METRIC RECOVERABILITY

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Abstract. We consider the recoverability of the Riemannian metric of a graph given information about the geodesic distance between boundary vertices on the graph. In the continuous case, when the metric is placed under certain restrictions, this information is enough to uniquely determine the metric up to an isometry which is the identity on the boundary [1]. In this paper we explore the discrete case and attempt to identify when the metric is uniquely determined on a discrete graph.

1. Introduction

The motivation for studying metric graphs comes from a paper by Leonid Pestov and Gunther Uhlmann titled *Two dimensional compact simple Riemannian manifolds are boundary distance rigid*. The geodesic distance between any two points, x and y, on a Riemannian manifold (M, g) with boundary ∂M is denoted by Pestov and Uhlmann as $d_g(x, y)$. The path which determines this distance is assumed to be unique on the manifold. The authors question whether the metric of a Riemannian manifold can be determined knowing only $d_g(x, y)\ \forall\ x, y \subset \partial M$. A Riemannian manifold is simple if it is simply-connected, any geodesic has no conjugate points, and the boundary is strictly convex. The following theorem is proved by Pestov and Uhlmann:

Theorem 1.1. *Let $(M, g_i), i = 1, 2$ be two dimensional simple compact Riemannian manifolds with boundary. Assume*

$$d_{g_1}(x, y) = d_{g_2}(x, y)\ \forall (x, y) \in \partial M \times \partial M.$$

Then there exists a diffeomorphism $\psi : M \to M, \psi\mid_{\partial M}= Id$, so that

$$g_2 = \psi^* g_1$$

Here we consider the application of these results to discrete metric graphs.

Definition 1.2. A discrete *metric graph* is a graph, $G(V, E, d)$ with vertices, V, boundary vertices $\partial V \subset V$, edges, $E \subset V \times V$, a distance function, $d : V \times V \to \mathbb{R}^{\geq 0}$, and an edge function $c : E \to \mathbb{R}^+$.

We denote $e_{k,l}$, $k, l \in V$, $k \in N(l)$, where $N(l)$ is the set of neighboring vertices to l, as the value of the function c on the single edge between neighboring vertices k and l.

Definition 1.3. Let $i, j \in V$. A *path*, P, from i to j is a set of edge values $(e_{k_0,l_0}, e_{k_1,l_1}, \ldots e_{k_n,l_n})$ with $k_0 = i$, $l_n = j$, and $l_a = k_{a+1}\ \forall\ 0 \leq a < n$.

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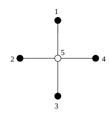

FIGURE 1.

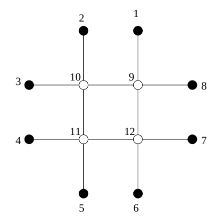

FIGURE 2. The 2×2 lattice

Definition 1.4. Given a set of paths $P_{i,j}$ from i to j in V, the *geodesic distance* $d_{i,j}$ between the vertices is defined as the minimum sum of the edge values in each path.

The distance function d on metric graphs is defined as the geodesic distance between vertices.

Definition 1.5. The discrete *metric* of a graph, G, is the column vector of edge distances, $e_{k,l}$, $k, l \in V$ in the graph, where $k \in N(l)$.

We attempt to recover the graph's metric given geodesic distances between all vertices, $v \in \partial V$ of a metric graph. Just as in the continuous case, this is not always enough information to determine the metric of the graph. This paper will investigate the additional necessary tools for metric recoverability and the obstructions to unique recoverability.

2. Recoverable Metric Graphs

Consider the simple case of the four star graph, composed of four edges, four boundary vertices, and one interior vertex shown in figure 1. The information we have is:

$$\begin{aligned}
d_{1,2} &= e_{1,5} + e_{5,2} \\
d_{2,3} &= e_{2,5} + e_{5,3} \\
d_{3,4} &= e_{3,5} + e_{5,4} \\
d_{1,4} &= e_{1,5} + e_{5,4} \\
d_{1,3} &= e_{1,5} + e_{5,3} \\
d_{2,4} &= e_{2,5} + e_{5,4}
\end{aligned}$$

Given the geodesic distances, we can use this system of equations to solve for each edge distance. The metric for this graph is recoverable.

For more complicated graphs, metric recoverability becomes more difficult because there are multiple paths available between boundary vertices. Consider the 2×2 square lattice in figure 2.

$$d_{1,2} = min((e_{1,9} + e_{9,10} + e_{10,2}), (e_{1,9} + e_{9,12} + e_{12,11} + e_{11,10} + e_{10,2})) \tag{1}$$

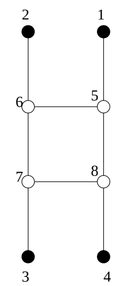

FIGURE 3.

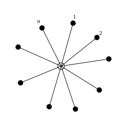

FIGURE 4. The n-star

The system of equations from the geodesic distances of this graph is not linear as in the first example and it is easy to see that more information is needed to determine the metric. If $d_{1,2} = e_{1,9} + e_{9,12} + e_{12,11} + e_{11,10} + e_{10,2}$ then it is possible that no geodesic lies on the edge between vertices 9 and 10. In this case, it is impossible to recover the distance of this edge. Another problem that will be of relevance later is the possibility that these two paths have the same geodesic distance–in such a case we are unable to determine which path is taken by the geodesic and therefore we can't translate the information we have. Similar problems in the continuous case led Pestov and Uhlman to impose restrictions on the metrics taken into consideration, considering only simple Riemannian manifolds. We attempt to alter these restrictions and apply them to the discrete case in order to define the problem of discrete metric recoverability.

3. Simple Discrete Metric Graphs

3.1. **Restrictions on Metric.** We limit the possible paths between boundary vertices by placing restrictions on the discrete metrics we consider. The metric of a discrete graph is *simple* if:

1. Each edge $e \in E$ lies on some geodesic.
2. The path corresponding to the geodesic distance is unique.

Taking into consideration only metrics which satisfy these conditions, we proceed to investigate metric recoverability.

3.2. **Restrictions on Graph Structure: Parameterizing metric response matrices.** It is also important that the graph provides enough information to recover the metric. This is not an issue in the continuous case, but in the discrete case it is possible that there is not enough independent information provided from the boundary information to recover the metric. For example, the graph in figure 3 has four boundary vertices and thus $\binom{4}{2} = 6$ pieces of information from boundary to boundary geodesic distances. But the graph is composed of eight edges, so even if the metric satisfies the above restrictions, it is impossible to determine the distances

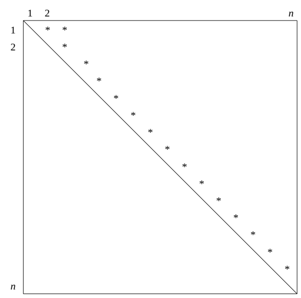

FIGURE 5. Parameterization for the n-star

	1	2	3	4	5	6	7	8
1		*	*			*	*	*
2			*					
3				*	*			
4					*			
5						*		
6							*	
7								*
8								

FIGURE 6. Parameter Positions for the response of the 2×2 lattice

of each edge given only this information. Also, even if a graph provides the right amount of information, some of the information may be redundant. For example, in the case of the four-star described above, all of the edges can be determined by only knowing four of the six boundary to boundary distances. In fact, all entries in the response matrix for any n-star can be determined by the indicated entries in figure 5. For this reason, we require that the metric graph have at least as many parameters as edges.

The remainder of this paper considers simple metric graphs with at least as many parameters as edges.

4. THE 2×2 LATTICE IS TOAST

Restricting the metric with the above conditions makes the metric recoverability of the 2×2 lattice very simple. First we notice that there is enough independent information to recover the metric. A parameterization for the 2×2 lattice is shown in figure 6; there are 12 parameters in the response matrix and 12 entries to recover. Next we make use of the fact that the metric is simple.

Definition 4.1. A *boundary antenna* is a pair of boundary spikes that share a common vertex. [2]

The 2×2 lattice contains four boundary antennas.

Theorem 4.2. *The edge distances of any boundary antenna are uniquely recoverable.*

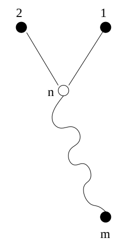

Figure 7. Boundary Antenna

Proof. Refer to the boundary antenna in figure 7. We know that the geodesic distance $d_{1,2}$ is the sum of $e_{1,n}$ and $e_{n,2}$; these edges must lie on the geodesic because there are no other edges connected to vertices 1 and 2, and the shortest distance from n to n is zero. Since there are no negative distances, any path besides the path which uses only these edges will be larger than $e_{1,n} + e_{n,2}$. We assume that there are some other boundary vertices on the graph because if not, then we would only have $\binom{2}{2} = 1$ piece of information and two edges to recover. Create a boundary vertex m anywhere on the graph. We know the geodesic distances $d_{1,m}$ and $d_{2,m}$. The first edge travelled by the geodesic $d_{1,m}$ must be the edge $e_{1,n}$ and similarly, the first edge travelled by the geodesic $d_{2,m}$ must be the edge $e_{2,n}$. By the uniqueness of geodesics, there is only one shortest distance from point n to point m. Therefore,

$$d_{1,m} - e_{1,n} = d_{2,m} - e_{2,n}.$$

Since we also know that $d_{1,2} = e_{1,n} + e_{n,2}$, we have a system of linear equations which allows us to solve for $e_{1,n}$ and $e_{2,n}$:

$$e_{1,n} = \frac{d_{1,m}+d_{1,2}-d_{2,m}}{2}$$

$$e_{2,n} = \frac{d_{2,m}+d_{1,2}-d_{1,m}}{2}$$

□

Theorem 4.3. *The shortest distance between neighboring vertices is the edge adjoining those vertices, i.e.* $d_{kl} = e_{kl} \ \forall \ k, l \in V$ *and* $k \in N(l)$.

Proof. Consider the vertices 9 and 10 on the 2×2 lattice in figure 2. We know by restrictions on the metric that $e_{9,10}$ lies on some geodesic of the graph. Let $m, n \in V$ and assume $e_{9,10}$ lies on the shortest path between these vertices. Therefore $d_{m,n} = x + e_{9,10}$ where x is the sum of all edges besides $e_{9,10}$ which make up the geodesic. Now, assume that the shortest distance between these vertices is not the edge $e_{9,10}$. Then there exists some path between vertex 9 and vertex 10 such that $d_{9,10} < e_{9,10}$ and so $x + d_{9,10} < x + e_{9,10}$. But $x + d_{9,10}$ also connects the vertices m and n, contradicting the fact that $d_{m,n}$ is the geodesic distance between m and n. Therefore, the shortest distance between neighboring vertices is the edge adjoining those vertices. □

Now that the 2×2 lattice is a simple metric graph, we know more information about which paths are taken by the geodesics. We can immediately determine the edge distances $e_{1,9}, e_{2,10}, e_{3,10}, e_{4,11}, e_{5,11}, e_{6,12}, e_{7,12}, e_{8,9}$ because these are all

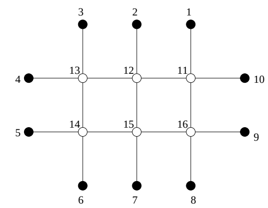

Figure 8. The 2×3 Lattice

	1	2	3	4	5	6	7	8	9	10
1		*	*	*				*	*	*
2			*				*			
3				*						
4					*	*				
5						*		*		
6							*			
7								*		
8									*	
9										*
10										

Figure 9. Parameter Positions for the response of the 2×3 lattice

boundary antenna edges. We know that $d_{1,2} = e_{1,9} + e_{9,10} + e_{10,2}$ because $e_{9,10} < e_{10,11} + e_{11,12} + e_{12,9}$. From this we can determine $e_{9,10}$ and similarly, $e_{10,11}, e_{11,12}$ and $e_{12,9}$. Note that there are still some geodesics which have ambiguous paths. We do not know from our restrictions which path is taken by $d_{2,6}$ for example. But from the known geodesic paths, we can recover the metric of the 2×2 lattice.

When the metric is uniquely determined everywhere on the graph as with the four star and the 2×2 lattice, the analogue to a diffeomorphism in the continuous case is the identity matrix:

Let G, G' be two simple discrete metric graphs with boundary. If the metric is uniquely recoverable on G and $d_{i,j} = d_{i',j'} \forall i, j \subset \partial V, \forall i', j' \subset \partial V'$, then

$$e = Ie' \tag{2}$$

where e, e' are the metrics of G, G' respectively. We label the matrix which is analagous to the diffeomorphism in the continuous case ψ.

5. Graphs Which are Not Uniquely Recoverable

5.1. **The 2×3 lattice.** Consider the slightly more difficult example of the 2×3 lattice shown in figure 8. We see that the graph structure satisfies the restrictions. The parameters for the 2×3 lattice are shown in figure 9; there are 17 parameters and 17 edges in the graph. So we can assume that the metric is simple and proceed to recover the metric. By theorem 4.2 we can determine the edge distances $e_{1,11}, e_{3,13}, e_{4,13}$, $e_{5,14}, e_{6,14}, e_{8,16}, e_{9,16}$, and $e_{10,11}$, and by theorem 4.3 we can determine $e_{13,14}$ and $e_{16,11}$. We do not know, however, which path is taken by the geodesics through vertices 1 and 3 or through vertices 6 and 8 and so we don't have enough information to isolate the middle edges. However, we can determine which path is taken by these geodesics based on the information provided. First, since

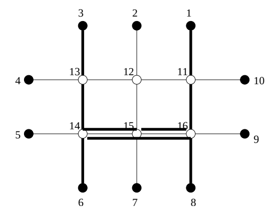

FIGURE 10.

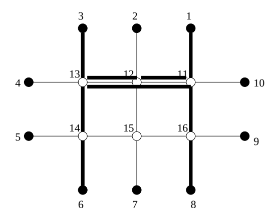

FIGURE 11.

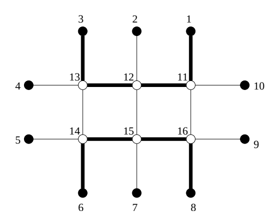

FIGURE 12.

$e_{1,11}$ and $e_{13,3}$ must lie on the geodesic from vertex 1 to 3, we can subtract these known edge lengths and deal only with the distance from vertex 11 to vertex 13, $d_{11,13}$. We do the same for the geodesic from 6 to 8 to get the distance $d_{14,16}$. Now, since we also know $e_{13,14}$ and $e_{11,16}$, subtract these edges from both of these new distances. If $d_{14,16} = d_{11,13} - e_{13,14} - e_{16,11}$, then both the geodesics from 1 to 3 and from 6 to 8 take paths which include the edges $e_{14,15}$ and $e_{15,16}$ (see figure 6). If $d_{11,13} = d_{14,16} - e_{14,13} - e_{11,16}$, then both of these geodesics take paths which include the edges $e_{11,12}$ and $e_{12,13}$ (see figure 11). If neither of these equalities holds, then the paths taken are as shown in figure 12. This gives three equivalence classes of paths taken by the boundary to boundary geodesic distances in the 2×3 lattice. If the paths are as shown in figure 10, then instead of the identity matrix,

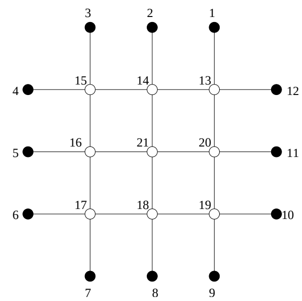

FIGURE 13. The 3×3 Lattice

ψ looks something like this:

$$
(3) \qquad \begin{pmatrix} e_{1,11} \\ e_{3,13} \\ e_{4,13} \\ e_{5,14} \\ e_{6,14} \\ e_{7,15} \\ e_{8,16} \\ e_{9,16} \\ e_{10,11} \\ e_{13,14} \\ e_{14,15} \\ e_{15,16} \\ e_{11,16} \\ e_{2,12} \\ e_{11,12} \\ e_{12,13} \\ e_{12,15} \end{pmatrix} = \begin{pmatrix} 1 & 0 & . & . & . & & & & \\ 0 & . & & & & & & & \\ . & & . & & & & & & \\ . & & & . & & & & & \\ . & & & & 1 & & & & \\ & & & & & a_1 & a_2 & a_3 & a_4 \\ & & & & & b_1 & b_2 & b_3 & b_4 \\ & & & & & c_1 & c_2 & c_3 & c_4 \\ & & & & & d_1 & d_2 & d_3 & d_4 \end{pmatrix} \begin{pmatrix} e'_{1,11} \\ e'_{3,13} \\ e'_{4,13} \\ e'_{5,14} \\ e'_{6,14} \\ e'_{7,15} \\ e'_{8,16} \\ e'_{9,16} \\ e'_{10,11} \\ e'_{13,14} \\ e'_{14,15} \\ e'_{15,16} \\ e'_{11,16} \\ e'_{2,12} \\ e'_{11,12} \\ e'_{12,13} \\ e'_{12,15} \end{pmatrix}
$$

where the columns preserve certain linear relations;

$$3(a_1) + b_1 + c_1 + d_1 = 3$$
$$3(a_2) + b_2 + c_2 + d_2 = 1$$
$$3(a_3) + b_3 + c_3 + d_3 = 1$$
$$3(a_4) + b_4 + c_4 + d_4 = 1$$

If the paths are as shown in figure 11, then the matrix will be similar; the submatrix of the four rows which differ from the identity matrix will shift to correspond to edges $e_{7,15}, e_{15,16}, e_{6,14}$ and $e_{12,15}$. If the paths are as shown in figure 12, the above matrix can be used with $a_2, a_3, a_4, b_1, b_3, b_4, c_1, c_2, c_4, d_1, d_2, d_3 = 0$ and $a_1, b_2, c_3, d_4 = 1$(The identity matrix).

5.2. **The 3×3 lattice.** The 3×3 lattice is shown in figure 13. Again, we can immediately determine the edge lengths $e_{1,13}, e_{3,15}, e_{4,15}, e_{6,17}, e_{7,17}, e_{9,19}, e_{10,19}, e_{12,13}$. Just as with the 2×3 lattice, we can also determine which paths are taken by the geodesic distances $d_{2,11}, d_{2,5}, d_{5,8}$, and $d_{8,11}$. Consider the geodesic with distance $d_{2,11}$–the path from vertex 2 cannot leave vertex 14 along the edge $e_{14,15}$. If it did, then no boundary to boundary geodesic would go through the middle edges. Therefore it must take edge $e_{14,21}$ or $e_{14,13}$. Since we know that $d_{1,2}$ is the sum of the edges $e_{1,13}, e_{13,14}, e_{14,2}$ and $d_{11,12}$ is the sum of the edges $e_{11,20}, e_{20,13}, e_{13,12}$, we can determine $e_{2,14} + e_{14,13}$ and $e_{13,20} + e_{20,11}$ from the known edges. Therefore, if $d_{2,11} = e_{2,14} + e_{14,13} + e_{13,20} + e_{20,11}$ then the path travels along these edges. If not, then the path takes the edges $e_{14,21}$ and $e_{21,20}$ instead of $e_{14,13}$ and $e_{13,20}$. This gives us a number of different possibilities up to symmetry for the paths of these four geodesics, and in each case, other geodesic paths are necessarily determined. We cannot, however, necessarily determine which paths are taken by the geodesics that determine the distances $d_{1,3}, d_{4,6}, d_{7,9}, d_{10,12}$. So even though we know some of the summation relations between edge lengths of the 3×3 lattice, we can't create an exact matrix ψ.

6. Further investigation

Although the matrix relations between metrics approximate the discoveries in the continuous case, the generality of these relations is somewhat disappointing. We have limitations on the column sums of the matrix, but rarely can we determine the complete matrix unless it happens to be the identity matrix. We have considered making more restrictions on simple graphs. Pestov and Uhlmann noted that metric determination is impossible if there is a point $x_0 \in M$ so that $d_g(x_0, \partial M) > sup_{x,y \in \partial M} d_g(x, y)$ because in this case d_g is independent of a change of g in a neighborhood of x_0 [1]. This scenario is not possible for discrete simple metric graphs, but it is possible for $d_{i,j}, i, j \in \partial M$ to be independent of a change in the metric in the interior of the graph. It might be impossible to come up with a complete analogue to the continuous case. Pestov and Uhlmann define a scattering relation on Riemannian manifolds and use this to define the Dirichlet to Neumann map for metrics. The scattering relation takes any boundary point and any direction and determines a unique resulting boundary point by following the unique geodesic in that direction. There is no scattering relation in the discrete case. The boundary vertices in the lattice examples have only one possible starting direction. We considered looking at graphs without spikes to allow for more direction possibilities, but unless the graph is well-connected, there are never enough directions to allow for a unique boundary to boundary geodesic in each direction. Graphs without spikes lead to another issue with discrete metric graphs because in these cases geodesics may lie on boundary to boundary edges. If a geodesic went along the boundary in the continuous case, the surface would fail to be strictly convex. This is why all of the examples in this paper dealt with graphs with all boundary spikes. But the introduction of other graphs leads to more possibilities for the forward problem. We might also require that we know the direction from boundary vertices taken by the geodesics. There is room for adjustment to the problem as it is posed here, but in any case we hope to make more concrete discoveries about discrete metric determination.

References

[1] Pestov, Leonid; and Uhlmann, Gunther. "Two dimensional compact simple Riemannian manifolds are boundary distance rigid." 2005.
[2] Goff, Michael. "Recovering Networks With Signed Conductivities." 2003.