CHARACTERIZING SEMI-RECOVERABLE GRAPHS

JACOB DANTON

ABSTRACT. This paper discusses the application of the method described in by Nick Addington in *A Method for Recovering Arbitrary Graphs* [1] to a known class of two-to-one graphs to recover the characterizing quadratic. It also looks into the different conditions for when a network is two-to-one and when it is one-to-one. Lastly, it discusses the application of the method to all semi-recoverable graphs in order to characterize their recoverability.

CONTENTS

Date: August 11, 2005.

1. Introduction

In [1], Nick Addington describes a method for recovering arbitrary graphs. It is known to recover many different recoverable graphs(for examples, see [1]). However, it is unknown if this method can recover all recoverable graphs. For example, it is unknown if this method recovers graphs that are two-to-one. This paper deals with the application of this method on the triangle in triangle graph, which was shown to be two-to-one in [4].

2. Definitions

This paper assumes the reader is familiar with the terminology, definitions, and notation found in the papers referenced [1, 2, 3, 4, 5]. Additionally, the following are used.

Definition 2.1. The *shape* of a matrix, as used in this paper, is used to denote the pattern of the zero and non-zero entries, independent of signs. It does not refer to isomorphisms.

Definition 2.2. An entry in a matrix R_i is said to be *known* if that same entry is equal to zero in K_i.

Definition 2.3. A network that is neither recoverable ($1 \to 1$) nor nonrecoverable ($\infty \to 1$) is said to be *semi-recoverable*. For example, the N-gon-in-N-gon networks which are $2 \to 1$.

Definition 2.4. Semi-recoverable graphs are shown to be so through a polynomial or a system of polynomials which characterize it. These are refered to as *characteristic* polynomials.

Remark 2.5. The notation $M(i, j)$ denotes the entry of M at the i^{th} row, j^{th} column. The notation $M(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_m)$ denotes the $n \times m$ submatrix consisting of the entries found at the intersection of each a_i row and b_j column. Thus, $\det M(a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_m)$ denotes the determinant of the submatrix as defined.

3. The Recovery Method

The method makes use of the following lemma and proposition.

Lemma 3.1. *If an $n \times n$ matrix M is singular, we know all but one entry m_{ij}, and the cofactor M_{ij} is invertible, we can recover the unknown entry.*

Proposition 3.2. *Let M be the submatrix of R^{14}_{15} consisting of rows $r_1, \ldots, r_n$ by columns $c_1, \ldots, c_n$. Let N be the submatrix of Z^{14}_{15} consisting of rows $r_1, \ldots, r_n, 14, 15$ by columns $c_1, \ldots, c_n, 14, 15$. Then M is singular if and only if N is.*

Proof. For the proofs, see [1]. □

The method as described by Nick Addington in [1]:

(1) Write down the signs of all the entries of K_N, which we know from the graph. From these, determine the signs of all the entries of $K_M, \ldots, K_{N-1}$. We will use these to test submatrices for singularity using Proposition 3.2.

(2) Make empty matrices $K_M, \ldots K_N$, $R_{M+1}, \ldots, R_N$, and R^m_n, $M < m < n \leq N$ of the appropriate sizes: K_n is $n \times n$, and R_n and R^m_n are $n-1 \times n-1$. Fill in the zeros of all these, which can be derived from the zeros of K_N. Fill in the entries of K_M, the response matrix.

(3) Whenever we know two of three entries from something of the form $R^{13}_{16} = \underline{R^{15}_{16}|} + R^{13}_{14}$ or
$\overline{K_{13}} = \underline{K_{15}|} + R^{14}_{15}$, recover the third.

(4) Whenever we know all but one entry of a submatrix of any matrix, if the submatrix is singular and the cofactor of the unknown entry is invertible, recover the unknown entry using Lemma 3.1.

(5) Whenever we know all but one entry in a row of a K_n, recover it using the fact that the rows of Kirchhoff matrices sum to zero.

(6) If at any point no more entries can be recovered but some are still missing, parameterize an unknown entry. The first single layer R_n with unknown entries (first in the sense that n is least) seems to be the best place to parameterize.

4. Applying the Method to the Triangle-in-Triangle Graph

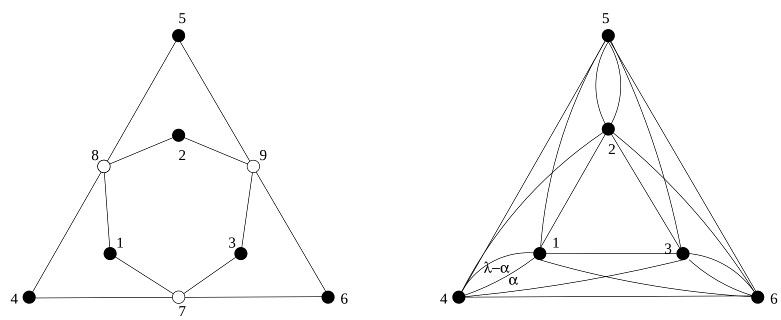

Figure 1. the Triangle in Triangle

To begin, we will apply the method to the simplest graph, the triangle-in-triangle graph. This graph was first shown to be two-to-one by Ernie Esser in [3] and later by Jennifer French and Shen Pan in [4].

4.1. **The Matrices.** The first step in recovering the graph is to find all zero entries in the K and R matrices. In this case, the shapes of the K matrices are as follows.

$$K_9 = \begin{bmatrix} + & 0 & 0 & 0 & 0 & 0 & - & - & 0 \\ 0 & + & 0 & 0 & 0 & 0 & 0 & - & - \\ 0 & 0 & + & 0 & 0 & 0 & - & 0 & - \\ 0 & 0 & 0 & + & 0 & 0 & - & - & 0 \\ 0 & 0 & 0 & 0 & + & 0 & 0 & - & - \\ 0 & 0 & 0 & 0 & 0 & + & - & 0 & - \\ - & 0 & - & - & 0 & - & + & 0 & 0 \\ - & - & 0 & - & - & 0 & 0 & + & 0 \\ 0 & - & - & 0 & - & - & 0 & 0 & + \end{bmatrix} \qquad K_8 = \begin{bmatrix} + & 0 & 0 & 0 & 0 & 0 & - & - \\ 0 & + & - & 0 & - & - & 0 & - \\ 0 & - & + & 0 & - & - & - & 0 \\ 0 & 0 & 0 & + & 0 & 0 & - & - \\ 0 & - & - & 0 & + & - & 0 & - \\ 0 & - & - & 0 & - & + & - & 0 \\ - & 0 & - & - & 0 & - & + & 0 \\ - & - & 0 & - & - & 0 & 0 & + \end{bmatrix}$$

$$K_7 = \begin{bmatrix} + & - & 0 & - & - & 0 & - \\ - & + & - & - & - & - & 0 \\ 0 & - & + & 0 & - & - & - \\ - & - & 0 & + & - & 0 & - \\ - & - & - & - & + & - & 0 \\ 0 & - & - & 0 & - & + & - \\ - & 0 & - & - & 0 & - & + \end{bmatrix} \qquad K_6 = \Lambda = \begin{bmatrix} + & - & - & - & - & - \\ - & + & - & - & - & - \\ - & - & + & - & - & - \\ - & - & - & + & - & - \\ - & - & - & - & + & - \\ - & - & - & - & - & + \end{bmatrix}$$

From these, we can find the shape of the residue matrices as well as the known and unknown entries. In this case, the single layer residue matrices are as follows

$$R_7 = \begin{bmatrix} ? & 0 & X & ? & 0 & X \\ 0 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & ? & X & 0 & ? \\ ? & 0 & X & ? & 0 & X \\ 0 & 0 & 0 & 0 & 0 & 0 \\ X & 0 & ? & X & 0 & ? \end{bmatrix}$$

$$R_8 = \begin{bmatrix} ? & X & 0 & X & X & 0 & 0 \\ X & ? & 0 & X & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & 0 & ? & X & 0 & 0 \\ X & ? & 0 & X & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_9 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & ? & X & 0 & X & X & 0 & 0 \\ 0 & X & ? & 0 & X & X & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X & X & 0 & ? & X & 0 & 0 \\ 0 & X & X & 0 & X & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Where each "?" denotes an unknown entry and each "X" an entry that follows directly from the K matrix before it. For example, since $K_7(1,3) = 0$, $R_7(1,3) = K_6(1,3)$, so that entry is an "X".

4.2. **Solving.** As described above, the next step to recovering the Kirchoff matrix would be to find all entries in R_7. From this, we could find K_7, R_8, etc. until we had recovered R_9, at which point we could get the characteristic polynomial. However, since we are only interested in finding the characteristic polynomial, we will begin at R_9, and by looking at the determinant of $R_9(1,2;5,6)$, we can get the quadratic using an easier method which can be easily extended to any n-gon in n-gon network.

Since every 2×2 determinant of R_9 is equal to zero, we have:

$$det R_9(2,3;5,6) = 0$$

Which gives use the equation

$$R_9(2,5)R_9(3,6) - R_9(2,6)R_9(3,5) = 0$$

Since each of the corresponding terms in K_9 is zero, we can rewrite this as

$$K_8(2,5)K_8(3,6) - K_8(2,6)K_8(3,5) = 0$$

We can now use the fact that $K_N = \underline{K_{N+1}|} + R_{N+1}$ to rewrite all the remaining K matrices in terms of R matrices and $\overline{K_6}$ (ie Λ). Giving us

$$[K_7(2,5) - R_8(2,5)][K_7(3,6) - R_8(3,6)] - [K_7(2,6) - R_8(2,6)][K_7(3,5) - R_8(3,5)] = 0$$

and

$$[K_6(2,5) - (R_7(2,5) + R_8(2,5))][K_6(3,6) - (R_7(3,6) + R_8(3,6))]$$
$$-[K_6(2,6) - (R_7(2,6) + R_8(2,6))][K_6(3,5) - (R_7(3,5) + R_8(3,5))] = 0$$

From the above matrices, we know which entries are zero and which are known, allowing us to simplify this to

$$[\lambda_{2,5} - R_8(2,5)][\lambda_{3,6} - R_7(3,6)] - \lambda_{2,6}\lambda_{3,5} = 0$$

As $R_8(2,5)$ is an unknown entry, we need to rewrite by looking at $detR_8(1,2;4,5)$ and continuing to reduce each following unknown entry in this manner. Since $R_7(3,6)$ is an unsolvable, unknown entry and we must introduce a parameter into R_7 to solve, we can arbitrarily make this the parameterized entry t. Thus, we get

$$[\lambda_{2,5} - \frac{R_8(1,5)R_8(2,4)}{R_8(1,4)}][\lambda_{3,6} - t] - \lambda_{2,6}\lambda_{3,5} = 0$$

$$[\lambda_{2,5} - \frac{\lambda_{1,5}\lambda_{2,4}}{K_7(1,4)}][\lambda_{3,6} - t] - \lambda_{2,6}\lambda_{3,5} = 0$$

$$[\lambda_{2,5} - \frac{\lambda_{1,5}\lambda_{2,4}}{K_6(1,4) - R_7(1,4)}][\lambda_{3,6} - t] - \lambda_{2,6}\lambda_{3,5} = 0$$

And finally end up with

$$[\lambda_{2,5} - \frac{\lambda_{1,5}\lambda_{2,4}}{\lambda_{1,4} - \frac{\lambda_{1,6}\lambda_{3,4}}{t}}][\lambda_{3,6} - t] - \lambda_{2,6}\lambda_{3,5} = 0$$

After simplification, this yields the quadratic

$$(\lambda_{1,5}\lambda_{2,4} - \lambda_{1,4}\lambda_{2,5})t^2 + (\lambda_{1,4}\lambda_{2,5}\lambda_{3,6} - \lambda_{1,4}\lambda_{2,6}\lambda_{3,5} - \lambda_{1,5}\lambda_{2,4}\lambda_{3,6} + \lambda_{1,6}\lambda_{2,5}\lambda_{3,6})t$$
$$+(\lambda_{1,6}\lambda_{2,6}\lambda_{3,4}\lambda_{3,5} - \lambda_{1,6}\lambda_{2,5}\lambda_{3,4}\lambda_{3,6}) = 0$$

In this case, we can rewrite this as

$$-At^2 + (\lambda_{1,4}C - \lambda_{2,5}B + \lambda_{3,6}A)t - \lambda_{1,6}\lambda_{3,4}C = 0$$

Where $A = \det\Lambda(1,2;4,5)$, $B = \det\Lambda(1,3;4,6)$, and $C = \det\Lambda(2,3;5,6)$.

Thus, we get

$$t = \frac{(\lambda_{1,4}C - \lambda_{2,5}B + \lambda_{3,6}A) \pm \sqrt{(\lambda_{1,4}C - \lambda_{2,5}B + \lambda_{3,6}A)^2 - 4\lambda_{1,6}\lambda_{3,4}AC}}{2A} \tag{1}$$

4.3. **The Solutions.** For both of these solutions to work, we first need the discriminant of the quadratic to be positive. Also, it needs to be that the values of t yield valid Kirchoff matrices. In other words, the resulting matrix must be positive and follow the correct sign convention. Notice, this method does not assume either one (positive or negative diagonal entries). Thus, Λ can be a valid response matrix only if the following hold.

$$(\lambda_{1,4}C - \lambda_{2,5}B + \lambda_{3,6}A)^2 \geq 4\lambda_{1,6}\lambda_{3,4}AC$$

and

$$0 < \lambda_{3,6}\left(\lambda_{3,6} - \frac{(\lambda_{1,4}C - \lambda_{2,5}B + \lambda_{3,6}A) \pm \sqrt{(\lambda_{1,4}C - \lambda_{2,5}B + \lambda_{3,6}A)^2 - 4\lambda_{1,6}\lambda_{3,4}AC}}{2A}\right)$$

since both solutions must have the correct sign (ie the same as each λ).

5. Extending it to N-gon-in-N-gon Graphs

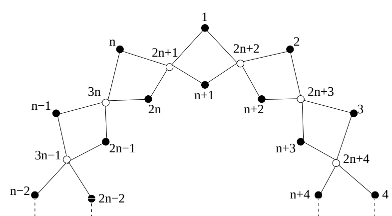

Figure 2. The N-gon-in-N-gon Graph

Now, we can use these same steps to solve for the general *N-gon-in-N-gon* case.

5.1. **The Matrices.** For all *N-gon-in-N-gon* networks, the R matrices have the following shape:

$$R_{2n+1} = \begin{bmatrix} ? & 0 & \dots & 0 & X & ? & 0 & \dots & 0 & X \\ 0 & & & & 0 & 0 & & & & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & & & & 0 & 0 & & & & 0 \\ X & 0 & \dots & 0 & ? & X & 0 & \dots & 0 & ? \\ ? & 0 & & 0 & X & ? & 0 & & 0 & X \\ 0 & & & & 0 & 0 & & & & 0 \\ \vdots & & & & \vdots & \vdots & & & & \vdots \\ 0 & & & & 0 & 0 & & & & 0 \\ X & 0 & \dots & 0 & ? & X & 0 & \dots & 0 & ? \end{bmatrix}$$

where all nonzero entries are in the intersection of the rows and columns from the set {1, n, n+1, 2n}. And for every R matrix after until R_{3n-1}

$$R_{2n+i+1} = \begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & \dots & ? & X & \dots & X & X & \dots \\ 0 & \dots & X & ? & \dots & X & ? & \dots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & \dots & X & X & \dots & ? & X & \dots \\ 0 & \dots & X & ? & \dots & X & ? & \dots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}$$

where all nonzero entries are in the intersection of the rows and columns from the set {i,i+1,n+i,n+i+1}. And lastly

$$R_{3n} = \begin{bmatrix} 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & \dots & ? & X & \dots & X & X & \dots \\ 0 & \dots & X & ? & \dots & X & X & \dots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \\ 0 & \dots & X & X & \dots & ? & X & \dots \\ 0 & \dots & X & X & \dots & X & ? & \dots \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \end{bmatrix}$$

where all nonzero entries are in the intersection of the rows and columns from the set {n-1,n,2n-1,2n}.

Again, the "?" indicates an unknown non-zero entry and the "X" an entry that follows directly from the K matrix before it.

5.2. **Solving.** Like before, we will start at the last residue matrix, R_{3n}. Since every 2×2 determinant of the R matrices is equal to zero, we have:

$$det R_{3n}(n-1, n; 2n-1, 2n) = 0$$

Which gives us the equation

$$R_{3n}(n-1, 2n-1) R_{3n}(n, 2n) - R_{3n}(n-1, 2n) R_{3n}(n, 2n-1) = 0$$

Since each of the corresponding terms in K_{3n} are zero, we can rewrite this as

$$K_{3n-1}(n-1, 2n-1) K_{3n-1}(n, 2n) - K_{3n-1}(n-1, 2n) K_{3n-1}(n, 2n-1) = 0$$

We can now use the fact that $K_N = \underline{K_{N+1}|} + R_{N+1}$ to rewrite all the remaining K matrices in terms of R matrices and $\overline{K_{2n}}$ (ie Λ). So we get

$$\begin{aligned} &[K_{3n-2}(n-1, 2n-1) - R_{3n-1}(n-1, 2n-1)][K_{3n-2}(n, 2n) - R_{3n-1}(n, 2n)] - \\ &[K_{3n-2}(n-1, 2n) - R_{3n-1}(n-1, 2n)][K_{3n-2}(n, 2n-1) - R_{3n-1}(n, 2n-1)] = 0 \end{aligned}$$

and continuing on until we reach K_{2n}, we get

$$[K_{2n}(n-1,2n-1)-\sum_{i=1}^{n-1}R_{2n+i}(n-1,2n-1)][K_{2n}(n,2n)-\sum_{i=1}^{n-1}R_{2n+i}(n,2n)]-$$
$$[K_{2n}(n-1,2n)-\sum_{i=1}^{n-1}R_{2n+i}(n-1,2n)][K_{2n}(n,2n-1)-\sum_{i=1}^{n-1}R_{2n+i}(n,2n-1)]=0$$

From the above matrices, we know which entries are zero and which are known, allowing us to simplify all of this to

$$[\lambda_{n-1,2n-1}-R_{3n-1}(n-1,2n-1)][\lambda_{n,2n}-R_{2n+1}(n,2n)]-\lambda_{n-1,2n}\lambda_{n,2n-1}=0$$

Once again, since $R_{2n+1}(n,2n)$ is an unknown, unsolvable entry and we must parameterize a single entry in R_{2n+1}, we can let $R_{2n+1}(n,2n)=t$, our parameter. Which gives us

$$[\lambda_{n-1,2n-1}-R_{3n-1}(n-1,2n-1)][\lambda_{n,2n}-t]-\lambda_{n-1,2n}\lambda_{n,2n-1}=0 \tag{2}$$

Now, all that is left to do is simplify the term $R_{3n-1}(n-1,2n-1)$. We can do this through a series of rewrites which we get from the following lemma.

Lemma 5.1. *For all $i<n-1$,*

$$R_{2n+j+1}(j+1,n+j+1)=\frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}}{\lambda_{j,n+j}-R_{2n+j}(j,n+j)}$$

Proof. Since all 2×2 determinant of R_{2n+j+1} are equal to zero, we have

$$detR_{2n+j+1}(j,j+1;n+j,n+j+1)=0$$

so

$$R_{2n+j+1}(j+1,n+j+1)R_{2n+j+1}(j,n+j)-R_{2n+j+1}(j,n+j+1)R_{2n+j+1}(j+1,n+j)=0$$

thus

$$\begin{aligned}
R_{2n+j+1}(j+1,n+j+1) &= \frac{R_{2n+j+1}(j,n+j+1)R_{2n+j+1}(j+1,n+j)}{R_{2n+j+1}(j,n+j)} \\
&= \frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}}{K_{2n+j}(j,n+j)} \\
&= \frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}}{K_{2n+j-1}(j,n+j)-R_{2n+j}(j,n+j)} \\
&\vdots \\
&= \frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}}{K_{2n}(j,n+j)-\sum_{i=1}^{j} R_{2n+i}(j,n+j)} \\
&= \frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}}{\lambda_{j,n+j}-R_{2n+j}(j,n+j)}
\end{aligned}$$

□

From this, we can write $R_{3n-1}(n-1,2n-1)$ in terms of $\lambda_{i,j}$ and $R_{2n+1}(1,n+1)$. Since

$$\begin{aligned}
R_{3n-1}(n-1,2n-1) &= \frac{\lambda_{n-2,2n-1}\lambda_{n-1,2n-2}}{\lambda_{n-2,2n-2}-R_{3n-2}(n-2,2n-2)} \\
&= \frac{\lambda_{n-2,2n-1}\lambda_{n-1,2n-2}}{\lambda_{n-2,2n-2}-\frac{\lambda_{n-3,2n-2}\lambda_{n-2,2n-3}}{\lambda_{n-3,2n-3}-R_{3n-3}(n-3,2n-3)}} \\
&\vdots \\
&= \frac{\lambda_{n-2,2n-1}\lambda_{n-1,2n-2}}{\lambda_{n-2,2n-2}-\frac{\lambda_{n-3,2n-2}\lambda_{n-2,2n-3}}{\lambda_{n-3,2n-3}-\dots\frac{\lambda_{1,n+2}\lambda_{2,n+1}}{\lambda_{1,n+1}-R_{2n+1}(1,n+1)}}}
\end{aligned}$$

And, by 5.1, since $R_{2n+1}(1,n+1)$ can be written in terms of $\lambda_{i,j}$ and t, every $R_{2n+j+1}(j+1,n+j+1)$ term can as well.

Lemma 5.2. *For all $i<n-1$, $R_{2n+i}(i,n+i)$ can be written in the form $\frac{a_it+b_i}{c_it+d_i}$*

Proof. From $det\Lambda(1,2n;n,n+1)$, we have $R_{2n+1}(1,n+1)=\frac{\lambda_{1,n}\lambda_{n+1,2n}}{t}$ Now suppose there exists a j such that $R_{2n+j}(j,n+j)=\frac{a_jt+b_j}{c_jt+d_j}$ then, from 5.1, we know

$$R_{2n+j+1}(j+1,n+j+1)=\frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}}{\lambda_{j,n+j}-R_{2n+j}(j,n+j)}$$

so

$$\begin{aligned}
R_{2n+j+1}(j+1,n+j+1) &= \frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}}{\lambda_{j,n+j}-\frac{a_jt+b_j}{c_jt+d_j}} \\
&= \frac{\lambda_{j,n+j+1}\lambda_{j+1,n+j}c_jt+d_j}{\lambda_{j,n+j}c_jt+d_j-a_jt+b_j} \\
&= \frac{(\lambda_{j,n+j+1}\lambda_{j+1,n+j}c_j)t+(\lambda_{j,n+j+1}\lambda_{j+1,n+j}d_j)}{(\lambda_{j,n+j}c_j-a_j)t+(\lambda_{j,n+j}d_j-b_j)}
\end{aligned}$$

therefore, $R_{2n+j+1}(j+1,n+j+1)$ can also be written in the form $\frac{a_{j+1}t+b_{j+1}}{c_{j+1}t+d_{j+1}}$ $\square$

Furthermore, we can see that

$$\begin{aligned}
a_{j+1} &= \lambda_{j,n+j+1}\lambda_{j+1,n+j}c_j \\
b_{j+1} &= \lambda_{j,n+j+1}\lambda_{j+1,n+j}d_j \\
c_{j+1} &= \lambda_{j,n+j}c_j-a_j \\
d_{j+1} &= \lambda_{j,n+j}d_j-b_j
\end{aligned}$$

Thus, we can write all $R_{2n+j}(j,n+j)$ entries in the form $\lambda_{j-1,n+j}\lambda_{j,n+j-1}\frac{c_{j-1}t+d_{j-1}}{c_jt+d_j}$ with c_j and d_j defined recursively as:

$$\begin{aligned}
c_{j+1} &= \lambda_{j,n+j}c_j-\lambda_{j-1,n+j}\lambda_{j,n+j-1}c_{j-1} \\
c_1 &= 1 \\
c_2 &= \lambda_{1,n+1} \\
&\mathit{and} \\
d_{j+1} &= \lambda_{j,n+j}d_j-\lambda_{j-1,n+j}\lambda_{j,n+j-1}d_{j-1} \\
d_1 &= 0 \\
d_2 &= -\lambda_{1,n}\lambda_{n+1,2n}
\end{aligned}$$

Thus equation (2) now becomes

$$[\lambda_{n-1,2n-1}-\lambda_{n-2,2n-1}\lambda_{n-1,2n-2}\frac{c_{n-2}t+d_{n-2}}{c_{n-1}t+d_{n-1}}][\lambda_{n,2n}-t]-\lambda_{n-1,2n}\lambda_{n,2n-1}=0$$

This can be further reduced to produce the quadratic

$$\begin{aligned}
c_nt^2+(d_n+\lambda_{n,2n-1}(\lambda_{n-1,2n}-\lambda_{n,2n})c_{n-1}+\lambda_{n-2,2n-1}\lambda_{n-1,2n-2}\lambda_{n,2n})t \\
+(\lambda_{n,2n}(\lambda_{n,2n-1}-\lambda_{n-1,2n-1})-d_{n+1})=0
\end{aligned}$$

5.3. **The Solutions.** For both of these solutions to work, we first need the discriminant of the quadratic to be positive. Also, it needs to be that the values of t yield valid Kirchoff matrices. In other words, the resulting matrix must be positive and follow the correct sign convention (ie it must have the same sign as $\lambda_{n,2n}$). Notice, this method does not assume either one (positive or negative diagonal entries). Thus, Λ can be a valid response matrix only if the following hold.

$$(d_n+\lambda_{n,2n-1}(\lambda_{n-1,2n}-\lambda_{n,2n})c_{n-1}+\lambda_{n-2,2n-1}\lambda_{n-1,2n-2}\lambda_{n,2n})^2 \geq 4(\lambda_{n,2n}(\lambda_{n,2n-1}-\lambda_{n-1,2n-1})-d_{n+1})c_n$$

and

$$0 < \lambda_{n,2n}(\lambda_{n,2n} - t)$$

Also note that for $i \leq n-1$, $detR_{2n+i}(i, n+1; i+1, n+i+1) = 0$ gives us

$$\lambda_{i,i+1}\lambda_{n+i,n+i+1} = \lambda_{i+1,n+i}\lambda_{i,n+i+1}$$

Notice that these are very similar to the conditions given in [4] to determine whether Λ is a valid response matrix. In fact, the definitions of the c_i and d_i are nearly identical to the definitions of the p_i and q_i in [4] both can be written in terms of the other. Additionally, since we assumed neither sign convention, some of the requirements discussed in [4] are irrelevant. Thus, the conditions stated above are actually the same as those in [4].

6. Extending to Generalizations of the N-gon-in-N-gon Graph

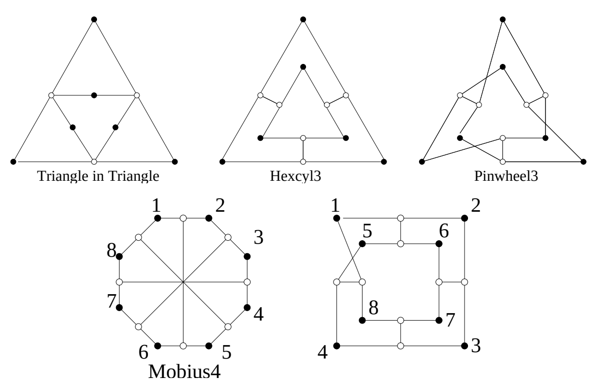

Figure 3. Some Generalizations of the Triangle in Triangle

In *Mobius Strips, Pinwheels, and Other Two-to-one Generalizations of N-gon-in-N-gon Graphs* [5], Nick Reichert describes some generalizations of the *N-gon-in-N-gon* graphs, the $Hexcyl_n$, $Pinwheel_n$, and $Mobius_n$ graphs, and shows them to be at most two-to-one. However, it is not known what conditions are applicable to these (ie what characterizations remain throughout the different generalizations). We will now examine these three generalizations.

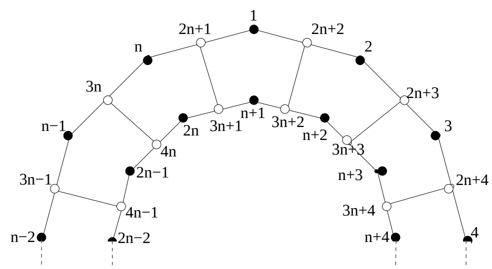

FIGURE 4. The $Hexcyl_n$ Graph

6.1. **The $Hexcyl_n$ Graph.** If we look at the residue matrices for the $Hexcyl_n$ graph, we see that for all $2n+1 < i < 3n$, R_i has the same shape as the R_i matrix for the *N-gon-in-N-gon* graph. Therefore, the method of recovery is the same as the one for the *N-gon-in-N-gon* graphs as shown above. Thus, the same conditions described above apply for all $Hexcyl_n$ graphs. The reason for this can easily be seen by looking at K_{3n}.

Each $Hexcyl_n$ graph starts with $4n$ vertices and the matrix K_{4n}. If we interiorize the n inner interior vertices ($3n$+1, $3n$+2,…,$4n$), it results in the matrix K_{3n}. Graphically, we can see that this network has the same connections as the n-gon-in-ngon with an additional n edges connecting adjacent, inner boundary vertices (n+1,n+2,…,$2n$) shown by dotted lines in the figure.

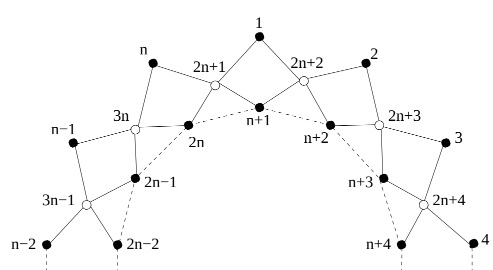

FIGURE 5. The $Hexcyl_n$ Graph after Interiorizing

Each interiorization of a node is a simple $Y-\Delta$ transformation, so all the steps up to this point are recoverable. Also, since the extra connections do not effect the $R_{2n+j+1}[j,j+1;n+j,n+j+1]$ submatrices, it does not effect the method or the result above. Thus, the conclusions are the same as they are in the *N-gon-in-N-gon* case. This confirms that the $Hexcyl_n$ graph is at most two-to-one.

6.2. **The $Mobius_n$ Graph.** Similarly to the $Hexcyl_n$ graphs, the $Mobius_n$ graphs can be shown to have similar shaped R_i matrices to the ones for the *N-gon-in-N-gon* graphs. So, again, the same method of recovery can be used and the same conditions for the solutions hold.

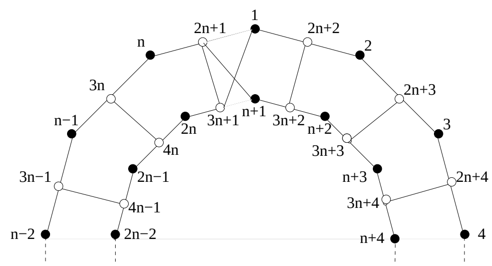

FIGURE 6. The $Mobius_n$ Graph

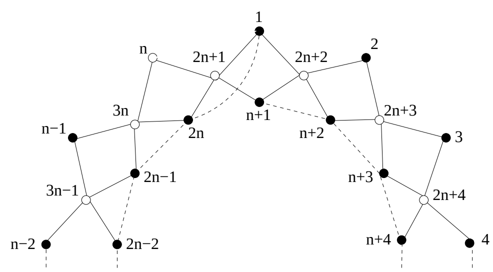

FIGURE 7. The $Mobius_n$ Graph After Interiorizing

Again, this is easier to see from looking at the corresponding K_{3n} network. Each interiorization is a recoverable $Y-\Delta$ transformation. So after interiorizing all the inner interior vertices ($3n$+1, $3n$+2,…,$4n$), we get a network which contains all the connections found in the *N-gon-in-N-gon* network plus some extra boundary to boundary connections (shown by dotted lines in the figure). Since the extra connections do not effect the $R_{2n+j+1}[j, j+1; n+j, n+j+1]$ submatrices, we can use the same method and, therefore, get the same results as in the *N-gon-in-N-gon* case. Again, this confirms that the $Mobius_n$ graph is at most two-to-one.

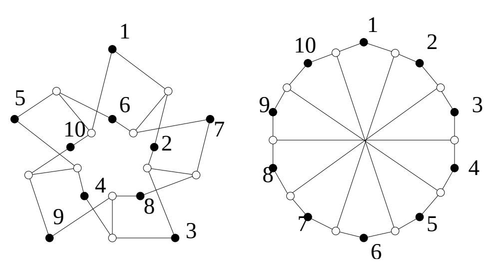

FIGURE 8. Examples of the $Pinwheel_n$ to $Mobius_n$ Isomorphism

6.3. **The** $Pinwheel_n$ **Graph.** As seen in the figure, the $Pinwheel_n$ graph is isomorphic to either the $Mobius_n$ graph or the $Hexcyl_n$ graph, depending on the parity of n [5]. Thus, $Pinwheel_n \cong Mobius_{2n+1}$ and $Pinwheel_{2n} \cong Hexcyl_{2n}$. So, it is enough to show the conditions for both the $Mobius_n$ and $Hexcyl_n$ graphs

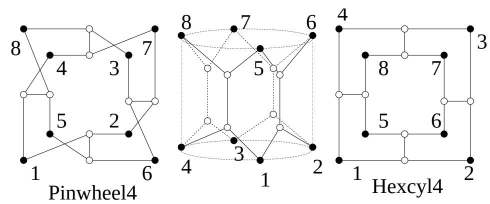

FIGURE 9. Examples of the $Pinwheel_n$ to $Hexcyl_n$ Isomorphism

and apply these to the $Pinwheel_n$. So, the conclusions are the same as for the *N-gon-in-N-gon* graphs.

7. Extending to Single Parameter, Semi-Recoverable Graphs

Instead of looking at specific cases, we now want to look at how these different steps of recovery can be applied to all single perameter semi-recoverable graphs. In other words, what restrictions or conditions can be found for all semi-recoverable graphs.

7.1. **Solving.** One of the most important results from the method above is the fact that the characterizing equation comes from one of the 2×2 determinants in a single-layered residue matrix. This fact will help us recover it for all semi-recoverable graphs.

Theorem 7.1. *The characteristric polynomial of a single parameter semi-recoverable network is recovered from a 2×2 determinant of a single layer residue matrix. For some indices a_1, a_2, b_1, and b_2, each term in $R_i(a_1, a_2; b_1, b_2)$ is non-zero and $detR_i(a_1, a_2; b_1, b_2)$ yields a polynomial which characterizes the network.*

Proof. The characteristic polynomial must be recovered from some submatrix determinant which is composed entirely of known entries in a R_n^m matrix. Since all multi-layered R matrices can be written as the sum of single-layered R matrices, all known entries in the multi-layered are known in the single-layered. Thus, if such a determinant exists in a multi-layered, it must exist in a single-layered. $\square$

This is an important result since it now tells us the first step in the method above is a valid step for all semi-recoverable graphs. Thus, we can state the following.

For some indices a_1, a_2, b_1, and b_2 all of which are known and non-zero

$$detR_i(a_1, a_2; b_1, b_2) = 0 \tag{3}$$

Similarly to what we did in the *N-gon-in-N-gon* case we can rewrite this as

$$\begin{aligned}
detR_i(a_1,a_2;b_1,b_2) &= R_i(a_1,b_1)R_i(a_2,b_2) - R_i(a_1,b_2)R_i(a_2,b_1) \\
&= K_{i-1}(a_1,b_1)K_{i-1}(a_2,b_2) - K_{i-1}(a_1,b_2)K_{i-1}(a_2,b_1) \\
&= [K_{i-2}(a_1,b_1) - R_{i-1}(a_1,b_1)][K_{i-2}(a_2,b_2) - R_{i-1}(a_2,b_2)] \\
&\quad -[K_{i-2}(a_1,b_2) - R_{i-1}(a_1,b_2)][K_{i-2}(a_2,b_1) - R_{i-1}(a_2,b_1)] \\
&= [K_{i-3}(a_1,b_1) - R_{i-1}^{i-2}(a_1,b_1)][K_{i-3}(a_2,b_2) - R_{i-1}^{i-2}(a_2,b_2)] \\
&\quad -[K_{i-3}(a_1,b_2) - R_{i-1}^{i-2}(a_1,b_2)][K_{i-3}(a_2,b_1) - R_{i-1}^{i-2}(a_2,b_1)] \\
&\vdots \\
&= [K_k(a_1,b_1) - R_{i-1}^{k+1}(a_1,b_1)][K_k(a_2,b_2) - R_{i-1}^{k+1}(a_2,b_2)] \\
&\quad -[K_k(a_1,b_2) - R_{i-1}^{k+1}(a_1,b_2)][K_k(a_2,b_1) - R_{i-1}^{k+1}(a_2,b_1)] \\
&= [\Lambda(a_1,b_1) - R_{i-1}^{k+1}(a_1,b_1)][\Lambda(a_2,b_2) - R_{i-1}^{k+1}(a_2,b_2)] \\
&\quad -[\Lambda(a_1,b_2) - R_{i-1}^{k+1}(a_1,b_2)][\Lambda(a_2,b_1) - R_{i-1}^{k+1}(a_2,b_1)]
\end{aligned}$$

Thus, we have

$$[\Lambda(a_1,b_1)-R_{i-1}^{k+1}(a_1,b_1)][\Lambda(a_2,b_2)-R_{i-1}^{k+1}(a_2,b_2)]-[\Lambda(a_1,b_2)-R_{i-1}^{k+1}(a_1,b_2)][\Lambda(a_2,b_1)-R_{i-1}^{k+1}(a_2,b_1)] = 0 \tag{4}$$

as the characterizing equation.

7.2. **The Characteristic Polynomial.** Before we continue determining the characteristic polynomial, we need to find what form it is in. To do this, we will examine a number of important facts about the single and multi-layered residue matrices.

Lemma 7.2. *If an entry in K_i is equal to zero, the same entry is zero in all K_j entries where $j \geq i$.*

Proof. This is easily shown in terms of connections. If a connection exists before the interiorization of a node, it exists after. In other words, if an entry in K_{i+1} is nonzero, it is nonzero in K_i. Thus, if an entry in K_i is zero, it must be zero in all K_j entries where $j > i$. □

Lemma 7.3. *If an entry in R_i is known, the entry is equal to zero for all R_j where $j > i$.*

Proof. For an entry in R_i to be known, the corresponding entry must be zero in K_i. Thus, as we showed above, it is zero in all K_j matrices where $j > i$. So all the entries in the R_j entries are known. Which means they have the same value as the previous K_j matrix which is exactly zero. □

Lemma 7.4. *A diagonal entry in a R matrix, single-layered or multi-layered, is non-zero $\Leftrightarrow$ at least one off-diagonal entry is non-zero. In other words, if a diagonal entry is zero, all the entries in that column and row are zero.*

Proof. This comes directly from the fact that the residue matrices are the difference (or residue) of two Kirchhoff matrices. Therefore, if you change at least one connection (ie an off-diagonal entry) you must change the sum of connections (ie the diagonal entry). This argument works both ways. □

Lemma 7.5. *All zero entries in a single-layer R matrix are part of a row or column of zeros. In other words, if an entry is zero, so is every entry in either the column or the row.*

Proof. Again, this can easily be seen by looking at the residue matrix in terms of changing connections. Suppose for some residue matrix R_i, two diagonal entries $R_i(a_1, a_1)$ and $R_i(a_2, a_2)$ are non-zero. Then, the connection between node a_1 and a_2 has changed. Thus, $R_i(a_1, a_2)$ and $R_i(a_2, a_1)$ are non-zero.

So, if an off-diagonal entry is zero, at least one of its corresponding diagonal entries is zero and, therefore, the entire row or column is zero. □

Lemma 7.6. *For a single-layered residue matrix R_i to be recoverable, every non-zero entry must be part of a 2×2 submatrix with at least two known entries.*

Proof. At least one unknown entry must be part of a 2×2 with three known entries in order for it to be recoverable. Then, all the following matrices must be part of a 2×2 submatrix with all its entries known or already recovered. This means, the second recovered entry must be part of a submatrix with at least two known entries and the third may be part of one with only one. However, since the ones before were part of submatrices with at least two, the third can also be written as part of a submatrix with at least two known entries. This is easier to see by example.

$$\begin{bmatrix} \vdots & & \vdots & \\ X & \dots & ? & \dots \\ \vdots & & \vdots & \\ X & \dots & X & \dots \end{bmatrix} \to \begin{bmatrix} \vdots & & \vdots & \\ X & \dots & ? & \dots \\ \vdots & & \vdots & \\ X & \dots & ? & \dots \\ \vdots & & \vdots & \\ X & \dots & X & \dots \end{bmatrix} \to \begin{bmatrix} \vdots & & \vdots & & \vdots & \\ X & \dots & ? & \dots & ? & \dots \\ \vdots & & \vdots & & \vdots & \\ X & \dots & ? & \dots & X & \dots \\ \vdots & & \vdots & & \vdots & \\ X & \dots & X & \dots & ? & \dots \end{bmatrix} or \begin{bmatrix} \vdots & & \vdots & & \vdots & \\ X & \dots & ? & \dots & ? & \dots \\ \vdots & & \vdots & & \vdots & \\ X & \dots & ? & \dots & ? & \dots \\ \vdots & & \vdots & & \vdots & \\ X & \dots & X & \dots & X & \dots \end{bmatrix}$$

Although the entry in the upper right corner can be written as part of a 2×2 submatrix that has only one known entry (or none in the second case), it also can be written as part of a submatrix with at least two. The same is true for the entry in the lower right.

This is true for all following unknown entries in the single-layered residue matrix and, thus, all entries are part of a 2×2 submatrix with at least two known entries. □

Lemma 7.7. *Every recoverable, single-layered residue matrix must have a $j \times j$ submatrix where $j > 2$ with no zero entries and at least one known entry in every column and row.*

Proof. First, suppose the largest non-zero submatrix of some R_i is 2×2. Then, two of the entries are diagonal entries which cannot be known. Thus, we have a 2×2 submatrix that is not recoverable. So, by 7.6, R_i is not recoverable. Thus, every recoverable residue matrix must have a non-zero submatrix of size greater than 2.

Also, to be recoverable, the matrix must have at least one known entry per column and row.

□

Lemma 7.8. *Every entry in a single-layered residue matrix for a single parameter semi-recoverable graph must be part of a 2×2 determinant which has two known entries which come directly from Λ.*

Proof. In the first residue matrix, all known entries must be λ's since they come directly from Λ.

Now, suppose R_i is a recoverable residue matrix or is once a single parameter is used. Let j denote the size of the largest non-zero submatrix. Then, it must have at least $2j$ known entries (including the parameter in the parameterized layer). If $2(j-1)$ entries are λ's, then, from the lemmas above, these entries coorespond to zero rows/columns in every R_l matrix where $i < l$. Thus, at least $(j-2)$ of the row/columns are zeroed out, leaving exactly four uneffected entries, two of which are diagonal entries. Thus, there are only 2 entries left that can be in the following residue matrices. But, from 7.7, we know each single-layer residue matrix is recoverable only if it has a non-zero submatrix of size greater than 2. So, the next R matrix must have $2(m-1)$ known entries which do not come from any of the unknown entries in the previous residue matrices. Thus, they must be from Λ.

We also know that all 2×2 submatrices have at least two known entries. So, for all single-layered residue matrices, at least two of the known entries in each 2×2 submatrix come directly from Λ. □

Corollary 7.9. *Every entry in a single-layered residue matrix for a semi-recoverable graph with one parameter can be written in the form $\frac{at+b}{ct+d}$ where t is the parameter and a, b, c, and d are polynomials in Λ.*

Proof. Since at least two entries in each 2×2 submatrix are λ's and the third known is linear fractional, solving for an unknown entry is done by a linear rational operation. Thus, every entry is linear rational.

Alternatively, we can show this algebraically. Let x denote the unknown entry and $f(t)$ denote a linear fractional entry. Then

$$x \cdot f(t) - \lambda_{i,j}\lambda_{k,l} = 0 \Rightarrow x = \frac{\lambda_{i,j}\lambda_{k,l}}{f(t)}$$

and

$$\lambda_{i,i}x - \lambda_{i,j}f(t) = 0 \Rightarrow x = \frac{\lambda_{i,j}}{\lambda_{i,i}}f(t)$$

Either way, x retains linear rationality. So every entry can be written in the form $\frac{at+b}{ct+d}$.

□

Theorem 7.10. *The characteristic polynomial of all single variable, semi-recoverable graphs is a quadratic. In other words, they are $2 \rightarrow 1$.*

Proof. This follows immediately from 7.1, 7.8, and 7.9. Since $\det R_i(a_1, a_2; b_1, b_2)$ has at two λ entries and two linear fractional entries, $\det R_i(a_1, a_2; b_1, b_2) = 0$ will yield either a linear or quadratic equation. However, since we assumed the graph is semi-recoverable, it cannot be a linear equation, since that would make the parameter spurious. Thus, it is a quadratic and the graph is $2 \to 1$.

□

7.3. **The Polynomial.** We now return to solving the equation as we did for the *N-gon-in-N-gon* case.

We now know we can rewrite 4 in one of the following ways

$$[\Lambda(a_1,b_1) - R_{i-1}^{k+1}(a_1,b_1)][\Lambda(a_2,b_2) - R_{i-1}^{k+1}(a_2,b_2)] - [\Lambda(a_1,b_2)][\Lambda(a_2,b_1)] = 0$$

or

$$[\Lambda(a_1,b_1) - R_{i-1}^{k+1}(a_1,b_1)][\Lambda(a_2,b_2)] - [\Lambda(a_1,b_2)][\Lambda(a_2,b_1) - R_{i-1}^{k+1}(a_2,b_1)] = 0$$

Both remaining R_{i-1}^{k+1} entries must be linear fractionals dependant on the parameter t.

We also know that each non-zero entry in R_{i-1}^{k+1} comes from some single residue matrix R_j where $k+1 < j < i-1$.Additionally, from 7.8, each term can be written recursively as

$$\begin{aligned} R_j(a_1,b_1) &= \frac{\lambda_{a_1,b_2}\lambda_{a_2,b_1}}{R_j(a_2,b_2)} \\ &= \frac{\lambda_{a_1,b_2}\lambda_{a_2,b_1}}{K_{j-1}(a_2,b_2)} \\ &= \frac{\lambda_{a_1,b_2}\lambda_{a_2,b_1}}{\Lambda(a_2,b_2) - K_{j-1}^{k+1}(a_2,b_2)} \\ &= \frac{\lambda_{a_1,b_2}\lambda_{a_2,b_1}}{\lambda_{a_2,b_2} - R_l(a_2,b_2)} \end{aligned}$$

or

$$\begin{aligned} R_j(a_1,b_1) &= \frac{\lambda_{a_2,b_1}}{\lambda_{a_2,b_2}} R_j(a_1,b_2) \\ &= \frac{\lambda_{a_2,b_1}}{\lambda_{a_2,b_2}} K_{j-1}(a_1,b_2) \\ &= \frac{\lambda_{a_2,b_1}}{\lambda_{a_2,b_2}} (\Lambda(a_1,b_2) - K_{j-1}^{k+1}(a_1,b_2)) \\ &= \frac{\lambda_{a_2,b_1}}{\lambda_{a_2,b_2}} (\lambda_{a_1,b_2} - R_l(a_1,b_2)) \end{aligned}$$

for some indices a_1, a_2, b_1, and b_2 where $l < j$.

So, we see that a continued fraction (or other recurrance relation) similar to the one in the *N-gon-in-N-gon* case is typical of the characteristic quadratics for all semi-recoverable graphs.

8. Extending to All Semi-Recoverable Graphs

We will now look at what this result says for semi-recoverable graphs in general, not just ones that are recoverable with a single parameter.

Theorem 8.1. *All semi-recoverable graphs are* $2^n \to 1$.

Proof. This follows almost directly from 7.10.

Suppose K is a semi-recoverable graph with n parameters. Then, since each yields a quadratic, every parameter gives 2 solutions. So we have n parameters, each yielding 2 solutions, which means the graph has 2^n solutions. Thus, it is $2^n \to 1$. □

Corollary 8.2. *All networks K associated with a semi-recoverable graph G have 2^k solutions for some integer k.*

Proof. Since G is semi-recoverable, we know that it is $2^n \to 1$. However, this does not mean that all its associated networks K have 2^n solutions since any of the characteristic quadratics can have double roots. Since each double root reduces the number of solutions by a factor of two, we can see that there are 2^k solutions where $1 \leq k \leq n$. □

9. Conclusion

Not only have we shown that the method for recovering arbitrary graphs described in [1] can recover the conditions for multiple solution graphs, we also see that it also recovered all conditions that make Λ a valid response for a *N-gon-in-N-gon* graph that were shown in [4]. Also, we have confirmed the $Hexcyl_n$, $Mobius_n$, and $Pinwheel_n$ are at most two-to-one and have shown that all known generalizations of the *N-gon-in-N-gon* graphs have the same conditions for when they are two-to-one and when they are one-to-one as the *N-gon-in-N-gon* itself.

More importantly, through use of methods similar to the ones used in the *N-gon-in-N-gon* case, we have shown that all semi-recoverable graphs and their corresponding networks are $2^n \to 1$. This greatly restricts the semi-recoverability of graphs. However, this paper makes no argument for why that is except for through the method used. In other words, there is no other explanation for why there does not exist say a $3 \to 1$ graph other than the fact that it is impossible to get a cubic characteristic polynomial.

References

[1] Addington, Nicolas. *A Method for Recovering Arbitrary Graphs*, 2003.
[2] Curtis, Edward B., and James A. Morrow. "Inverse Problems for Electrical Networks". Series on applied mathematics – Vol. 13. World Scientific, ©2000.
[3] Esser, Ernie. *On Solving the Inverse Conductivity Problem for Annular Networks*, 2000.
[4] French, Jennifer and Shen Pan. *2^n to 1 Graphs*, 2004.
[5] Reichert, Nick. *Mobius Strips, Pinwheels, And Other Two-to-one Generalizations of N-gon-in-N-gon Graphs*, 2004.