Circular-planar graph embedding

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1 Critical Graph Embedding

1.1 Introduction

This paper is motivated by the following informal question: when is there only one way to draw a graph?

We first must translate this question into a well-posed problem. We confine ourselves to a discussion of planar graphs. Which planar graphs have the property that all their embeddings are topologically isomorphic?

We begin by presenting an inductive argument that critical circular-planar graphs have a unique embedding given an ordering of the boundary nodes. We perform a double induction on the number of interior vertices and the number of adjacent boundary-to-boundary edges. We then proceed through another induction, this time arguing that the interior faces of critical circular-planar graphs are embedding-invariant regardless of the ordering of the boundary nodes. This second induction is a double induction on the number of interior vertices and the number of interior faces. For the purposes of brevity, when we use the word "graph" we mean a graph that is critical circular-planar unless otherwise indicated.

1.2 Results

Definition. An *embedding* of a graph G in a surface S is a one-to-one map $f : G \to S$ such that vertices map to points in S and edges map to simple disjoint curves in S that connect their boundary points. A *face* is a surface, bounded by edges in $f(E)$ and the circular boundary of the graph, that is topologically equivalent to a disc.[1]

Theorem 1. A critical circular-planar graph with ordered boundary vertices is uniquely embedded in the plane.

For the following proof, an ordering of the boundary vertices is assumed when the word "graph" is used.

Interior Vertex Base Case. Our base case is a graph with no interior vertices.

Claim: A graph with no interior vertices is embedding-invariant.

Proof. Given an embedding, the faces of a graph with no interior vertices will fall into three types: those bounded by an adjacent boundary-to-boundary ($\partial - \partial$) edge and the circular boundary of the graph; those bounded by at least one non-adjacent $\partial - \partial$ edge and the circular boundary of the graph; and those bounded by $\partial - \partial$ edges only. The first of these three types is embedding-invariant since the boundary vertices are guaranteed to be adjacent in any embedding by our ordering of the boundary vertices. The ordering also ensures preservation of faces of the second type. The Jordan

[1] This definition is taken from Jeff Eaton and Owen Biesel's paper, "Circular-planar self-dual graphs."

curve theorem gives us embedding-invariance of the third type of face. Thus we find that a critical circular-planar graph with no interior vertices equipped with an ordering of the boundary vertices has only one embedding.

Interior Vertex Inductive Hypothesis. We now assume there is a unique embedding for a graph with n interior vertices.

Given a graph with $n+1$ interior vertices, we can induct on the number of adjacent $\partial - \partial$ edges.

Adjacent Boundary-to-Boundary Base Case. Our base case is a graph with no adjacent $\partial - \partial$ edges.

Claim: A graph with $n+1$ interior vertices and no adjacent $\partial - \partial$ edges is embedding-invariant.

Proof. By Lemma 8.6 [1], there is at least one boundary spike on a graph with no adjacent $\partial - \partial$ edges, and we can contract this boundary spike to form an n-interior-vertex graph. Contracting the spike shortens an edge of the the faces it borders. Note that this n-interior-vertex graph is both critical and electrically equivalent to the $n+1$-interior-vertex graph, so we can apply the inductive hypothesis: By construction, the n-interior-vertex graph has identical faces to that of the $(n+1)$-interior-vertex graph, so we conclude that the $(n+1)$-interior-vertex graph is embedding-invariant.

Adjacent Boundary-to-Boundary Inductive Hypothesis. Now assume that there is a unique embedding for a graph with $n+1$ interior vertices and m adjacent $\partial - \partial$ edges.

Claim: A graph with $n+1$ interior vertices and $m+1$ adjacent $\partial - \partial$ edges is embedding-invariant.

Proof. Given a graph with $m+1$ adjacent $\partial - \partial$ edges, we can reduce to a graph with m adjacent $\partial - \partial$ edges by deleting a $\partial - \partial$ edge. This operation deletes an embedding-invariant face, as shown previously, so we can conclude that a graph with $m+1$ adjacent $\partial - \partial$ edges is embedding-invariant.

This concludes the proof.

Now we wish to show a slightly stronger result.

Definition. Given a circular-planar graph, an *interior face* is one that is bounded by edges of the graph only (not the circular boundary).

Theorem 2. Given a critical circular-planar graph G, the interior faces of G are embedding-invariant regardless of the ordering of the boundary vertices.

Again we use double induction.

Interior Vertex Base Case. Our base case is again a graph with no interior vertices.

Claim: The interior faces of a graph with no interior vertices are embedding-invariant.

Proof. We are concerned with polygons bounded by boundary-to-boundary edges. The Jordan curve theorem guarantees that these polygons will exist in any embedding; it remains to show that if the polygons are empty in one embedding, that they are empty in all other embeddings (and thus are embedding-invariant). This follows immediately from the fact that all of the vertices in the polygons lie on a circular boundary.

Interior Vertex Inductive Hypothesis. We now assume that the faces of an n-interior-vertex graph are embedding-invariant.

Given a graph with $n+1$ interior vertices, we can induct on the number of faces:

An $n+1$-interior-vertex graph with one face trivially has the desired property. We now assume that for a graph with m faces and $n+1$ interior vertices, the m faces are embedding-invariant. There are two ways to reduce the $n+1$-interior-vertex $m+1$-face graph: reduce the number of interior vertices or reduce the number of faces. The first case (reducing the number of interior vertices) proceeds easily by deleting a boundary spike, if such a spike exists in the graph. If no such spike exists, we are forced into the second form of reduction. By deleting adjacent boundary-to-boundary connections, eventually either we will produce a boundary spike or we will reduce the number of faces in the graph by one. What we want to now show is that when reduce the number of faces in the graph, we do so in such a way as to secure the embedding-invariance of the face we lose. Note that deleting a boundary-to-boundary edge always merges two faces, either two interior faces, or an interior face and a face adjacent to the so-called north pole when we consider the graph on a sphere. So we can proceed by contradiction: Assume that deleting a boundary-to-boundary edge merges different faces in different embeddings. Then there were two different ways to embed the reduced graph, contradicting our induction hypothesis.

2 Whitney's Theorem

Although the section above was devoted to the study of critical circular-planar graphs, the question posed at the beginning is a more general one. Whitney proved a theorem on the embedding-invariance of 3-connected graphs. We attempt here to present an exposition of this theorem. A promising problem for future research is to formulate a converse to Whitney's theorem for circular-planar graphs, since critical circular-planar graphs are by no means the only circular-planar graphs with unique embeddings.

2.1 Definitions

Cycle. The cycle C_n is a graph with vertices $\{v_1, v_2, ..., v_n\}$ and n edges $v_i v_{i+1}$ for $1 \leq i \leq n$, where $v_{n+1} = v_1$.

Non-separating. The cycle C is non-separating if $G - V(C)$ is connected. ($G - V(C)$ consists of G minus all vertices in C and those edges incident to these vertices.

Chord. An edge which joins two vertices of a cycle but is not itself an edge of the cycle.

Induced Cycle. A chord-less cycle.

C-Bridge. A C-bridge in G is a subgraph of G, which is (i) An edge not in C but with both ends in C, or, (ii) A connected component of $G - V(C)$ together with all edges which have one end in this component and the other end in C. The first type of C-bridge is equivalent to a chord.

Overlap. Let C be a cycle in a graph G. Two C-bridges B_1 and B_2 overlap if at least one of the following is satisfied: (i) B_1 and B_2 have three vertices of attachment in common, (ii) C contains distinct vertices a, b, c, d (in this cyclic order) such that a and c are vertices of attachment of B_1 and b and d are vertices of attachment of B_2. This second case is referred to as a skew-overlap.

3-connected. G is 3-connected if $|G| \geq 4$ and for any set of vertices S, $S \subseteq V(G)$, $|S| \leq 2$, $G - S$ is connected.

2.2 Proof of Whitney's Theorem

Lemma. C is induced and nonseparating if and only if there is at most one C-bridge in G.

Proof. We prove the forward direction by contrapositive. Suppose there are two C-bridges in G. Then either one is a chord, in which case C is not induced, or both are type (ii), in which case it is separating. (To see why this is true, recall that C-bridges are connected components, so if there are two of them, when the cycle is deleted, the graph is separated since it has more than one connected component.) The remaining direction goes as follows: If there is at most one C-bridge in G, then C is nonseparating since there is at most one connected component of $G - V(C)$.

Proposition. The face boundaries in a 3-connected planar graph are precisely its nonseparating induced cycles.

Proof. Let G be a 3-connected plane graph, and let C be a nonseparating induced cycle in G. Since C is a cycle, by the Jordan curve theorem, it divides the plane into a bounded and an unbounded region. Non-separating guarantees that $G - V(C)$ must either lie in the bounded or the unbounded region aside from chords. Because it is induced, C bounds exactly one face – either the bounded or the unbounded face.

The converse is proved by contradiction. Let C be a facial cycle of G and suppose that it is not an induced nonseparating cycle. Then by our lemma, there are at least two C-bridges, call them H_1 and H_2. They lie by definition in the exterior of the cycle. We want to show that they do not overlap.

Suppose they do. Then we can obtain a homeomorph of a K_5 or a $K_{3,3}$ by adding to $C \bigcup H_1 \bigcup H_2$ a vertex in $int(C)$ and joining it to all vertices of C. As a consequence of Euler's formula, neither K_5 nor $K_{3,3}$ are planar-embeddable, which draws a contradiction.

Now we claim that some pair of consecutive vertices of attachment of H_1 on C separates G, which is a contradiction to 3-connectedness.

H_1 and H_2 do not overlap implies that (i) they have fewer than three vertices in common, and (ii) there do not exist four vertices a, b, c, d in cyclic order such that a, c are vertices of attachment of H_1 and b, d are vertices of attachment of H_2. There are then three cases:

(1) H_1 and H_2 have two vertices of attachment in common. Removing these two vertices will produce the desired disconnection. (2) H_1 and H_2 have one vertex of attachment in common. Removing it and another one (arbitrarily) will produce the desired disconnection. (3) They have no vertices of attachment in common. Remove all vertices of attachment of H_1 from C.

This concludes the proof.

Whitney's theorem follows directly from this proposition.

Theorem (Whitney, 1932). Any two planar embeddings of a 3-connected graph are equivalent.[4]

References

[1] Edward B. Curtis and James A. Morrow, *Inverse Problems for Electrical Neworks* 2000: World Scientific Publishing Co. Pte. Ltd., Singapore.

[2] Bojan Mohar and Carsten Thomassen, *Graphs on Surfaces* 2001: Johns Hopkins University Press, Baltimore.

[3] Reinhard Diestel, *Graph Theory* 1997: Springer-Verlag New York, Inc.

[4] Hassler Whitney, "Congruent Graphs and the Connectivity of Graphs", *American Journal of Mathematics*, pp. 150-61, 1932.