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$, and suppose $M(P+D;Q+D)$ is a submatrix of M where P is a set of row indices in A and Q is a set of column indices in A, and $|P|=|Q|=k$. Then*

$$\det M(P+D;Q+D) = (-1)^k \sum_{\tau\in S_k} \operatorname{sgn}(\tau)\Bigg(\sum_{\phi,\tau_\phi=\tau} \Big(\prod_{(i,j)\in\mathrm{E}_\phi} -m_{i,j}\Big)U_\phi\Bigg) \tag{5}$$

where

- *τ is a permutation from starting indices P to ending indices Q*
- *ϕ is a set of disjoint, directed paths from P to Q*
- *E_ϕ is the set of edges in $\mathcal{G}_M$ used in ϕ*
- *U_ϕ is the determinant of the submatrix of D corresponding to all rows (or columns) not used in D*

We will now define what we mean by a connection from P to Q using the connection-determinant formula as a guide. For any valid ϕ, we must have a set of disjoint directed paths from P to Q, but we do not need P and Q to be distinct. This motivates the following definitions:

Definition 1. *A* connection *$(P;Q)$ is a set of disjoint, directed paths from starting boundary vertices P to ending boundary vertices Q on a directed graph G with boundary.*

This is exactly what our old definition of a connection was, except that we now allow the possibility of P and Q not being distinct. In addition, it is useful to make a further distinction between connections:

Definition 2. *A* well-behaved connection *is a connection on a weighted, directed graph with boundary (which is associated to a matrix A) such that* $\det A(P+I;Q+I) \neq 0$. *A* strongly well-behaved connection *is a connection on a graph such that for every valid choice of weights,* $\det A(P+I;Q+I) \neq 0$.

3. The Boundary Edge Formula

It was shown in [1] that under certain conditions, it is possible to recover a boundary edge of an electrical network from the network's response matrix by taking Schur complements in the response matrix. In fact, this procedure is more general: it will work for recovering all "boundary edges" (including the newly-defined boundary loops) of any square matrix K from its Schur complement Λ, as long as certain determinantal conditions are satisfied. Using the connection-determinant formula, we can translate these conditions into connection-breaking conditions similar to those specified in [1].

Before proving the boundary edge formula, we will need one key fact about Schur complements, which we state without proof (see [1] for details):

Proposition 1. *Given any square matrix K decomposed into $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, we denote the* Schur complement *$A - BD^{-1}C$ by K/D. For any matrix K and its Schur complement K/D,*

$$\det K/D \cdot \det D = \det K \tag{6}$$

We now state and prove the boundary edge formula using purely algebraic terminology:

Theorem 2. Boundary edge formula. *Let K be a $k \times k$ matrix decomposed into $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, and define $\Lambda = K/D$. Suppose that $P' = p + P$ and $Q' = q + Q$ are two sequences of indices in A. Now create a new matrix K' obtained from K by zeroing out the entry κ_{pq}. Define $\Lambda' = K'/D$. Suppose that* $\det \Lambda'(P';Q') = 0$, *but* $\det \Lambda'(P;Q) \neq 0$. *Then*

$$\kappa_{pq} = \frac{\Lambda(P';Q')}{\Lambda(P;Q)} = \frac{\det \Lambda(P';Q')}{\det \Lambda(P;Q)} \tag{7}$$

Proof. Since $\Lambda = A - BD^{-1}C$, if κ_{pq} is subtracted from an entry of A, the only change in Λ is that κ_{pq} is subtracted from λ_{pq}. Therefore, by the conditions above, we have that

$$\det \Lambda'(P';Q') = \det \begin{bmatrix} \lambda_{pq} - \kappa_{pq} & \Lambda(p;Q) \\ \Lambda(P;q) & \Lambda(P;Q) \end{bmatrix} = 0$$

Temporarily denote this matrix as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We perform block Gaussian elimination to zero out b. To do this, we subtract $bd^{-1}c$ from a, and $bd^{-1}d$ from b. Note that

$$a - bd^{-1}c = \lambda_{pq} - \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;q) - \kappa_{pq} = \frac{\Lambda(P';Q')}{\Lambda(P;Q)} - \kappa_{pq}$$

Because performing Gaussian elimination in this way does not change the determinant of a matrix, we have

$$\det \begin{bmatrix} \frac{\Lambda(P';Q')}{\Lambda(P;Q)} - \kappa_{pq} & 0 \\ \Lambda(P;q) & \Lambda(P;Q) \end{bmatrix} = 0 \tag{8}$$

Since $\det \Lambda(P;Q) \neq 0$, (8) can be satisfied only if

$$\kappa_{pq} = \frac{\Lambda(P';Q')}{\Lambda(P;Q)},$$

which proves the boundary edge formula.

Note that since $\kappa_{pq} \neq 0$ by hypothesis, we have shown that $\det \Lambda(P';Q') \neq 0$ as well. In fact, we could have used this as a hypothesis instead of $\det \Lambda'(P;Q) \neq 0$. One could then see that since zeroing out κ_{pq} changed the value of $\det \Lambda(P';Q')$, the cofactor of κ_{pq} must have been non-zero, and this cofactor is exactly $\Lambda'(P;Q)$. We will now use this alternative hypothesis to find a sufficient condition for using the boundary edge formula in terms of connections on the associated graph $\mathcal{G}_K$.

As we have just seen, if we have that $\det \Lambda(P';Q') \neq 0$ but zeroing out κ_{pq} zeroes out $\det \Lambda(P';Q')$, then we can use the boundary edge formula. If K is an arbitrary square matrix, it is quite difficult to guarantee both of these conditions for any weights on the edges of $\mathcal{G}_K$. What is required is that there is a single way of making the connection from P to Q on $\mathcal{G}_K$. Moreover, this single connection must extend to the interior nodes of the graph; in other words, D_ϕ must collapse to a single term. (These restrictions do not make recovery impossible; Michael Goff in [3] shows how to overcome them under even stronger restrictions, as he needed D_ϕ to disappear completely, not just collapse to a single term.)

When dealing with Kirchhoff matrices, the restrictions are eased quite a bit, when P and Q are disjoint. Then (see [1]) we only need all permutations τ from P to Q on the electrical network to be of the same sign, and deleting edge pq breaks all of these connections. However, when P and Q are not disjoint, we are back to having very strict requirements on the connection. This is because we are now making connections which mix positive (diagonal) and negative (off-diagonal) entries in the product of entries $m_{i,j}$. Therefore we must again require that there be a single way of making the connection $(P;Q)$. Since principal proper submatrices of K are positive definite, however, we do not need D_ϕ to collapse to a single term. To summarize, we state again the sufficient connection-breaking properties for using the boundary edge formula:

Corollary 1. *Let K be a square matrix decomposed into $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where indices in A are denoted boundary indices, and indices in D are denoted interior indices. Assume that the location of all zeroes in K are known. Let $P' = p + P$ and $Q' = q + Q$ be two sets of boundary indices in A. Form the associated graph $\mathcal{G}_K$ of K. Then we can use the boundary edge formula to recover κ_{pq} if:*

- *(No restrictions on K) there is a single connection $(P';Q')$ on $\mathcal{G}_K$ and D_ϕ collapses to a single term.*
- *(K is a Kirchhoff matrix) there is a single connection $(P';Q')$ on $\mathcal{G}_K$.*
- *(K is a Kirchhoff matrix and P' and Q' are disjoint) the permutation of all connections $(P';Q')$ are of the same sign, and after deleting edge pq, there is no connection $(P';Q')$.*

3.1. **Boundary spikes.** In [1], a formula is given which will recover the conductivity γ_{pr} of a boundary spike under certain conditions (here p denotes the boundary node and r the interior node). This formula has no analogue in the recovery method presented by Nick Addington in [2], which purports to be a general recovery algorithm. In Addington's method, to recover a boundary spike conductivity γ_{pr}, one must first recover the diagonal entry κ_{pp} of the Kirchhoff matrix, and then use the relation $\kappa_{pp} = \gamma_{pr}$ to recover the boundary spike. In fact, we will see that this is essentially what the boundary spike formula in [1] does.

In the proof of the boundary spike formula, it is first noted that if the edge pr is a boundary spike, then within the Kirchhoff matrix, there is a submatrix of the form

$$K(p, r; p, r) = \begin{bmatrix} \kappa_{pp} & -\kappa_{pp} \\ -\kappa_{pp} & \sigma \end{bmatrix}$$

where the remaining entries in row p and column p are all 0. The next step is to expand $K(P+p+I; Q+p+I)$ (corresponding to the connection $(P+p; Q+p)$ on the graph) along the row corresponding to node p, which results in the formula

$$\begin{aligned} \det\ K(P+p+I; Q+p+I) = \kappa_{pp}\det\ K(P+I; Q+I)\ - \\ \kappa_{pp}^2\det\ K(P+I-r; Q+I-r). \end{aligned}$$

The conditions required to use the formula then imply that $\det K(P+I-r; Q+I-r) = 0$, so we can use the same Schur complement identity (Proposition 1) to show that

$$\det\ \Lambda(P+p; Q+p) = \kappa_{pp}\ \det\ \Lambda(P; Q)$$

which recovers the diagonal entry κ_{pp} and therefore γ_{pr}.

Since the boundary spike formula is essentially recovering the diagonal entry of the Kirchhoff matrix rather than the boundary spike conductivity (they just happen to be equal), it is not surprising to find that whenever one can apply the boundary spike formula, one could have applied the boundary edge formula instead to find κ_{pp}. To do so, we directly consider the connection $(P+p; Q+p)$ (which the boundary spike formula does implicitly). Since the connection $P; Q$ exists and $\kappa_{pp} \neq 0$, the extended connection also exists using the loop edge from p to itself. Furthermore, p must loop to itself; any other connection would necessarily use interior node r twice. Therefore, each possible $(P; Q)$ connection's contribution in the connection-determinant formula is simply multiplied by a non-zero factor κ_{pp}, and so if $(P; Q)$ is well-behaved, so is $(P+p; Q+p)$. To use the boundary edge formula, we must know that zeroing out κ_{pp} breaks the connection $(P+p; Q+p)$. Zeroing out the entry κ_{pp} means that any possible connection $(P+p; Q+p)$ with non-zero weight must have the set of paths from P to Q not use interior node r, since the connection from p to p must now use r. This is exactly the same restriction on $(P; Q)$ which results from contracting pr. Therefore, if contracting pr breaks a connection $(P; Q)$ (allowing us to use the boundary spike formula), then zeroing out the entry κ_{pp} will break the connection $(P+p; Q+p)$, allowing us to use the boundary edge formula. The boundary spike formula, as it turns out, can be subsumed by the boundary edge formula.

4. The Addington Recovery Method

The Addington recovery method, presented in [2], is currently the most versatile recovery algorithm we have. Central to the algorithm is examining a set of residue

(R) matrices to recover entries in the upper left corner of the Kirchhoff matrix. In this section we will examine the R matrix method of recovering information and compare this to using the boundary edge formula. To begin with, we will define the four matrices we will be considering, denoted K, Λ, Z, R:

Definition 3. *Given a $k \times k$ Kirchhoff matrix K decomposed into $\left(\begin{smallmatrix} A & B \\ B^\top & D \end{smallmatrix}\right)$ where A and D are square, let Λ denote the Schur complement K/D. The Z matrix is obtained from K by replacing A with a matrix of zeroes; that is, $Z = \left(\begin{smallmatrix} 0 & B \\ B^\top & D \end{smallmatrix}\right)$. The R matrix is then defined as the Schur complement Z/D.*

(Note that it is not necessary to restrict K to be a Kirchhoff matrix. To avoid confusion, however, K will denote a Kirchhoff matrix in this section and the rest of the paper.)

When tackling the inverse problem, we are given Λ and asked to construct K. Nick Addington's method actually attempts to recover R, and then uses the fact that $A = \Lambda - R$ to recover A. To recover an entry of R, we must know all but one of the entries in a particular submatrix of R, we must know that the particular submatrix is singular, and we must know that the cofactor of the unknown entry is non-singular. We can then solve for the unknown entry. Clearly, we must have some method of determining whether a determinant of R will be zero or non-zero. To do this, we note that since $R = Z/D$, for any submatrix $R(P;Q)$ of R,

$$\det\, R(P;Q) \cdot \det\, D = \det\, Z(P + D; Q + D)$$

Since D is non-singular, det $R(P;Q) = 0$ if and only if det $Z(P + D; Q + D) = 0$. The connection-determinant formula tells us that we can inspect determinants of submatrices of Z by looking at Z's associated graph $\mathcal{G}_Z$. We can easily obtain $\mathcal{G}_Z$: it is the graph which results from removing all boundary edges from the associated graph of K (which itself is the electrical network we are studying with loops representing diagonal entries added in). If a connection on $\mathcal{G}_Z$ cannot be made, then the corresponding submatrix of R is singular. Conversely, if a connection can be made in precisely one way on $\mathcal{G}_Z$, then the corresponding submatrix of R is non-singular.

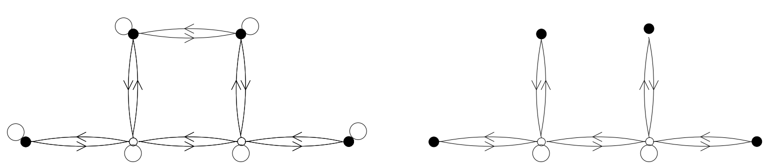

FIGURE 4. $\mathcal{G}_K$ and $\mathcal{G}_Z$ for the well-connected graph on 4 nodes

As we will now see, looking at the residue matrices will not give us any new information. All of the information we recover by looking at R matrices could have been recovered directly using the boundary edge formula, if we remove each boundary edge from the graph as we recover it. We state this as a theorem:

Theorem 3. *If an entry in R is recoverable, then the corresponding entry in A, the upper left corner of K is recoverable directly by using the boundary edge formula, if each boundary edge is removed from the graph as soon as it is recovered.*

Proof. Assume that we are recovering an entry r_{pq} of R. To recover r_{pq}, there must be some submatrix of R in which r_{pq} is the only unknown entry. Then the corresponding entries in the Kirchhoff matrix are known and so are 0, by hypothesis. Let the submatrix of R we are inspecting correspond to rows $P' = p + P$ and $Q' = q + Q$. Then $R(P;Q)$ (the cofactor of r_{pq}) is non-singular by hypothesis, and since $R = Z/D$, $Z(P + D; Q + D)$ and $K(P + D; Q + D)$ are non-singular as well ($K(P + D; Q + D) = Z(P + D; Q + D)$ since all of $K(P;Q)$ has been zeroed out). We now consider the submatrix $K(P' + D; Q' + D)$. Assume that κ_{pq} is the upper left entry of this submatrix. We consider $K(P' + D; Q' + D)$ as a linear function $F(z)$ of its first column. This first column can be represented as $z = x + y$, where $x = \left[\begin{smallmatrix}\kappa_{pq}\\ 0\end{smallmatrix}\right]$ and $y = \left[\begin{smallmatrix}0\\ a\end{smallmatrix}\right]$. Since κ_{pq} is the only non-zero entry in $K(P';Q')$, $F(y) = \det\, Z(P' + D; Q' + D) = 0$. Then

$$\begin{aligned}\det\,\mathrm{K}(P' + D; Q' + D) &= F(x) + F(y)\\ &= F(x)\\ &= \kappa_{pq}\,\det\,\mathrm{K}(P + D; Q + D)\end{aligned}$$

We have already shown that $K(P + D; Q + D)$ is non-singular, so we can proceed to solve for κ_{pq} by the boundary edge formula. (Author's note: To finalize the proof, a discussion of the square root trick should be given, since this is a way of recovering an entry in R. Perhaps in a later version this case will be handled).

It is not known at present whether the other direction of this theorem holds; that is, it is not known that one can always recover everything from the R matrix that one could recover from the boundary edge formula. Neither a proof nor a counter-example seems to be in sight.

5. Connections and Recoverability

5.1. **Circular Planar Graphs.** For circular planar graphs it has been shown that connections are intimately related to the recoverability of a network. In particular, a circular planar graph is recoverable if and only if removing or contracting any edge in the graph breaks a connection between disjoint sets of boundary nodes (for more information see [1]). A natural question to ask is whether this characteristic of circular planar graphs extends to the new loops representing diagonal entries. That is, is it true for critical circular planar graphs that zeroing out a diagonal entry breaks some connection in the graph? Unfortunately, the answer is no. The simplest recoverable graph which violates this property is the kite graph (the well-connected graph on 4 nodes) shown below on the left:

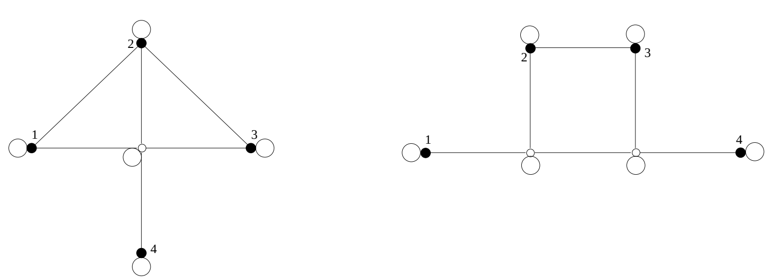

Figure 5. The kite graph and its wye-delta equivalent, the top hat graph

In fact, this graph is wye-delta equivalent to the "top hat" graph (depicted above on the right), which does have the property that zeroing out any entry in its Kirchhoff matrix breaks a connection on its associated graph. This shows that not even wye-delta transformations preserve the connection-breaking property (for edges on the associated graph) which was used to characterize the recoverability of circular planar graphs. While diagonal entries can sometimes be used to expedite the recovery of circular planar graphs, they certainly do not make the recovery process conceptually simpler.

5.2. **Arbitrary Graphs.** The situation is exacerbated when we look at non-circular planar graphs. Here even edges between disjoint vertices (non-loops) do not require the connecton-breaking property to be recoverable. For example:

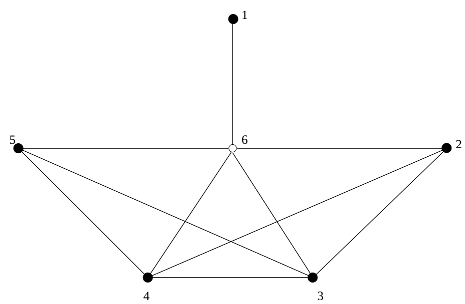

FIGURE 6. A graph without the connection-breaking property which characterizes circular planar recoverability

This graph is clearly recoverable, as we can use the (1,2;1,5) connection to recover κ_{11} and thereby the boundary spike, which we can then contract to complete the recovery process. However, deleting the (3,4) boundary edge does not break any connection in the graph. Every 2-connection involving the boundary edge (3,4) exists in 2 different permutations. Breaking the (3,4) edge only breaks one of these permutations. But if we recover, for example, the (2,3) edge (deleting this edge breaks the (1,2;3,5) connection) and then delete it, then we create a new electrical network in which deleting the (3,4) edge breaks the (1,3;2,4) connection, and so we can use the boundary edge formula to recover this edge. Therefore we can see that in the general case, there are situations in which a boundary edge is not immediately recoverable (by the boundary edge formula), but becomes so after recovering one or more boundary edges which do break connections in the graph.

This does not happen for "normal" boundary edges in the circular planar case, as it appears to require that connections $(P;Q)$ with two or more different permutations τ from starting vertices to ending vertices must exist in the graph, one of which is broken by deleting the boundary edge in question, and the others by deleting other boundary edges which are recoverable by looking at entirely different connections. By the Jordan Curve theorem, in the circular planar case there can only exist one permutation from starting vertices to ending vertices when examining any possible connection $(P;Q)$, if P and Q are disjoint. If P and Q are not disjoint, then this is not necessarily true, as we can see in the "kite" graph above. In fact, the five boundary node graph depicted above and the kite graph are quite similar. Zeroing out the diagonal entry κ_{11} in the kite graph will not break a connection for much the same reason as deleting the (3,4) boundary edge did not: every 2-connection on the kite graph involving the (1,1) edge exists in 2 permutations, only one of which is broken by deleting the (1,1) edge, and the other by deleting some (recoverable) boundary edge in the graph.

This makes it much more difficult to say if a particular boundary edge is recoverable just by looking at the topology of the graph that results by deleting that edge. It seems conceivable that some such description exists, at least for boundary edges recoverable by the boundary edge formula. It seems that for a boundary edge to be recoverable, deleting it must break one permutation τ of a particular $(P;Q)$ connection. The difficulty is in determining whether the other permutations will be broken by deleting other boundary edges. It may be that one must consider some property of all of the boundary edges in the graph as a whole, rather than inspecting them one by one. Clearly, there is much left to do in this area.

6. Future Research

(1) The boundary edge formula is a very general formula which works for recovering entries in the upper left corner of arbitrary square matrices. Is there a formula for entries outside of this corner, that is, boundary-interior or interior-interior connections? The formula would be much more complicated than the boundary edge formula, since the variation of Λ with respect to entries in A ($\Lambda = A - BC^{-1}B^{\top}$) is simpler than the variation with respect to the rest of K. On the other hand, Λ is still a linear function of entries in B or $B^{\top}$, so it seems like there could be some formula which allows us to recover entries in these submatrices without resorting to special properties of Kirchhoff matrices (specifically, that rows and columns add to 0).

(2) When we use the boundary edge formula, we are actually taking the Schur complement of $\Lambda(P';Q')$ mod $\Lambda(P;Q)$, where $P' = P + p$ and $Q' = Q + q$. It is interesting that we calculated Λ by taking a Schur complement in K, and then got back an entry κ_{pq} in K by taking a Schur complement in Λ. In a sense, the Schur complement seems to be its own inverse function (under very special conditions). Can we explain the boundary edge formula in a different way using this view of the Schur complement? What happens if we take the Schur complement of some rows in Λ when the conditions of the boundary edge formula do not apply? Is it purely a coincidence that we used the Schur complement twice to get back an entry in K, or is something else going on here?

(3) The question brought up at the end of the previous section: is there a way to characterize (by edge-breaking properties) whether or not a boundary edge will be recoverable by the boundary edge formula?

(4) The other direction of the theorem in Section 4: Will inspecting R matrices work to recover a boundary edge whenever that edge is recoverable by the boundary edge formula? This would certainly be very convenient, since it is much easier to inspect the Z matrix and its associated graph for connections that do/do not exist rather than K itself.

7. References

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