Links to Total Positivity

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1 Introduction

This report does not contain any new results due to me. Rather it summarizes a paper of Fomin's [1] that shows some connection between the electrical network theory developed in this REU seminar over the years, and the existing body of work on total positivity of matrices and their association to walks on directed graphs. (In fact, Fomin's paper was sparked by a conversation between him and former REU student David Ingerman.)

There are interesting parallels or analogies between these two bodies of theory. In both, we have associations between edge-weighted graphs and matrices satisfying certain properties. In both, the geometry (or topology, or connectivity) of certain graphs (e.g. circular planar in the case of electrical networks) results in constraints on the associated matrices, namely in the sign (+,0, or -) of various minors of the matrix. In both, these constraints correspond to the (non-)existence of sets of disjoint paths in the graph between equinumerous sets of distinguished nodes (boundary nodes in the case of electrical networks, sources and sinks in the case of directed graphs), and corresponding formulas for the relevant minors.

These similarities might give one hope that, more than being merely analogous, theory from one field could be in fact applied to the other, but there are obstacles to direct application. Much of the work on total positivity (see [2], for example) associates a totally positive matrix to a weighted *acyclic directed* graph, whereas the graph underlying an electrical network is undirected. There is a straightforward way to convert an undirected graph to a directed graph by replacing each undirected edge with a pair of oppositely directed arcs, but the resulting graph is not acyclic. Also, it is not obvious how the response matrix of an electrical network could be related to walks on the graph of the network. Thus while some of the REU students in past years have studied inverse problems related to random walk networks, as far as I know they have treated these as problems analogous to, but separate from, the inverse problems related to electrical networks which are the standard fare for this seminar. Fomin's paper gives a (modest) connection between these types of problems.

For brevity of this talk, I would direct you to [1] for more on all the parallels and for combinatorial proofs of the analogous determinantal identities. Here I will cut to the chase and report Fomin's connection.

2 Walks on Graphs

Let G be an arc-weighted directed graph, not necessarily acyclic. Given a walk π (directed path) between nodes a and b,

$$\pi : a = v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_k} v_k = b$$

define the *weight* of the walk, denoted $w(\pi)$, to be the product of the weights of the arcs in the walk. The weights may be numbers or, more generally, formal variables. By convention, the weight of the zero-length path (i.e. no arcs) from a node to itself is 1. Given a family $\pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ of walks (not necessarily disjoint), define the weight of the family to be the sum of the weights of the member walks, i.e.

$$w(\pi) = \sum_{i=1}^{n} w(\pi_i)$$

We define the *walk matrix* W of the graph to be such that the entry $W(i,j)$ is the generating function for the weights of all of the walks in the graph from node i to node j, i.e. $W(i,j)$ is the weight of the family of all of the walks from node i to node j. For a finite acyclic graph, each $W(i,j)$ has a finite number of terms. For example, if each edge is given unit weight, then $W(i,j)$ is simply the number of distinct walks from node i to node j.

Let Q be the weighted adjacency matrix, i.e. such that the entry $Q(i,j)$ is the weight of the arc from node i to node j (or zero if there is no such arc). We can view the entry $Q(i,j)$ as the generating function for all walks of length 1 (i.e. one arc) from node i to node j. Similarly, the entry of $Q^k(i,j)$ is the generating function for all walks of length k from node i to node j. Then we have the formula

$$W = (I - Q)^{-1} = I + Q + Q^2 + Q^3 + \cdots$$

where I is the appropriately sized identity matrix, which can also be expressed as $W = QW + I$.

Assuming V is the vertex set of the graph, suppose we distinguish a non-empty subset $\partial V \subseteq V$, which we will call the *boundary* of the graph, and set $intV$ to be the complement, which we will call the *interior*. Similar to the walk matrix, define the *hitting matrix* X of the graph to have entries $X(a,b)$, where $b \in \partial V$ and $a \in V$, such that $X(a,b)$ is the generating function for all walks of length greater than zero from node a to node b such that no node in the walk is in ∂V except the last one, b, and perhaps the first if $a \in \partial V$ as well.

If the weights of the arcs are all positive and are normalized so that Q is a transition probability matrix for a Markov chain, i.e. such that row sums of Q are all 1, then $X(a,b)$ is the probability that a random walk beginning at a will "hit" node b after leaving a and do so before hitting any other node of ∂V. In this case we have a formula for the submatrix $X_{\partial V, \partial V}$, namely

$$I - X_{\partial V, \partial V} = (I - Q)/(I - Q_{intV, intV})$$

where the expression on the right hand side is the Schur complement, and by common abuse of notation, I represents various appropriately sized identity matrices. To see this, let

$$Q = \begin{bmatrix} Q_{\partial V,\partial V} & E \\ F & Q_{intV,intV} \end{bmatrix}$$

Then the Schur complement $(I - Q)/(I - Q_{intV,intV})$ is

$$(I - Q_{\partial V,\partial V}) - E(I - Q_{intV,intV})^{-1}F$$

As noted above, $(I - Q_{intV,intV})^{-1} = W_{intV,intV}$. Then the way to interpret the matrix $Q_{\partial V,\partial V} + EWF$ is that the (a, b) entry gives the probability of transitioning directly from node a to node b (from $Q_{\partial V,\partial V}$) plus the probability of transitioning from node a to the interior (E), then walking around in the interior (W), then finally walking back onto the boundary (F), which is precisely $X(a, b)$.

3 Electrical Networks

We have learned in the REU seminar about the Kirchoff matrix and response matrix associated with an electrical network, and that the relationship is given by

$$\Lambda = A - BC^{-1}B^T \tag{1}$$

where the Kirchoff matrix in block form is

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

and where the submatrix A is indexed by the boundary nodes and C by the interior nodes.

Now given an electrical network Γ, we can canonically construct a Markov chain M from it as follows. Let the graph of M be the directed graph obtained from the graph of Γ by having the same set of vertices and replacing every edge with a pair of oppositely directed arcs. To the arc from node i to node j, assign the weight

$$Q(i, j) = \frac{\gamma(ij)}{\sum\limits_{i\to k} \gamma(ik)}$$

where $\gamma(ik)$ is the conductance in Γ of the edge ik. In other words, we can randomly walk on our electrical network where the probability of moving from one node to neighboring nodes is proportional to the respective conductances of the edges connecting the given node to its neighbors. Let D be the diagonal portion of the Kirchoff matrix for Γ, i.e. $D(i, j) = K(i, j)\delta_{ij}$. Then it is apparent that $Q = I - D^{-1}K$.

Now let X be the corresponding hitting matrix of this Markov chain as described above. We now have the relation

$$X_{\partial V,\partial V} = I - D_0^{-1}\Lambda$$

where Λ is the response matrix for the network, and $D_0 = D_{\partial V,\partial V}$.

4 Is it useful?

While Fomin's paper gives a relationship between the response matrix of a network and the hitting matrix associated to the network, the formula also involves a part of the original Kirchoff matrix (for normalization). It is not obvious to me how this formula in itself would be useful to calculations for the standard inverse problem for electrical networks.

Fomin's paper does, however, show that there is some kind of connection between these two views of an electrical network, as given by the formula. Moreover, there are determinantal formulas for walk matrices of directed graphs (even ones with cycles) analogous to those for electrical networks, as shown in [1]. One never knows when such an alternative view may yield some new insight or new approach to tackling existing problems.

References

[1] Sergei Fomin. *Loop-Erased Walks and Total Positivity*, Transactions of the AMS, vol.353, no. 9, 3563-3583 (2001).

[2] Fomin and Zelevinsky. *Total Positivity: Tests and Parameterizations*, Math Intelligencer vol. 22, no. 1, 22-23(2000).

[3] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks* Series on applied mathematics - Vol. 13. World Scientific, ©2000.