All Recoverable Graphs with Five Boundary Nodes or Fewer

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Abstract

The case of 5 boundary nodes is examined in depth as a model for larger graphs, and to establish the set of recoverable 5 boundary node graphs. Unrecoverable graphs of 5 nodes or less are noted as well, including the unusual case of the "triple firecracker" graph.

Contents

1 Background

Electrical networks can be represented as graphs with edges and vertices. The vertices are designated as either boundary or interior nodes. From the boundary nodes, and these nodes only, it is assumed some data can be taken. This data can be stored in an n by n response matrix, where n represents the number of boundary nodes. The discrete Dirichlet problem presents a response matrix, from which the original Kirchoff matrix of edge conductivities is desired. If it is possible to obtain the Kirchoff matrix from the response matrix, the graph is called recoverable. There are currently many ways to approach the problem of recoverability, however, there is not yet any complete characterization of recoverable graphs. It is possible though, in cases of small numbers of boundary nodes, to entirely find all recoverable graphs. Examining all small graphs can provide a basis for examining larger graphs.

Definition 1 *A vertex of a graph has degree n if n edges meet at that vertex.*

2 Establishing Limits

Categorization by number of boundary nodes alone does not sufficiently limit the number of graphs to examine. As we are mainly concerned with recoverable graphs, it will be helpful to limit the set of graphs we examine through some elementary observations.

2.1 Maximum Number of Edges

Firstly, given an n by n response matrix, we have only $\frac{n(n-1)}{2}$ independent entries. This determines the maximum number of edges for recoverability of a graph with n boundary nodes. Thus in the cases of five boundary nodes or less this results in ten edges maximum for five boundary nodes, six for four, three for three, and one for two.

2.2 Minimum Degrees of Nodes

Only connected graphs are of interest so each node must have at least degree one.

2.2.1 Interior Nodes

A recoverable graph must always have at least three edges on each interior node, i.e. all interior nodes must have at least degree three. This can be seen easily by examining the degree one and two cases. In these cases it is always impossible to recover the edges adjacent to the interior node, thus rendering the entire graph unrecoverable.

2.3 The Limiting Equations

Given the limits on the number of total edges and the degree required for each type of node, it is possible to establish a limit on the total number of interior nodes allowed given the number of boundary nodes, and the total number of edges allowed in each case.

The condition that must be satisfied is:

$$\lceil (3 * (\#ofinteriornodes) + (\#ofboundarynodes))/2 \rceil = Maxedgesneeded$$

This function provides a lower limit on the number of edges in a recoverable graph given the number of interior and boundary nodes. As we also know the upper limit given by the examination of the response matrix, this formula gives us a bound on the number of interior nodes allowed.

$$\lfloor (2 * Maxedges - (\#ofboundarynodes))/3 \rfloor = Maxinteriornodes$$

Case 2 *Two Boundary Nodes*
Maximum Edges: 1   Maximum Interior: 0

Case 3 *Three Boundary Nodes*
Maximum edges: 3   Maximum Interior: 1
Minimum edges: 2

Case 4 *Four Boundary Nodes*
Maximum edges: 6   Maximum Interior: 2
Minimum edges: 3

Case 5 *Five Boundary Nodes*
Maximum edges:10 Maximum Interior: 5

Case 6 *N Boundary Nodes*
Maximum edges: $\frac{N(N-1)}{2}$ *Maximum Interior:* $\left\lfloor \frac{N^2-2N}{3} \right\rfloor$
Minimum edges: $N-1$

3 Methods of Discovery

3.1 Y-Delta Equivalency

The flow of electricity through a Y shaped graph and a delta shaped graph is equivalent, therefore any such transformation does not affect recoverability. Application of the Y-Delta transformation is an extremely useful tool in determining recoverability.

Particularly, if we have found all of the recoverable graphs with k interior nodes and x edges, then we may find all of the recoverable graphs with k-1 interior nodes and x edges that contain Deltas by performing Y-Delta transformations. Conversely, if we know all of the k interior node recoverable graphs with x edges then we may find all of the k+1 interior node recoverable graphs with x edges that have any degree three interior nodes.

Thus given a set of all k interior node recoverable graphs with x edges, it would only be necessary to examine the graphs with k+1 interior nodes such that these nodes have degree four or higher, and the graphs with k-1 interior nodes that contain only quadrilaterals or higher edged polygons.

This is incredibly useful for limiting the number of graphs to examine. Given k interior nodes and x edges, if it is not possible to construct graphs with degree ≥ 4 interior nodes, then all such graphs are Y-Delta equivalent to k-1 interior node, x edge graphs.

For five boundary nodes we can see that for k = 5, x = 10, that all graphs must be equivalent to k=4, x = 10. Additionally, all k = 4, x = 10 graphs, must be equivalent to k=3, x = 10 graphs. At this point we may make graphs that do not contain Y's to transform with degree four interior nodes, so we have found the source of all graphs where k = 4 or 5 and x = 10.

The Current State of Exhaustion.

	10 edges	9	8	7	6	5	4	TOTAL G
0int	K_5 (1 graph)	1 graph	3 graphs	4 graphs	5 graphs	5 graphs	3 graphs	22
1	(1 equiv)	?	(5 equiv)	(6 equiv)	(5 equiv)	(3 equiv)	-	
	?	?	16 graphs	14 graphs	9 graphs	5 graphs		
2	?	?	(9 equiv)	(8 equiv)	(3 equiv)	-	-	
	?	?	22 graphs	9 graphs				
3	?	(9 equiv)	(4 equiv)	(1 equiv)	-	-	-	
	? somenew	-	-	-				
4	all y-delta to above	(1 equiv.)	-	-	-	-	-	
5	(1 equiv graph)	-	-	-	-	-	-	1

3.2 Operations that Maintain Recoverability

3.2.1 Addition of Boundary-Boundary Edges

This does not maintain recoverability. See the counterexample in the figure.

Moreover, this does not ever improve the chances of recoverability since the new graph will always contain the old non-recoverable graph as a subgraph, redering it unrecoverable.

3.2.2 Removal of Boundary-Boundary Edges

If a graph is recoverable, then all of it's subgraphs must also be recoverable. Particularly, it is useful to consider removal of boundary to boundary edges. Given an x edge graph with k interior nodes, we can find x-1 edge graphs with k interior nodes by removing boundary boundary edges.

3.2.3 Addition of Boundary-Interior Edges

Addition of a boundary-interior edge does not improve recoverability

3.2.4 Removal of Boundary-Interior Edges

Spikes By the subgraph argument, contraction of a single boundary-interior spike on a recoverable graph must leave a recoverable graph. However, given a cluster of spikes all spikes from a particular interior node must be removed simultaneously to maintain recoverablity. Therefore, to maintain the same number of boundary nodes, only single spikes contraction is permitted.

4 Determining Recoverability

4.1 Circular Planar Graphs

The case of circular planar graphs has already been well characterized. See () For any circular planar graph a medial graph may be drawn. If this medial graph contains no lenses or loops, then the graph is recoverable. (Show example of recoverable, non recoverable)

4.2 Subgraphs

If a graph contains an unrecoverable subgraph, then the graph is not recoverable. See Ryan Card, and Amanda Cadieu for exposition on subgraphs. This method determines only unrecoverability. It is possible for a graph to have all recoverable proper subgraphs and still be unrecoverable. (check this statement)

4.3 Breaking One-Connections

If an edge breaks a one connection then the edge is recoverable. This fact will be particularly useful for non circular planar graphs. If enough edges can be discovered in this manner, then it may leave a smaller graph for which recoverability is known. This may or may not be a subgraph. This method can help determine both recoverability and unrecoverablility.

4.4 Other

Some graphs will not be circular planar, will have only recoverable subgraphs, and have no one connections that can be broken by the removal of an edge. These graphs will have to be treated on a case by case basis, using special flows, two connections and other more clever methods to attack the problem of recoverability. This set is currently the set of graphs that still needs to be characterized.

5 1, 2, 3, and 4 Boundary Nodes

It is simple to find all recoverable graphs of one to four boundary nodes.

Note that when grouped by Y-Delta equivalency, there are only 9 graphs for 4 boundary nodes instead of 14. Under further grouping, by removal of boundary-boundary edges, there are only two classes of graphs.

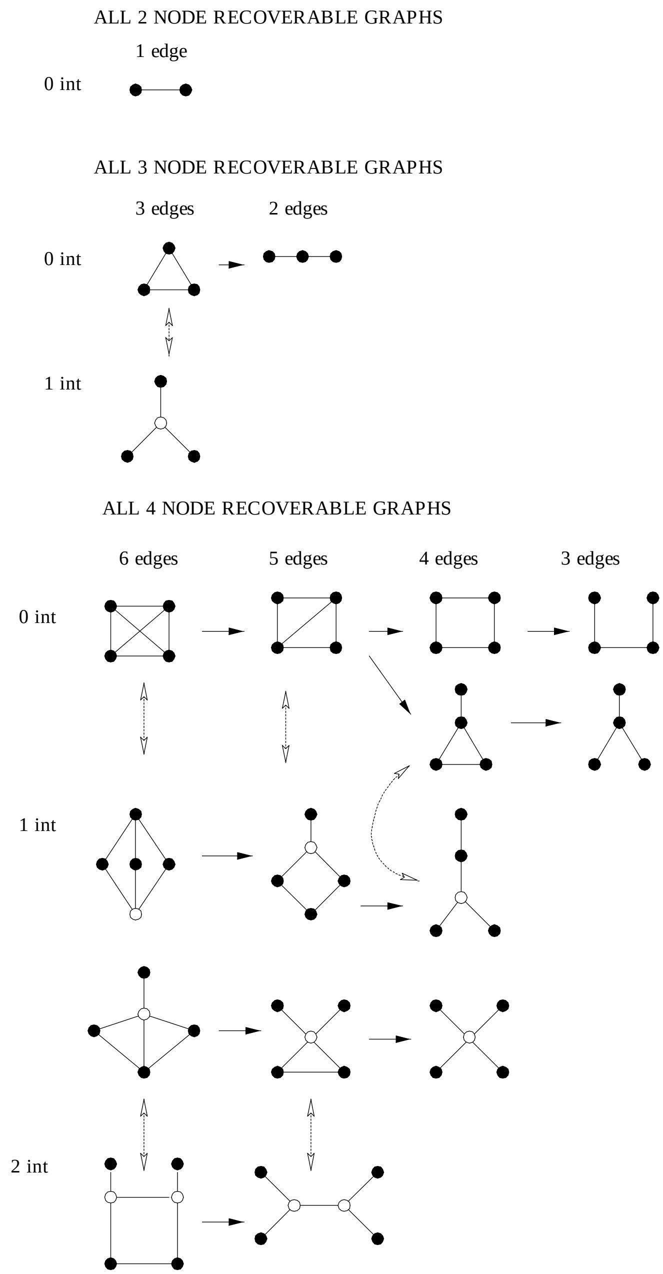

Figure 1: Every Recoverable Graph with 4 Nodes or Fewer

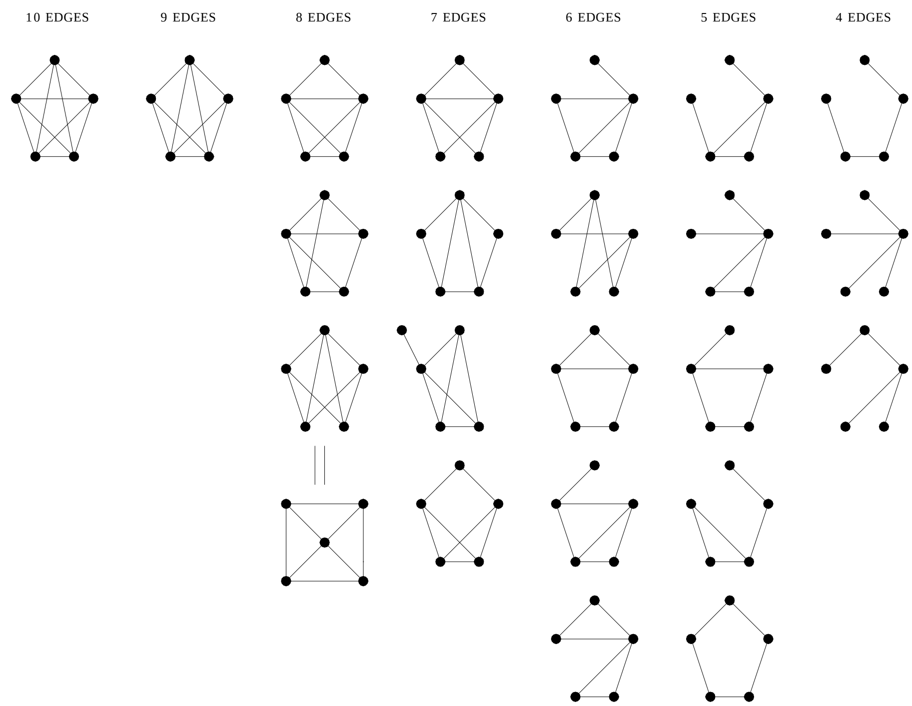

Figure 2: 0 INTERIOR NODE RECOVERABLE GRAPHS

6 5 Boundary Node Graphs

6.1 0 Interior Nodes

This case is trivial. All graphs with zero interior nodes can be found by deleting the boundary boundary edges from K_5. Additionally, all of these graphs will be recoverable as all of their nodes are boundary nodes.

(FIG)

6.2 1 Interior Node

So far complete for 5, 6, 7, 8 edges. 9, 10 edges remain.

To obtain these graphs, work up from 0 interior nodes, changing deltas to Y's. This gives all graphs with degree three interior nodes. Then examine all graphs with degree four or higher interior nodes.

6.3 2 Interior Nodes

So far complete for 6, 7, 8 edges. 9, 10 edges remain.

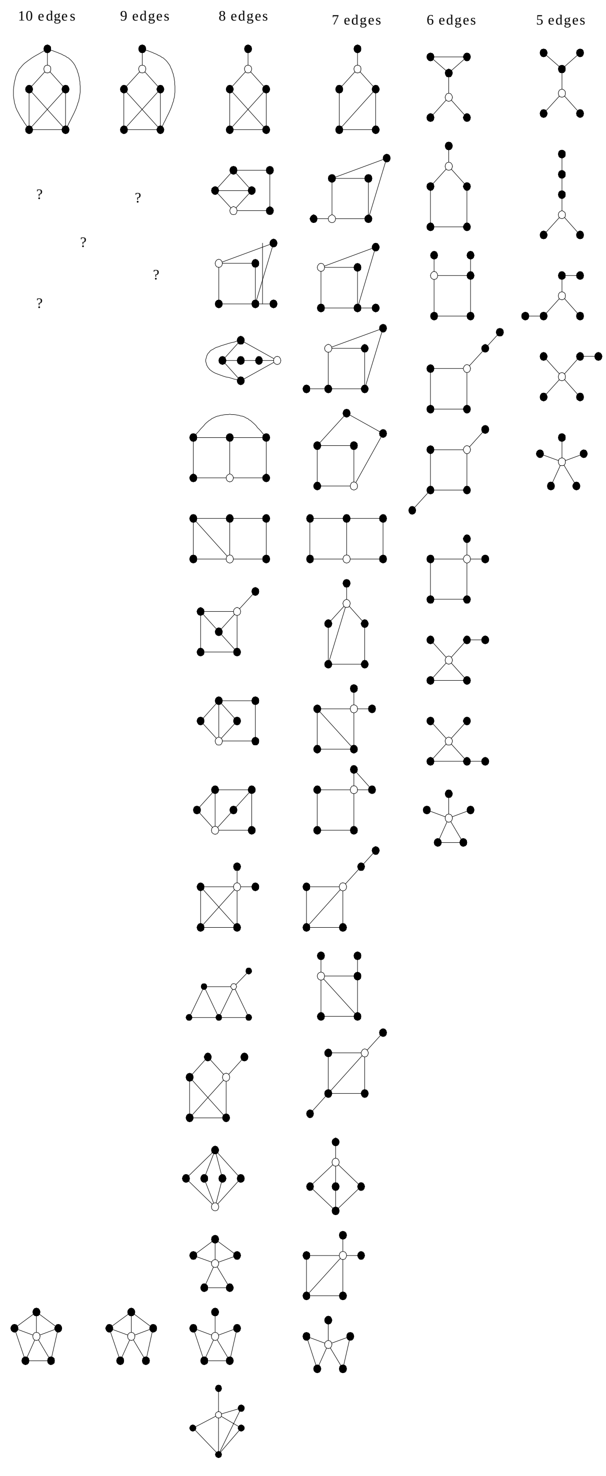

Figure 3: 1 interior node recoverable graphs

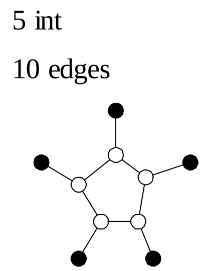

Figure 4: The 5 interior node recoverable graph

6.4 3 Interior Nodes

So far complete for 7, 8, 9 edges. 10 edges remains.

6.5 4 Interior Nodes

Complete for 9 edges. 10 edges remains.

Note that all graphs with 4 interior nodes must be Y-Delta equivalent to 3 interior node graphs as they must contain at least one degree three interior node.

6.6 5 Interior Nodes

Finished. 10 edges is the only possibility. There is only one recoverable graph and it is Y-Delta equivalent to a 4 interior node graph, which is equivalent to a 3 interior node graph.

7 Interesting Unrecoverable Graphs

Unrecoverable graphs can be interesting for a number of reasons. It is important to be able to instantly identify small unrecoverable graphs, since as subgraphs they cause the entire graph to be unrecoverable.

In the case of three boundary nodes, it is extremely obvious when graphs are not recoverable, as there are only three possible recoverable graphs. However, it is still worth taking note of the "kiss" graph, which is not recoverable because it has too many edges, and additionally, when Y-Delta transformed, contains a double edge. This graph frequently occurs as a subgraph, causing overall unrecoverablility, so it is expedient to name and specify this graph.

Things become slightly more interesting in the case of four boundary nodes. Here, all recoverable graphs are circular planar, except for the one which is Y-Delta equivalent to K_4. Thus all non-circular planar graphs besides this one are not recoverable. This includes the four-flower pictured. Additionally a few circular planar unrecoverable graphs are noteworthy. The "bowtie" graph frequently pops up as a subgraph, and is not recoverable by examining the medial graph, or by noticing that when both deltas are transformed to y's that there is a series connection. The "firecracker" graph also frequently occurs as a subgraph and is not recoverable by the lens in the medial graph or by noting the double edge created by Y-Delta transformations.

7.1 The Triple Firecracker

The triple firecracker has interior nodes of degree four, so there are no Y's to transform, and the smallest polygon in the graph is a square, so there are no deltas to transform. Additionally, the only subgraphs are four-stars which are trivial and recoverable. Also, contracting a spike does not break any one connections, so we cannot tell if they are recoverable or not. Thus this graph is very interesting and must be examined using other methods.

The first method is the "special flow method". This process is described in (ref) paper. As many conditions as there are edges may be placed. See (fig) for the set up. We begin with a current of 0 at node 4, a voltage of 0 at node 1, a voltage of 1 at node 2 and a current and voltage of 0 at node 5. We label unknown voltages a and b. Observe that several equations are gained from this set up.

$x(a-0)+y(a-1)+z(a-b)=0$

$xa-ya-y+za-zb=0$

This equation results from examining the interior node adjacent to node 4. This node must have a current of 0, as node 4 has a current of 0 and all of its current must come from the adjacent edges. We can then solve for a:

$a=\frac{zb+y}{x+y+z}$

For recovery to proceed we need a to be non-zero, since if a is 0 then there will be no current through the edge x, which will make this edge unrecoverable under this setup.

a $=0$ when $b=-\frac{y}{z}$

When will this condition on b hold? Given the above conditions we note that the node adjacent to node 5 must also have current and voltage of 0, for 5 to have 0 voltage and 0 current. Therefore we can form an equation about this node as well.

$u(0-0)+v(0-1)+w(0-b)=0$

$-v-bw=0$

$b=-\frac{v}{w}$

Therefore $b=-\frac{y}{z}$ precisely when $b=-\frac{v}{w}=-\frac{y}{z}$ or when $\frac{v}{w}=\frac{y}{z}$ which is equivalent to $vz=wy$.

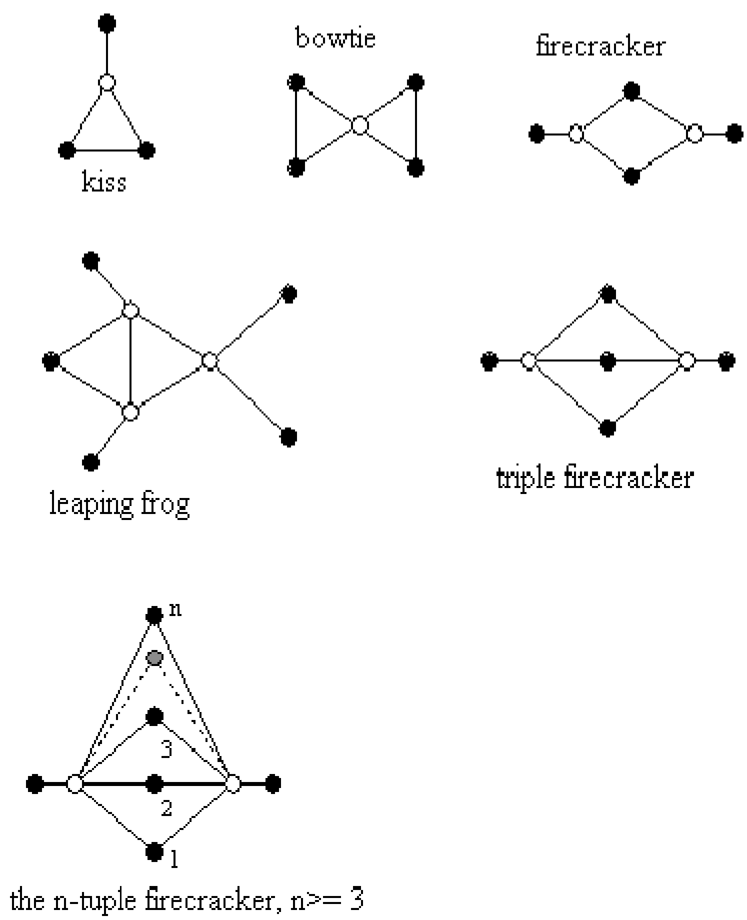

Figure 5: interesting unrecoverabale graphs

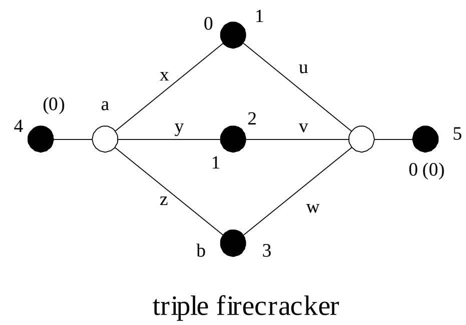

Figure 6: The triple firecracker graph with numbered nodes, labeled edges, and five conditions

However, even if this is true, we may be able to change the setup, choosing different nodes to place the 1 and 0 voltages on so that the edge x is recoverable. By symmetry we can interchange the firecracker pieces in the middle to obtain the related conditions.

$xv = yu$ and $xw = uz$.

It is not difficult to see that given two such conditions, the third is implied. Therefore if we have two such conditions, there will be no way to interchange the voltages to discover edge x.

If the conditions do not hold, then we can recover the entire graph.

The second method is an examination of two connections. Using the determinant connection formula we find that the sum of three 2 by 2 determinants must be 0.

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