Electrical Networks with Periodic Voltages and Complex Conductivities

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July 2005

Abstract

This paper studies electrical networks with coplex admittance and complex periodic generating voltage. We prove the uniqueness of the solution to the Dirichlet problem. The determinantal formulas do not work as nice as in real resistance networks in the sense that there are more conditions to be satisfied. Some algorithms are given to recover electrical networks made of well-connected graphs. Also, medial graphs can be used to recover a large group of critical circular planar graphs through a quite complicated algorithm.

Contents

1 Introduction

The motivation for this paper comes from the inverse problem of recovering an electrical network containing resistors, capacitors and inductors, knowing

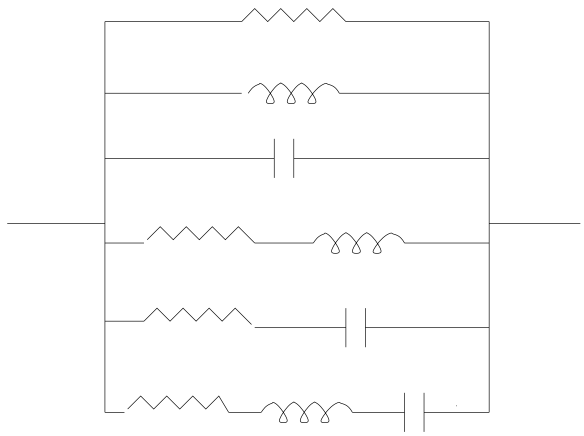


Figure 1: A simple parallel network

only its response to periodic input voltage. Consider an electrical network $\Gamma = (G, \gamma_e(w))$ where $G = (V, E)$ is a graph and $\gamma_e(\omega)$ is an complex valued ω-dependant edge function on all adages $e \in E$. To be more clear,$\gamma_e(\omega)$ for each $e \in$ E is the admittance of e corresponding to the voltage difference $V = V_0 e^{i\omega t}$ between its two ends. The relationship between the voltage, the admittance and the response current is determined by Ohm's law:$I(t) = \gamma(\omega) \cdot V_0 e^{i\omega t}$. So if we have $\gamma(\omega) = \gamma_0(\omega) e^{i\phi(\omega)}$,then $I(t) = I_0(\omega) e^{i\omega\prime t} = \gamma_0(\omega) \cdot V_0(\omega) \cdot e^{\phi(\omega)} \cdot e^{i(\omega t)}$.

A resistor has admittance $1/R$ where R is the resistance. This follows directly from Ohm's Law. When a periodic voltage applied,a capacitor has admittance $iC\omega$ where C is the capacitance of the capacitor defined as $C = Q/\Delta V$ where Q is the charge on the capacitor. And finally an inductor assumes the admittance of $1/iL\omega$ where L is the inductance.

2 Simple Networks

Definition 1. *A simple parallel network is a two-pole electrical network that is made up of a parallel combination of specific types of elements in series. The allowed type of elements in series are resistors, capacitors, inductors, or a combination of a resistor with either an inductor or a capacitor, or all three in series.*

If we compute the total admittance of a simple parallel network, we get a rational function which can be written in the following form, which we call a simple admittance:

Definition 2. *A simple admittance is an admittance of the form.*

$$\begin{aligned}\gamma(\omega) =& a_1\omega + a_0 + \left[\frac{1}{b_1(\omega-\omega_{b_1})} + \cdots + \frac{1}{b_l(\omega-\omega_{b_l})}\right]\\ &+ \left[c_1 + \frac{1}{d_1(\omega-\omega_{d_1})} + \cdots + c_m + \frac{1}{d_m(\omega-\omega_{d_m})}\right]\\ &+ \left[\frac{1}{f_1(\omega-\omega_{f_1})} + \frac{1}{f_1'(\omega-\omega_{f_1}')} + \cdots + \frac{1}{f_p(\omega-\omega_{f_p})} + \frac{1}{f_p'(\omega-\omega_{f_p}')}\right]\end{aligned}$$

where

- $\mathrm{Re}(a_1) = 0$
- $\mathrm{Re}(a_0) > 0$, $\mathrm{Im}(a_0) = 0$
- $\mathrm{Re}(b_i) = 0$, $\mathrm{Im}(b_i) > 0$,
  $\mathrm{Re}(\omega_{b_i}) = 0$, $\mathrm{Im}(\omega_{b_i}) \geq 0$, *for all* $1 \leq i \leq l$
- $\mathrm{Re}(d_i) = 0$, $\mathrm{Im}(d_i) < 0$,
  $\mathrm{Re}(\omega_{d_i}) = 0$, $\mathrm{Im}(\omega_{d_i}) > 0$,
  $c_i = 1/(d_i\omega_{d_i})$, *for all* $1 \leq i \leq m$
- $f_i' = -\overline{f_i}$, $\omega_{f_i}' = -\overline{\omega_{f_i}}$,
  $\mathrm{Re}(f_i) > 0$, $\mathrm{Im}(f_i) > 0$,
  $\mathrm{Re}(\omega_{f_i}) \geq 0$, $\mathrm{Im}(\omega_{f_i}) > 0$, *for all* $1 \leq i \leq p$

3 Uniqueness for the Dirichlet Problem

Now let consider an electrical network $\Gamma = (G, \gamma_e(w))$ where $\gamma_e(\omega)$ is a simple admittance for all $e \in E$. In other words, we're assuming that each edge of our graph is a simple parallel network. Also suppose that G is connected and the set of boundary nodes of Γ is nonempty.Then, we can write the corresponding Kirchhoff, matrix, K, defined as

$$k_{ij} = \begin{cases} -\gamma_{ij}(\omega)\,, \text{ if } i \neq j \\ \sum_l \gamma_{il}(\omega)\,, \text{ if } i = j \end{cases}$$

with corresponding block decomposition:

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where A is an $m \times m$ matrix, m being the number of boundary nodes.

Now we can raise the question of uniqueness of the Dirichlet problem in the following form: Given an $m\times 1$ vector of boundary voltages $\mathbf{v}_\partial e^{i\omega t}$, find a vector of interior voltages, $\mathbf{v}_{int}e^{i\omega t}$ such that the following equation is satisfied:

$$K\mathbf{v} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} \mathbf{v}_\partial \\ \mathbf{v}_{int}(\omega) \end{pmatrix} e^{i\omega t} = \begin{pmatrix} \mathbf{I}_\partial(\omega) \\ 0 \end{pmatrix} e^{i\omega t} \tag{1}$$

for some $m \times 1$ boundary current vector $\mathbf{I}_\partial(\omega)$.

Lemma 1. *If a simple parallel electrical network contains at least one resistor, then* $\mathrm{Re}\big(\gamma_e(\omega)\big) > 0$.

Proof. This follows from the form of simple admittance and the fact that for a complex function $f(\omega) = u(\omega) + iv(\omega)$, if $u(\omega) > 0$ for $\forall\omega$, then $\mathrm{Re}(\frac{1}{f}) = \frac{u(\omega)}{u^2(\omega)+v^2(\omega)} > 0$ for $\forall\omega$ too. □

Theorem 1. *Let $\Gamma = (G, \gamma_e(w))$ be an electrical network defined above with $\{\gamma_{ij}\}$ be defined on the edges of Γ such that the admittance of edge ij given by $\gamma_{ij}(\omega)$. Suppose that the simple parallel network defined on each $e \in E$ contains at least one resistor. Then the Dirichlet problem has a unique solution for all $\omega \in R^+$.*

Proof. We divide the proof in three cases, for ω s such that $\gamma_e(\omega) = 0$ for some e, ω s such that $\gamma_e(\omega) \neq 0$ but finite for all e,and ω s such that $\gamma_e(\omega) = \infty$ for some e:

Case1: Let $W_0 = \{\omega \in R_+ : \gamma_e(\omega) = 0 \text{ for some } e \in E\}$. Since we assumed that every edge in Γ contains at least one resister, then, by lemma 1, for $\forall e \in E$,$\mathrm{Re}((\gamma_e(\omega)) > 0$. Therefore W_0 is an empty set. In other words, this case never occurs.

Case 2: Let $W_1 = \{\omega \in R_+ : \gamma_e(\omega) = \infty \text{ for some } e \in E\}$. This set is empty too. One can easily verify this by looking at the denominators of the terms in equation 1; they are not zero for any ω.

Case 3: Let $W_2 = \{\omega \in R_+ : \gamma_e(\omega) \neq 0 \text{ but finite for } \forall e \in E\}$. In this case If the matrix C is non-singular, then the Dirichlet problem has the unique solution $v_{int}(\omega) = -C^{-1}B^T v_\partial$. To show that C is non-singular, we first show that the null space of K contains only constant vectors of functions. Consider a vector

$\mathbf{x}$ such that $K\mathbf{x} = 0$. Then, $\bar{\mathbf{x}}^T K\mathbf{x} = 0$. Thus,

$$\begin{aligned}
\bar{\mathbf{x}}^T K\mathbf{x} &= \sum_{i,j} \bar{x}_i(\omega) k_{ij}(\omega) x_j(\omega) \\
&= \sum_{i \neq j} \bar{x}_i(\omega) k_{ij}(\omega) x_j(\omega) + \sum_{i=1}^{n} k_{ii}(\omega) |x_i(\omega)|^2 \\
&= \sum_{i<j} k_{ij}(\omega) \big[\bar{x}_i(\omega) x_j(\omega) + \bar{x}_j(\omega) x_i(\omega)\big] + \sum_{i=1}^{n} k_{ii}(\omega) |x_i(\omega)|^2 \\
&= \sum_{i<j} k_{ij}(\omega) \big[x_i(\omega) - x_j(\omega)\big] \big[\bar{x}_j(\omega) - \bar{x}_i(\omega)\big] \\
&= -\sum_{i<j} k_{ij}(\omega) |x_i(\omega) - x_j(\omega)|^2 = 0.
\end{aligned}$$

Again by lemma 1, we know that for every edge $ij \in E$,$\mathrm{Re}\big(\gamma_e(\omega)\big) > 0$ and therefore $\mathrm{Re}(k_{ij}\big(\omega)\big) < 0$ for $i \neq j$ and $\forall \omega \in W_1$. This follows directly from the definitions of K and simple admittance. Since the terms are all non-positive, they must all be zero, so we conclude that $|x_i(\omega) - x_j(\omega)|^2 = 0$ if node i is a neighbor of node j, $\forall \omega \in W_1$. Because Γ is a connected graph, there exists a path between any two vertices. So, for every pair of vertices i and j, $x_i \equiv x_j$ on W_1. Thus, we conclude that $\mathbf{x}$ is a constant vector of functions.

Now, assume that there is a vector $\mathbf{y}$ of functions for which $C\mathbf{y} = 0 \; \forall \omega \in W_1$. Then, form the vector $\mathbf{z} = [0, \ldots, 0, y_1(\omega), \ldots, y_{n-m}(\omega)]$. So,

$$\bar{\mathbf{y}}^T C\mathbf{y} = \bar{\mathbf{z}}^T K\mathbf{z} = 0$$

which implies that $\mathbf{z}$ is a constant vector of functions. But, $\mathbf{z}$ has entries which are the zero function, so $\mathbf{z}$ is the constant vector of zero functions, which implies that $\mathbf{y}$ is also the vector of zero functions. Since $C\mathbf{y} = 0 \Leftrightarrow \mathbf{y} = 0$, we conclude that C is non-singular$\forall \omega \in W_1$. Having shown that C is non-singular, we have proved that the Dirichlet problem has a unique solution $\forall \omega \in W_1$.

Hence, the solution to Dirichlet problem is unique for all finite $\omega \in R^+$. □

4 Connections and Determinants

Suppose $G = (V, V_B, E,)$ is a connected graph with boundary. Let $I = V - V_B$ be the set of interior nodes. A *path* between two boundary nodes p and q is a sequence of edges $pr_1, r_1r_2, \ldots, r_mq$, where all of the r_j are distinct interior nodes. A *connection* between two sets of boundary nodes $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ is a set $\alpha = (\alpha_1, \ldots, \alpha_k)$ of disjoint paths. Let $\mathcal{C}(P; Q)$ be the set of all possible connections from P to Q. For every α in $\mathcal{C}(P; Q)$, define the following three objects:

- τ_α, the permutation of the vertices $(q_1, \ldots, q_k)$ that results at the end-points of $(\alpha_1, \ldots, \alpha_k)$;

- E_α, the set of edges present in the connection α;
- J_α, the set of interior nodes which are not endpoints of any of the edges in E_α.

Theorem 2. *Let $\Gamma = (G, \gamma_e(\omega))$ be a connected electrical network. Let P and Q be disjoint sets of k boundary nodes. Then,*

$$\det \Lambda(P;Q) \cdot \det K(I,I) = (-1)^k \sum_{\tau \in S_k} \operatorname{sign}(\tau) \left\{ \sum_{\substack{\alpha \in \mathcal{C}(P;Q) \\ \tau_\alpha = \tau}} \left[\det K(J_\alpha, J_\alpha) \cdot \prod_{e \in E_\alpha} \gamma_e(\omega) \right] \right\}$$

Proof. The proof of Lemma 4.1 in [**?**] is also valid in the case of admittance. ☐

In the case of circular planar networks, there is only one permutation τ possible in the above formula (the identity permutation) for a circular pair. and so we get the following formula for the sub-determinant of Λ corresponding to the circular pair, P and Q:

$$\det \Lambda(P;Q) = \frac{(-1)^k}{\det K(I;I)} \left\{ \sum_{\alpha \in \mathcal{C}(P;Q)} \left[\det K(J_\alpha, J_\alpha) \cdot \prod_{e \in E_\alpha} \gamma_e(\omega) \right] \right\}. \qquad (2)$$

In the case of resistor networks, since all of the admittance are positive real numbers, the only way that the determinant corresponding to two sets of vertices can be zero is if there is no connection between them. However, for the case we have complex admittance, it gets more complicated. To be more specific, the products $\prod_{e \in E_\alpha} \gamma_e(\omega)$ and complex determinants could make the whole sum to be zero for some values of ω. However, if this sum is zero only for some values of ω, it is still OK since we can recover the admittance using the values of ω for which this sum is not zero. We will elaborate on this later.

The serious problem would arise if $\det \Lambda(P;Q)(\omega) = 0$ identically. Even though one has to be very unlucky to get this sum to be identically zero, but it's totally possible. However, there are two ways to avoid this situation.

Lemma 2. $\det K(J_\alpha, J_\alpha)(\omega) \neq 0$ *for* $\forall J_\alpha$*.*

Proof. we prove this by showing that the null space of the matrix $K(J_\alpha, J_\alpha)$ is trivial. We remember that $K(J_\alpha, J_\alpha)$ is a submatrix of C in the Kirchhoff matrix. Let x be a column vector of appropriate size such that $Cx = 0$, This implies that $x^T C x = 0$. So,

$$\begin{aligned}
\bar{\mathbf{x}}^T C \mathbf{x} &= \sum_{i,j} \bar{x}_i(\omega) C_{ij}(\omega) x_j(\omega) \\
&= \sum_{i \neq j} \bar{x}_i(\omega) C_{ij}(\omega) x_j(\omega) + \sum_{i=1}^{n} C_{ii}(\omega) |x_i(\omega)|^2 \\
&= \sum_{i<j} C_{ij}(\omega) \left[\bar{x}_i(\omega) x_j(\omega) + \bar{x}_j(\omega) x_i(\omega) \right] + \sum_{i=1}^{n} C_{ii}(\omega) |x_i(\omega)|^2 \\
&= \sum_{i<j} C_{ij}(\omega) \left[x_i(\omega) - x_j(\omega) \right] \left[\bar{x}_j(\omega) - \bar{x}_i(\omega) \right] \\
&= -\sum_{i<j} C_{ij}(\omega) |x_i(\omega) - x_j(\omega)|^2 = 0.
\end{aligned}$$

But again since all elements of C have positive real parts, we conclude that $x_i(\omega) = x_j(\omega)$ for $\forall i, j, \omega$. Therefore, x is a constant vector of functions of ω. Now let y be a colum vector of functions of ω with appropriate size such that $K(J_\alpha, J_\alpha)(\omega) y = 0$. Then by adding zeros to y at appropriate places, we can expand y to build a new column vector z such that

$z^T C z = y^T K(J_\alpha, J_\alpha) y = 0$

This implies that z is a constant vector. But z contains some zeros and hence is the zero vector. Therefore y is the zero vector. SO, we prove that the null space of $K(J_\alpha, J_\alpha)(\omega)$ is trivial and so $\det K(J_\alpha, J_\alpha)(\omega) \neq 0$. □

Proposition 1. *Let $\Gamma = (G, \gamma_e(w))$ be an electrical network defined above. Also, suppose that G is circular planar graph. Let P and Q be a circular pair such that there is a connection between them. Then* $\det \Lambda(P; Q)(\omega) \neq 0$ *if at least one of the following conditions holds: 1) There is a unique connection between P and Q. 2) There is a connection between P and Q that goes through all the interior nodes in G.*

Proof. First suppose That the connection between P and Q is unique. Then the sum in equation 2 has only one term which is a product of bunch of admittance and $\det K(J_\alpha, J_\alpha)(\omega)$. By Theorem 1 and Lemma 2, none of the terms in the product are zero. Hence $\det \Lambda(P; Q)(\omega)$ can not be zero for $\forall \omega$.

Now suppose that there is a connection between P and Q that goes through all the interior nodes. Then by convention, $\det K(J_\alpha, J_\alpha) = 1$. So the sum will be over bunch of products of admittance. Now we show that this sum is not identically zero. Since $\det \Lambda(P; Q)(\omega)$ is a rational function of ω, it's enough to show that the sum is nonzero for some velure of ω. Let ω get very large; then we have $\gamma_e(\omega) \approx a_0 + a_1\omega$ where a_0 is the constant positive real part and $a_1\omega$ is the imaginary part of $\gamma(\omega)$. So the product of these conductivities will contain a constant term in it's real part. As we add these products we still get a positive constant term in the real part. Hence $\det \Lambda(P; Q)(\omega) \neq 0$ for large ω. □

Remark 1. *Let $\Gamma = (G, \gamma_e(\omega))$ be an electrical network and (P, Q) be a circular pair that satisfy the conditions in proposition 1. Then the Boundary Edge and Boundary Spike Formulas in [3] apply.*

This could be used to recover a large group of circular planar electrical networks.

5 Important Formulas in Recovery Process

In this section, we remind the reader the formulas that are used to recover electrical graphs using the determinantal formula. We need the boundary edge and boundary spike formulas. In a network Γ, Let $P = (p_1, \ldots, p_k)$, $Q = (q_1, \ldots, q_k)$, $P' = (p, p_1, \ldots, p_k)$, $Q' = (q, q_1, \ldots, q_k)$, and pq be a boundary edge. Suppose $\det \Lambda_\Gamma(P; Q) \neq 0$. Let Γ' be the network with the edge pq deleted, and suppose $\det \Lambda_{\Gamma'}(P; Q) = 0$. Then,

$$\gamma(pq) = -\Lambda(p; q) + \Lambda(p; Q) \cdot \Lambda(P; Q)^{-1} \cdot \Lambda(P; q) \tag{3}$$

Similarly, in a network Γ, let pr be a boundary spike between boundary node p and interior node r. Suppose there are disjoint sets of boundary nodes P and Q such that $\det \Lambda_\Gamma(P; Q) \neq 0$. Let Γ' be the network obtained after the contraction of pr, and suppose $\det \Lambda_{\Gamma'}(P; Q) = 0$. Then,

$$\gamma(pr) = \Lambda(p; p) - \Lambda(p; Q) \cdot \Lambda(P; Q)^{-1} \cdot \Lambda(P; q) \tag{4}$$

Suppose we delete a boundary edge of conductivity ξ between boundary nodes 1 and 2. If the old response matrix looks like this:

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & a \\ \lambda_{2,1} & \lambda_{2,2} & b \\ d & e & C \end{bmatrix} \tag{5}$$

The new response matrix looks like this:

$$\Lambda' = \begin{bmatrix} \lambda_{1,1} - \xi & \lambda_{1,2} + \xi & a \\ \lambda_{2,1} + \xi & \lambda_{2,2} - \xi & b \\ d & e & C \end{bmatrix} \tag{6}$$

Now, suppose we contract a boundary spike of conductivity ξ. If the old response matrix looks like this:

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & a \\ b & C \end{bmatrix} \tag{7}$$

The new response matrix looks like this, with $\delta = \lambda_{1,1} - \xi$:

$$\Lambda' = \begin{bmatrix} -\xi - \frac{\xi^2}{\delta} & -\frac{a\xi}{\delta} \\ -\frac{b\xi}{\delta} & C - \frac{ab}{\delta} \end{bmatrix} \tag{8}$$

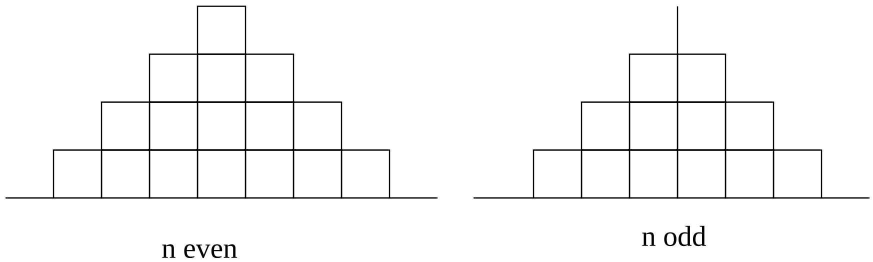


Figure 2: Standard H_n family of Graphs

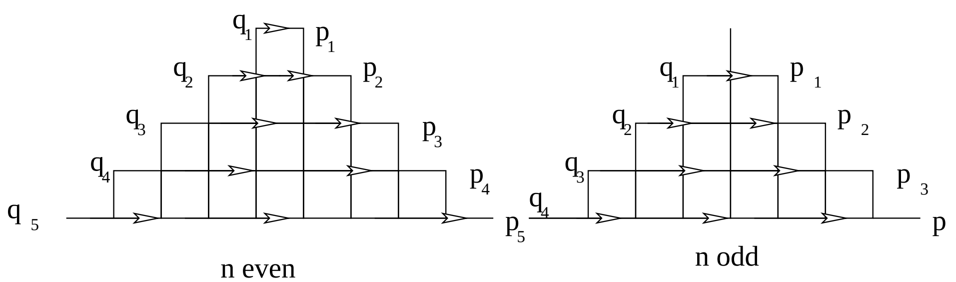


Figure 3: (p,Q) and the connection between them for recovering top boundary edges and spikes

6 Well-connected Graphs

As we remember, a graph is well-connected if for every circular pair P and Q, each containing k boundary nodes, there is a k-connection from P to Q.

In figure 2, we see the standard H_n family of well-connected graphs. We can use the proposition 1 to recover this type of electrical networks.

For recovering this family of electrical networks, we introduce a simple algorithm. We always start the recovery process from the top of the graph. This means that if n is even,we start by recovering the top boundary edge and if n is odd by recovering the top boundary spike. So, we have to look for P and Q such that there is a connection between P and Q that goes through all the interior nodes and also the connection is broken by contracting the top boundary spike or deleting the top boundary edge. These connections are shown in the figure 3. One can easily verify that this connections go through all the interior nodes and deleting the top boundary edge or contracting the top boundary spike break the connections.

In the next steps there are going to be more than one top boundary spike or boundary edge. In these steps, if there are more than one boundary spikes, we

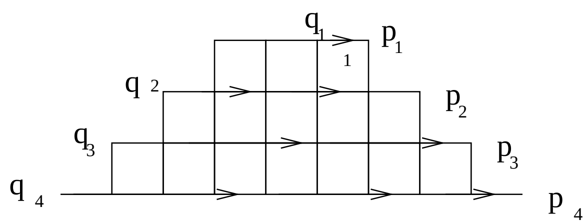
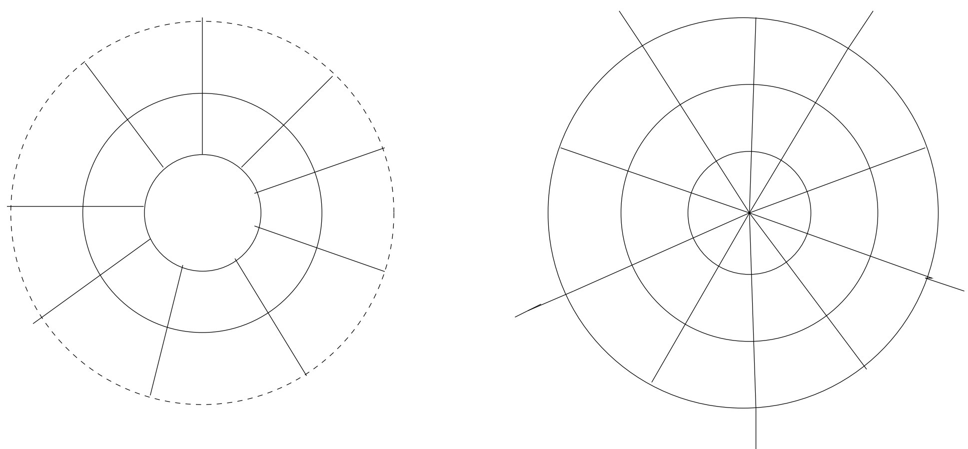


Figure 4: Recovering a boundary edge neighboring other boundary edges



G−9
Example of the type 4m+1

G_10
Example of the type 4m+2

Figure 5: Standard G_n family of Graphs

can recover them all at once using exactly the same circular pair and connection as above. If there are more than one top boundary edges, then each one can be recovered separately by choosing (P,Q) by ignoring the other top boundary edges. For example in figure 4, we see how to recover the boundary edge 1.

Doing this over and over, we'll end up with a graph that consists of only boundary vertices and edges. The response matrix of this graph is the same as its Kirchhoff's matrix and so is trivially recoverable.

Another group of well-connected graphs, G_n are easy to recover using the method above too. G_n graphs have four different shapes depending on n mod 4. The cases $n = 4m + 1$ and $n = 4m + 2$ are shown in figure 5.

Let look at the 4m+1 group of graphs. First, we start by recovering all the boundary spikes. After choosing which boundary spike we want to recover, we take the two boundary verticies neighboring this spike; we call them p_1 and q_1, and then draw the shortest path between them. Then we call the boundary vertex neighboring p_1, p_2 and them same way, we find q_2. Then, we draw the

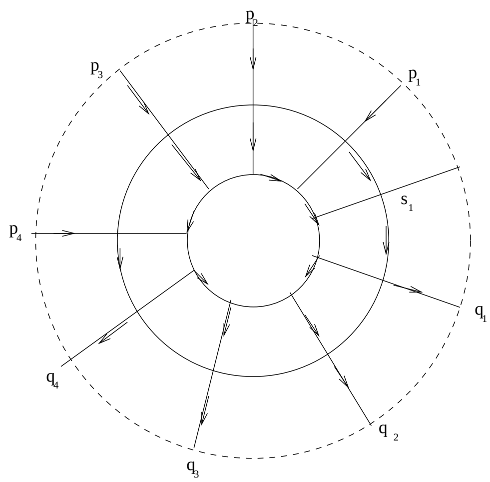


Figure 6: Recovering s_1

shortest path between them. The complete connection is shown in figure 6. This figure corresponds to recovery of boundary spike s_1.

After we are done with the boundary spikes, we are left with a new graph looking like the one in figure 7. Now, we can delete all the boundary edges one by one. The first step of this process is shown in figure 7.

7 Recovering General Critical Circular Planar Networks

Let say we want to recover a critical circular planar network using the boundary spike and boundary edge formulas. So, at each step we have to find two circular pairs P and Q such that we can find a connection between them which goes through all the interior nodes and the deletion of a boundary edge or spike breaks this connection. To do this, it is easier to work with medial graphs. First we have to decide which boundary edge or spike we want to recover.

The general procedure and rules are as follows:

- Draw the medial graph.
- Choose the boundary edge or spike that is to be recovered. The first cut is placed on this edge.
- Using Cut-Piont Lemma, find where the other cut goes so that there is a maximal connection.
- Make an appropriate coordinate system with a prefered direction.

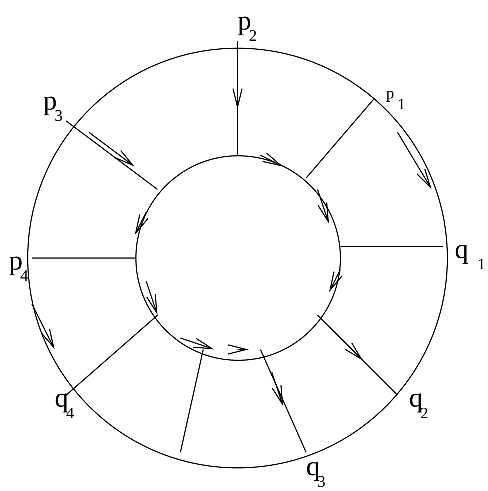


Figure 7: Recovering boundary edges p_1q_1 and p_4q_4

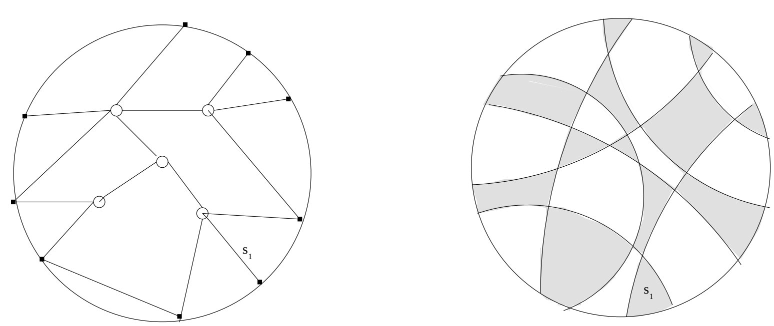


Figure 8: Example 1

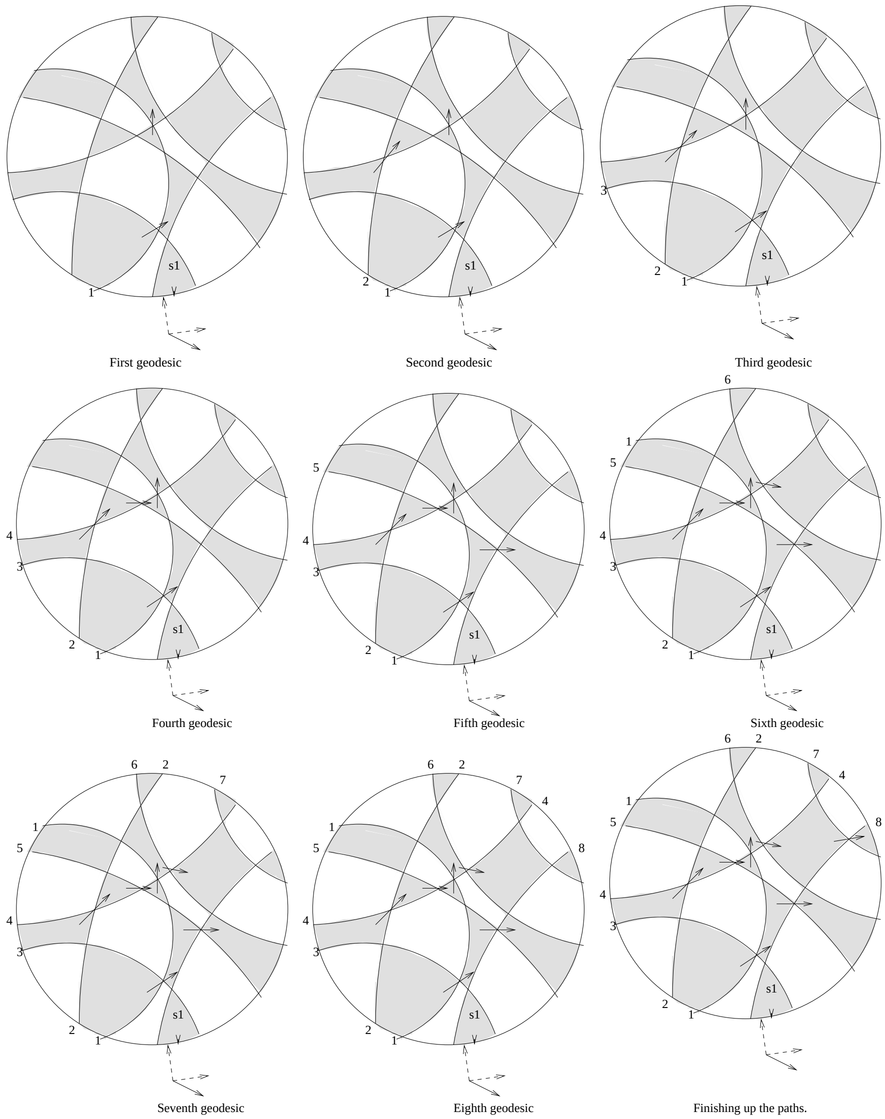


Figure 9: Finding the connection resulting in recovery of s_1

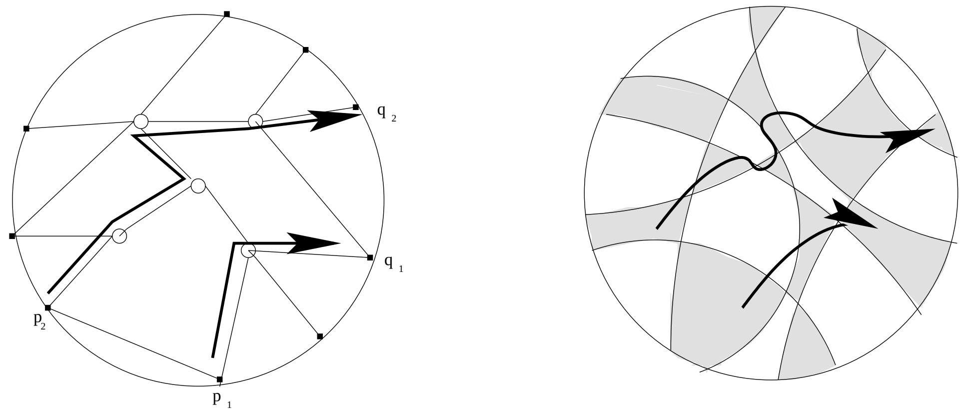


Figure 10: The result connection

- Start from the first cut, going counterclockwise.
- When you get to a geodesic, start going along the geodesic connecting the dark faces neighboring the geodesic preferably in the prefered direction using arrows.
- Going along a geodesic, you are allowed to use a dark face only once.
- Do not end any paths.
- In each dark region, only one arrow is allowed to enter and one to exit.
- The interior nodes that are connected to only one other interior node have the highest priority to be crossed by arrows.
- Now end the paths.

Let look at some examples:

Example 1. *In figure 8, we see a critical circular planar graph and its medial graph. Let say we want to recover the boundary spike* s_1*. We make our first cut at the spike on the medial graph. The coordinate system and preferable direction are shown in the figure 9. Next step is to start from the first geodesic. Each step is shown in the figure 9. Note that in each step we do not end any paths. The result connection is shown in figure 10.*

8 Future Research

There is still so much to do on this projects. Some of unanswered questions are:

- It is still an open question whether or not all critical circular planar graphs are recoverable.

- This paper does not provide a proof of why the algor him in section 7 works.
- For a critical circular planar network,can we always find a circular pair (P,Q) that satisfies the conditions in proposition 1?
- How do 2-1 networks behave with complex admittance and complex periodic applied voltages?

9 Acknowledgments

First of all, I would like to thank Professor James Morrow for the huge amount of time, energy and passion he put in this program. I would like to thank him for providing me such an amazingly "dense" summer.

Thanks to Jenny, Ernie and Sam for all their help and support both at work and social activities.

In the end, I would like to thank all of the REU students for making this group so united. Also, thanks for helping me in shading my graphs, making the printer to print my graphs,...

This program is founded by NSF.

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