THE CUTSET-RANK COROLLARY

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Abstract. This paper contains a generalization of the relationship outlined in [1] between the connections of an electrical network Γ and the determinants of $\Lambda(P;Q)$. Specifically, it will be shown that rank $\Lambda(P;Q) \leq |M|$ where M is the minimal cutset of the graph of Γ. In addition, when the graph is circular planar, rank $\Lambda(P;Q) = |M|$.

Let $\Gamma = (G, \gamma)$ be an electrical network, and let the vertices V of G be partitioned into $V = \partial V \cup \text{int } V$, where $\partial V \neq \emptyset$. The response matrix Λ of Γ is defined as the matrix for which $\Lambda\phi = \psi$, where ϕ is the colulmn vector of voltages defined on ∂V, and ψ is the vector of currents on ∂V. Let P and Q be two disjoint subsets of ∂V, where $|P| = |Q|$. Then $\Lambda(P;Q)$ is the submatrix of Λ with row indices P and column indices Q. In [1] it is proved that if det $\Lambda(P;Q) \neq 0$ then there exists a k-connection from P to Q, where $k = |P| = |Q|$. Moreover, in the circular planar case the existence of a k-connection implies that det $\Lambda(P;Q) \neq 0$. Our generalization of these two statements will require the following definitions:

Definition 1. (Cutset, Minimal Cutset): *Given a graph $G = \{V, E\}$ (where V is the set of vertices and E is the set of edges) and two disjoint sets of vertices $P, Q \subset V$ a* cutset *of the pair P, Q is a set $R \subset V$ such that all paths from a vertex $p \in P$ to a vertex $q \in Q$ intersect R. A* minimal cutset *of the pair P, Q is a cutset of P, Q of minimal cardinality among all such cutsets.*

We can now state Menger's Theorem:

Theorem 1. (Menger's Theorem [2]): *Let $G = \{V, E\}$ be a graph and A,$B \subseteq V$. Then the cardinality of the minimum cutset of the pair A,B in G is equal to the maximum number of disjoint paths from A to B in G.*

Definition 2. (Degree of Connectivity) *Given two disjoint subsets of ∂V P and Q, the* degree of connectivity *of (P, Q) is defined as the largest integer k such that there exists a k-connection between P' and Q', where $P' \subseteq P$ and $Q' \subseteq Q$.*

Our proof of the cutset-rank corrolary requires the following lemma.

Lemma 1. *Given a graph G, define a graph G' where $V' = V - \partial V + P + Q$ and $E' = E - E(\partial V)$ where $E(\partial V)$ are all edges incident on a deleted vertex. On this graph G', the degree of connectivity is equal to the cardinality of the minimum cutset.*

This follows directly from Menger's Theorem as the degree of connectivity is precisely the maximum number of disjoint paths from P to Q.

We now head on to the desired result.

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Corollary 1. (Cutset-Rank Corollary) *Given* $\Lambda(P;Q)$ *defined on* $\Gamma = (G, \gamma)$. *Let the minimal cutset of* (P,Q) *be denoted by* M. *In general,*

$$\text{rank } \Lambda(P;Q) \leq |M| \tag{1}$$

Moreover, if G *is circular planar,*

$$\text{rank } \Lambda(P;Q) = |M| \tag{2}$$

Proof. Let $\Lambda(P;Q)$ have rank m. This implies that the largest non-zero subdeterminant of $\Lambda(P;Q)$ is of size $m \times m$. Therefore there exists an m-connection between $P' \subseteq P$ and $Q' \subseteq Q$ where $|Q'| = |P'|$. Thus the degree of connectivity is at least m, the rank of $\Lambda(P;Q)$.

Moreover, if G is circular planar, since every subdeterminant larger than $m \times m$ of $\Lambda(P;Q)$ is zero then there does not exist any k-connection for $k > m$. This implies then that for the circular planar case the degree of connectivity is equal to m, the rank of $\Lambda(P;Q)$.

By Lemma 1 both of these statements can translated into the desired results. Q.E.D.

References

[1] Curtis, Edward B. & Morrow, James A., *Inverse Problems for Electrical Networks*, Series on Applied Mathematics - Vol. 13, World Scientifc, New Jersey, 2000.

[2] Diestel, Richard, *Graph Theory* , Springer-Verlag, New York, 1997.