Harmonic Graph Homomorphisms

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Abstract

We are interested in developing the theory of maps between resistor networks that preserve certain electrical properties. In the case of continuous potentials, there have been fairly extensive studies of maps between Riemannian manifolds that pull back locally defined functions that are harmonic on the codomain to locally defined harmonic functions on the domain. These maps, known as harmonic morphisms, have been discretized so that they may be applied to graphs. We study some basic properties of these maps and produce a few families of examples when sources and sinks are allowed in the networks.

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1 Introduction

The theory of functions that satisfy Laplace's equation $\triangle u = 0$ is of great importance in many fields of mathematics. Such functions are called *harmonic*. These functions represent electrical potentials, so they occur quite frequently in physics as well. It would be quite useful to understand maps that preserve Laplace's equation. Specifically, we are interested in maps f between two objects G and H such that if u is a harmonic function defined on some neighborhood $N \subset H$, then the composition $u \circ f$ is harmonic on $f^{-1}(N) \subset G$. Maps with this property are commonly referred to as *harmonic morphisms*.

1.1 A Quick Example from Complex Analysis

To help understand this concept, we prove that analytic maps are in fact harmonic morphisms:

Theorem: *Let $f : \Omega \to \Omega'$ be an analytic function, $u : \Omega' \to \mathbb{R}$ a harmonic function, then $u \circ f : \Omega \to \mathbb{R}$ is harmonic.*

Proof: Write

$$f = a + bi, \text{ so that } (u \circ f)(x, y) = u(a(x, y), b(x, y))$$

Let

$$z = x + iy \in \Omega, \text{ and } \zeta = \xi + i\eta \in \Omega'$$

By the chain rule we have

$$(u \circ f)_x = u_\xi a_x + u_\eta b_x,$$
$$(u \circ f)_{xx} = u_{\xi\xi} a_x^2 + u_{\xi\eta} a_x b_x + u_\xi a_{xx} + u_{\eta\xi} a_x b_x + u_{\eta\eta} b_x^2 + u_\eta b_{xx}$$

Writing a similar expansion for $(u \circ f)_{yy}$ and grouping certain terms we get that

$$\triangle(u \circ f) =$$
$$u_\xi(a_{xx} + a_{yy}) + u_\eta(b_{xx} + b_{yy}) + u_{\xi\xi}(a_x^2 + a_y^2) + u_{\eta\eta}(b_x^2 + b_y^2) + 2u_{\xi\eta}(a_x b_x + a_y b_y)$$

Both the real and imaginary parts of an analytic function are harmonic, thus

$$a_{xx} + a_{yy} = b_{xx} + b_{yy} = 0.$$

The Cauchy-Riemann equations state that

$$a_x = b_y, a_y = -b_x$$

So we are left with

$$\triangle(u \circ f) = u_{\xi\xi}(a_x^2 + a_y^2) + u_{\eta\eta}(b_x^2 + b_x^2) + 2u_{\xi\eta}(a_x b_x + a_y b_y) =$$
$$u_{\xi\xi}(a_x^2 + b_x^2) + u_{\eta\eta}(a_x^2 + b_x^2) + 2u_{\xi\eta}(a_x b_x - a_x b_x) = (u_{\xi\xi} + u_{\eta\eta})(a_x^2 + b_x^2) = 0$$

where the last equality holds since u was assumed to be harmonic. $\square$

Later in this paper we will use a few properties of analytic functions as motivations to develop analogous properties for harmonic morphisms.

1.2 Continuous Potentials and Riemannian Manifolds

In order to generalize the harmonic pull-back property of analytic functions on the complex plane, there has been much work to study harmonic morphisms between Riemannian manifolds. This work resulted in a nice characterization, listed on page 108-9 of [1]:

Theorem: *Let $M = (M, g)$ and $N = (N, h)$ be Riemannian manifolds, and let $\phi : M \to N$ be a smooth map between them. Then the following conditions are equivalent:*

- *ϕ is a harmonic morphism.*
- *ϕ is both harmonic and horizontally weakly conformal.*
- *For each smooth function $f : V \to \mathbb{R}$ defined on an open subset V of N with $\phi^{-1}(V)$ non-empty, we have $\triangle(f \circ \phi) = \Lambda\triangle(f)$ for some smooth function $\Lambda : M \to [0, \infty)$.*

To say that ϕ is *horizontally weakly conformal* means that for $x \in M$, either $\mathrm{d}\phi_x = 0$, or $\mathrm{d}\phi_x$ maps the horizontal space $H_x = \{ker(d\phi_x)\}^\perp$ conformally onto $T_{\phi(x)}N$, the tangent space of N at $\phi(x)$. It is not necessary to fully understand this theorem. We include it only to give a flavor of the results that are being obtained for the case of harmonic morphisms between manifolds.

1.3 Electrical Networks and the Discrete Laplacian

We now consider potentials defined on resistor networks $\Gamma = (G, \gamma)$. We adopt all the conventions of [2] when describing electrical networks. Namely,

$$G = (\mathrm{int}V \cup \partial V, E)$$

is a graph with a (possibly empty) set of vertices designated as boundary, so that the remaining vertices are referred to as interior, and γ is a positive conductivity function defined on all edges of G. The existence of an edge between vertices x and y is often denoted by $x \sim y$, and the set of all neighbors of a vertex z (that is, the set of all vertices adjacent to z) is denoted by $N(z)$.

We interpret the boundary vertices as possible sources and sinks and assume Kirchhoff's current rule at all interior vertices; that is, the sum of the current leaving each interior vertex is taken to be zero. It turns out that this condition is precisely the discrete analogue of the Laplacian. Thus, functions $u : V \to \mathbb{R}$ that satisfy

$$\sum_{q \in N(p)} \gamma_{pq}(u(p) - u(q)) = 0$$

at all interior vertices p are referred to as harmonic, or γ-harmonic, where the gamma is sometimes written to emphasize the dependence on the edge conductivities. There is an alternate way to write this expression, known as the *harmonic averaging principle*, which states that a function u is harmonic at a vertex p if and only if

$$u(p) = \frac{\sum\limits_{q \in N(p)} \gamma_{pq} u(q)}{\sum\limits_{q \in N(p)} \gamma_{pq}}$$

Thus we can determine if u is harmonic at p either by verifying that the net current out of p is zero, or by seeing if $u(p)$ is the weighted average of its neighboring vertices.

1.4 Harmonic Morphisms on a Graph

A *manifold* is a topological space that is locally Euclidean. A *Riemannian manifold* is a manifold endowed with a *metric*. This means that it is possible to calculate the distance between any two points on the manifold — typically by looking at the minimal length of geodesics joining these points. Another interpretation of the resistor networks described above is that of a *metric graph*, which is just a graph with positive edge weights. In other words, we simply interpret the conductivities of an electrical network to be the distances between vertices of the graph. This interpretation shows how a metric graph is actually a discretization of a Riemannian manifold.

With essentially the same definition for the discrete Laplacian mentioned earlier (i.e. Kirchhoff's current rule), the notion of harmonic morphism was extended to generic graphs in [3] and to metric graphs in [4]. In this context, the definition of harmonic morphism is exactly as one might imagine; namely, a map $\phi : V(G) \to V(H)$ between two electrical networks $\Gamma = (G, \gamma_G)$ and $\Gamma = (H, \gamma_H)$ is a harmonic morphism if it pulls back locally defined γ_H-harmonic functions on H to locally defined γ_G-harmonic functions on G. The same authors describe an analogue to the notion of horizontal conformality, reproduced here in the case of graphs with positive edge weights:

Definition: A map $\phi : (V_1, E_1, \gamma_1) \to (V_2, E_2, \gamma_2)$ is *horizontally conformal* if the following two conditions are satisfied:

1. For all $x, y \in V_1$ such that $x \sim y$, either $\phi(x) = \phi(y)$ or $\phi(x) \sim \phi(y)$.
2. There exists a function $\lambda : V_1 \to \mathbb{R}^+$ such that for all $p \in V_1$, $q' \in N(\phi(p))$,

$$\sum_{p' \in N(p), \phi(p') = q'} \gamma_1(pp') = \lambda(p)\gamma_2(\phi(p)q').$$

The authors then produce a result analogous to the characterization theorem for harmonic morphisms between Riemannian manifolds. Specifically,

Theorem: *A map $\phi : (V_1, E_1, \gamma_1) \to (V_2, E_2, \gamma_2)$ is a harmonic morphism if and only if it is horizontally conformal.*

Remark: The results from these papers are for networks in which harmonicity is required at *all* vertices of the graph. We are interested in studying the effect of introducing electrical sources and sinks by way of decreeing that the

Laplacian of u need not be zero at a set of vertices designated as the boundary.

To help clarify the distinction between locally and globally defined harmonic functions, it is useful to introduce some terminology:

Definition: Let $G = (V, E)$ be a graph, and let $f : W \to \mathbb{R}$ be a function defined on a subset $W \subset V$ of its vertices. Then the for each vertex $v \in W$ such that $N(v) \subset W$, we may define the *germ* of f at v to be the equivalence class of all functions equal to f at v and at all vertices in the neighborhood $N(v)$.

A globally defined harmonic function is necessarily harmonic on all neighborhoods contained in its domain of definition, thus harmonic morphisms pull back germs of harmonic functions to germs of harmonic functions.

1.5 Graph Homomorphisms

We would like harmonic morphisms on a graph with boundary to preserve some essential structure of the graph. Before we do this, it is necessary to define a graph homomorphism:

Definition: A *homomorphism* from a graph $G = (V, E)$ to another graph $G' = (V', E')$ is a function $\phi : V \to V'$ such that $a \sim b$ implies $\phi(a) \sim \phi(b)$.

Remark: This condition is not an "if and only if" relation; that is, we may have $\phi(a) \sim \phi(b)$ even when there is no edge joining a to b.

It is important to note that we do *not* allow loops (that is, an edge joining a vertex to itself). This imposes a significant restriction on the set of graphs that have a homomorphism to a given graph, since $a \sim b$ implies that $\phi(a) \neq \phi(b)$. It is easily seen that the existence of a homomorphism from a graph G to the complete graph on 2 nodes, denoted K_2, is equivalent to a proper 2-coloring of G. In fact, there exists a homomorphism sending G to K_n if and only if G is properly n-colorable [5]. Thus graph homomorphisms are in many ways a generalization of graph colorings.

We are now able to specify the two additional requirements of any harmonic morphism ϕ under consideration:

1. ϕ must be a graph homomorphism.
2. ϕ must map interior vertices to interior vertices and boundary vertices to boundary vertices.

The first condition is in some ways a discrete analogue to the condition that a map between manifolds is continuous. When we are in fact dealing with manifolds, it can be shown that a harmonic morphism is an open map — that is, it sends open sets to open sets. This is verified in [1]. We would like this

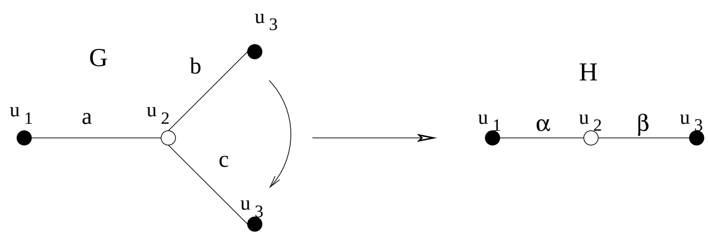

Figure 1: Merging two boundary spikes

same property to remain true on graphs. Later in this paper we verify that this second condition is sufficient to ensure that ϕ is an open map.

¿From the definition of horizontal conformality on graphs and the ensuing theorem, we see that the condition of a harmonic morphism being a graph homomorphism is automatic when there is no boundary — except that in the work by [3] and [4] a harmonic morphism is allowed to map two adjacent vertices to the same image vertex. As will be shown later, permitting this occurrence can cause violations of the openness of a harmonic morphism. Moreover, in the case of graphs with boundary it is easy to find examples of harmonic morphisms that are not graph homomorphisms (due to boundary-to-boundary edges). We therefore refer to harmonic morphisms that satisfy the two conditions listed above as *harmonic homomorphisms* throughout this paper to avoid any possible confusion.

2 Definition, Examples, and Basic Properties

Now that we have developed a heuristic understanding of harmonic homomorphisms, it is necessary to provide a precise, formal definition.

Definition: Let $\Gamma = (V, E, \gamma)$ and $\Gamma' = (V', E', \gamma')$ be two electrical networks, and let $\phi : G = (V, E) \to G' = (V', E')$ be a graph homomorphism such that $\phi(\partial V) \subset \partial V'$ and $\phi(\text{int}V) \subset \text{int}V'$. Then ϕ is a *harmonic homomorphism* if for every function $u : V' \to \mathbb{R}$ that is γ'-harmonic at some vertex $p' \in V'$, the composition $u \circ \phi : V \to \mathbb{R}$ is γ-harmonic at every vertex $p \in \phi^{-1}(p') \subset V$.

We now illustrate some basic examples of harmonic homomorphisms. In most of these cases it is clear that the maps in question are homomorphisms that preserve boundary and interior, so the main condition to check is the harmonic pull-back property.

2.1 Boundary Spikes and Cayley Trees

Let G and H be the graphs shown in **Figure 1**. We would like to choose appropriate conductivities on H so that harmonic functions on H pull back to harmonic functions on G. In other words,

Given: $\alpha(u_2 - u_1) + \beta(u_2 - u_3) = 0$,

Want: $a(u_2 - u_1) + b(u_2 - u_3) + c(u_2 - u_3) = 0$.

It is easily seen that if we choose $\alpha = a, \beta = b + c$ then the map ϕ will indeed be a harmonic homomorphism.

This example is readily generalized to produce a harmonic homomorphism from any n-star to any m-star, for $m < n$.

We can construct more sophisticated maps by stitching together graphs related to the previous two simple ones. Two examples of this construction are depicted in **Figure 2**. In both cases a cursory inspection reveals that the graphs involved are simply a collection of the 3-star graphs considered above, all identified at various boundary vertices. This allows us to map each 3-star to the same 2-star so that the map will be a harmonic homomorphism. The graph G_2 has the special property that it is a *tree* (meaning that there are no cycles in the graph) and all vertices, except the outer ones (referred to as *leaves*), have exactly three neighbors. Such a graph is commonly referred to as a *3-Cayley tree*. As with the example of boundary spikes and stars described above, it is possible to generalize this map somewhat:

Theorem: *Given any n-Cayley tree G such that the length of the (unique) path between any two leaves is even, it is possible to choose the boundary and interior according to a proper 2-coloring and choose conductivities on G so that there is a harmonic homomorphism $\phi : G \to H_n$, where H_n is an n-star with one interior vertex adjacent to n boundary vertices, regardless of the conductivities on H_n.*

Proof: With the common convention that black denotes boundary and white denotes interior, pick an arbitrary leaf (outer vertex) of G and color it black — that is, declare it a boundary vertex. This vertex has only one neighbor, which is necessarily white (i.e. an interior vertex). This white vertex has $n - 1$ neighbors that have yet to be considered, but they all must be colored black. For each of these $n - 1$ boundary vertices, either they are leaves or they have n neighbors that must necessarily be white. We can continue this coloring procedure to produce a proper 2-coloring on all of G, noting that all leaves will be black due to parity.

Number the vertices of H by $0, 1, \ldots, n$, where 0 corresponds to the interior vertex. Now let γ_i denote the conductivity on the edge joining vertex 0 to vertex i. With the coloring chosen above, each interior vertex of G has exactly n neighbors. For every such interior vertex, choose the conductivities on its n incident edges to be $\gamma_1, \ldots, \gamma_n$. Thus each interior vertex of G and its corresponding neighborhood is an isomorphic copy of H_n. The map ϕ is defined by sending each such n-star of G onto H_n. The only trick is that when we give

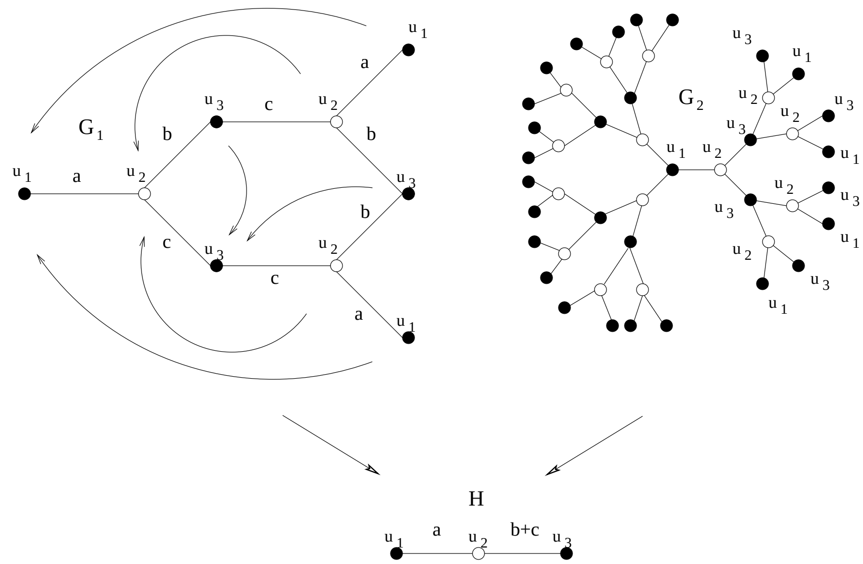

Figure 2: Harmonic homomorphisms on graphs constructed from 3-stars

conductivities to the edges incident to some interior vertex of G, we are implicitly determining where each of these boundary vertices will be mapped to. It is quite possible that some of these boundary vertices will neighbor other interior vertices, thus we may have already determined their image and produced an ambiguous definition for ϕ.

We can ensure that ϕ is well defined by starting with an arbitrary interior vertex v_i of G and mapping it along with its neighbors $b_1, \ldots, b_n$ to H_n in the obvious way. Now pick one of these boundary vertices b_j that is *not* a leaf (if all of them are leaves then the map is the identity). We know that b_j is adjacent to $n-1$ interior nodes aside from v_i. The conductivities of all edges incident to b_j must be equal, since we have already determined the image of b_j under ϕ. This means that for each of these n-stars that neighbor the one centered at v_i, exactly one edge conductivity will be chosen thus far. We are free to choose the remaining $n-1$ in any way so long as there is exactly one edge corresponding to each of the n edges of H_n. Because G is a tree, we know that this way of defining ϕ can extend radially outward without leading to any contradictions.

With the map ϕ constructed thus, it is trivial to verify that it is a harmonic homomorphism since it essentially maps multiple copies of H_n onto itself, hence every germ of a harmonic function on H_n pulls back to the identical germ of a harmonic function at each interior vertex of G. $\square$

Corollary: *For the preceding theorem, the map can take the n-Cayley tree to any m-star H_m, for $1 \leq m \leq n$.*

Proof: This is essentially a simultaneous application of the above theorem with the boundary spike merging example. □

All the examples of harmonic homomorphisms in this section are from some large graph G to an m-star with boundary at the outer vertices and interior at the central vertex. We choose the boundary at the end points so that we can imagine applying a voltage at these vertices and then measure the current that runs through the graph. However, because harmonic morphisms are dependent only on germs of harmonic functions, we can generalize these maps by allowing an arbitrary choice of boundary, as the next theorem illustrates:

Theorem: *Suppose there is a harmonic homomorphism $\phi : G \to H_n$ from a graph G to an n-star H_n for some choice of boundary on both graphs such that the center vertex of H_n is interior. Then we can redefine the boundary on H_n to be any of the 2^{n+1} possibilities and redefine the corresponding pull-backs on G without sacrificing the harmonicity of ϕ.*

Proof: It is sufficient to show that ϕ is still a harmonic homomorphism when we choose all vertices to be interior, because this will imply that any germ of a harmonic function on H_n pulls back to the germ of a harmonic function at every vertex of G — thus if we designate any collection of boundary vertices on H_n and the corresponding boundary on G, then the set of pulled back germs of functions that we must check for harmonicity will be a subset of the set of pulled back germs of functions that are harmonic given by the case that all vertices are interior.

Begin by numbering the vertices of H_n by $v_0, v_1, \ldots, v_n$, where v_0 is the center vertex, and v_1 through v_n are the neighbors of v_0. We are given that all germs of harmonic functions at v_0 pull back to germs of harmonic functions on G, so all we need to show is that the germ of a harmonic functions at v_i, for $1 \leq i \leq n$, pulls back to the germ of a harmonic function in G. Since v_i has only one neighbor, namely v_0, we know that the germ of any harmonic function at v_i is necessarily constant. Let $p_i \in \phi^{-1}(v_i)$. Then because ϕ is a graph homomorphism we also know that all neighbors of p_i get mapped to the neighborhood of v_i, which is simply v_0. Thus if we let u be the germ of a harmonic function at v_i, so that $u(v_i) = u(v_0) = c$, then $(u \circ \phi)(p_i) = (u \circ \phi)(q) = c$ for all $q \in N(p_i)$. Therefore u pulls back to the germ of a constant, and hence harmonic, function on G. □

When we have a map from a graph to an n-star, as in the map on the n-Cayley tree, we can choose all vertices to be interior. However, as the next section demonstrates, it is much more difficult in general to construct harmonic homomorphisms between networks with no boundary.

2.2 Identification of Interior Vertices

We now consider a map that sends one interior vertex to another and leaves the rest of the graph fixed. Let G and H be the graphs in **Figure 3**. For a

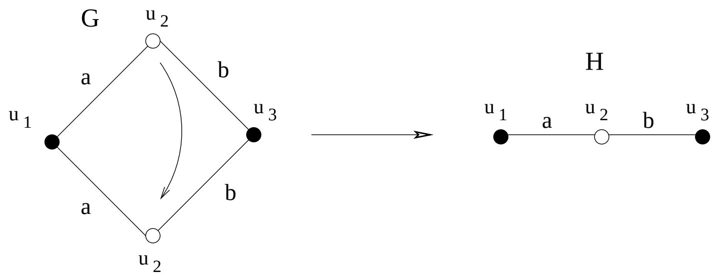

Figure 3: An identification of two interior vertices

pulled-back function to be harmonic, we see that the only condition needed is

$$a(u_2 - u_1) + b(u_2 - u_3) = 0$$

due to the symmetry of the conductivities chosen. However, this condition is precisely what is given for all harmonic functions on H, thus the map is a harmonic homomorphism.

As with the case of merging boundary spikes, we can apply this technique multiple times to produce more interesting harmonic homomorphisms. However, when dealing with graphs that have little or no boundary, it seems more difficult to construct harmonic homomorphisms since all the conditions required to ensure that germs of harmonic functions pull back to germs of harmonic functions become quite tangled and convoluted. In **Figure 4** we see two such maps. A careful inspection of each vertex in these graphs shows that the maps in question are indeed harmonic homomorphisms. For the map from G_2 to H_2 we show explicitly how all the conditions work out:

- The germ of a harmonic function at vertex 1 is necessarily constant (i.e. $u_1 = u_2$). For each of the two vertices in G_2 that get pulled back from this vertex, the neighboring vertices are all pulled back from vertex 2. Thus the pull-back is also constant and hence harmonic.

- For vertex 2 we are given that

$$a(u_2 - u_1) + b(u_2 - u_3) = 0$$

  At every vertex pulled back from this vertex, the condition to check is

$$a(u_2 - u_1) + \tfrac{b}{2}(u_2 - u_3) + \tfrac{b}{2}(u_2 - u_3) = 0$$

  But this is clearly identical to the given condition.

- The harmonicity at vertex 3 and its pull-backs are similar to the preceding case. Specifically, we are given

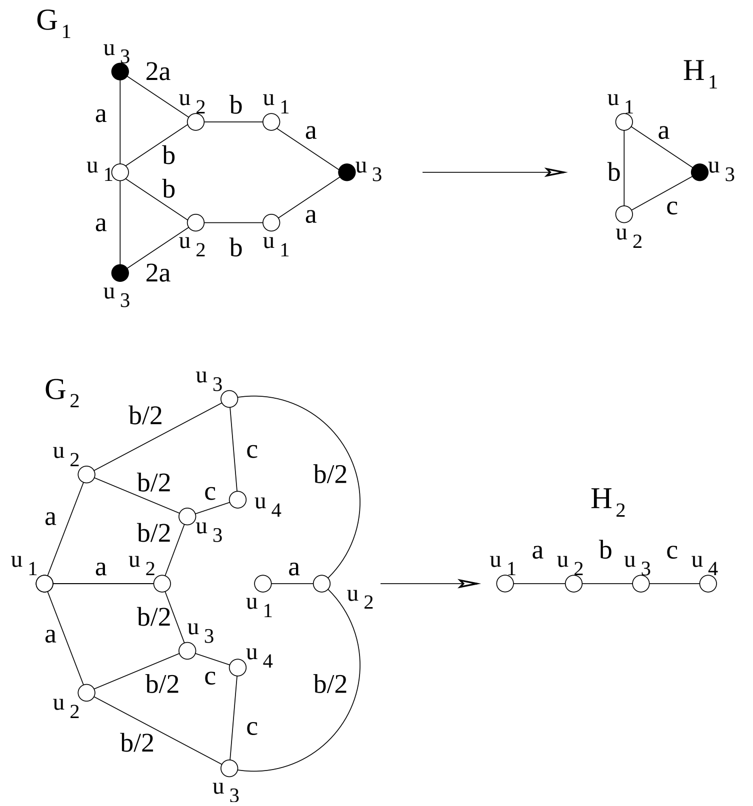

Figure 4: Two harmonic homomorphisms on graphs that are mostly interior

$$b(u_3 - u_2) + c(u_3 - u_4) = 0$$

and need to satisfy

$$\tfrac{b}{2}(u_3 - u_2) + \tfrac{b}{2}(u_3 - u_2) + c(u_3 - u_4) = 0$$

So again germs of harmonic functions pull back to germs of harmonic functions.

- Finally, as with the first case, every germ of a harmonic function at vertex 4 and its pull-back are constant and therefore harmonic.

It would be nice to have a deeper understanding of why these maps are harmonic homomorphisms, aside from a direct verification as above. Perhaps the key lies in horizontal conformality, but there do not appear to be any intuitive explanations of the definition offered by [3] and [4] in print anywhere to help elucidate this matter.

2.3 A Map Between Complete Bipartite Graphs

By combining the techniques of the above two simple examples, we are able to prove the following theorem:

Theorem: *Suppose the n boundary vertices and m interior vertices of a graph G form the complete bipartite graph $K_{n,m}$. Then the conductivities can be chosen so that there is a harmonic homomorphism ϕ sending G to $H = K_{p,q}$, for any $1 \leq p \leq n, 1 \leq q \leq m$.*

Proof: We start with the case that no boundary vertices are identified under the map ϕ. That is,

$$|\partial V(G)| = |\partial V(H)|$$

Number the boundary vertices of both G and H by $b_1, b_2, \ldots, b_n$. Let all edges incident to b_i have conductivity γ_i in both graphs. Then for every germ of a harmonic function u on H we are given that

$$\sum_{i=1}^{n} \gamma_i(u_j - u_i) = 0$$

for each interior vertex j of H. However, this is exactly the condition that is needed for each interior vertex of G to ensure that $u \circ \phi$ will be harmonic.

Now consider the case that no interior vertices are identified:

$$|\mathrm{int}V(G)| = |\mathrm{int}V(H)|$$

We will show how to solve this case when the n^{th} boundary vertex gets mapped to the $(n-1)^{th}$ boundary vertex. The rest of the case will follow by an iterative application of this result. Again number the boundary vertices of G by $b_1, b_2, \ldots, b_n$. Define the conductivity of all edges in G incident to b_i by γ_i.

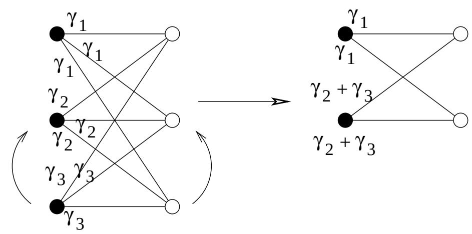

Figure 5: A harmonic homomorphism from $K_{3,3}$ to $K_{2,2}$

Then all we need to do is choose the conductivity for all edges incident to the first $n-2$ boundary vertices of H to be exactly as they were in G, and then for the last boundary vertex of H we choose the conductivity of all incident edges to be $\gamma_{n-1} + \gamma_n$.

It is fairly straightforward to stitch these two cases together to complete the proof of this theorem. The particular case in which $n = m = 3, p = q = 2$ is illustrated in **Figure 5**. $\square$

This theorem can be generalized somewhat by weakening the bipartite structure of both graphs in the following ways:

1. Creating boundary-to-boundary edges has no effect on the harmonicity of any functions, so we may permit an arbitrary collection of such edges.

2. If a graph consists of several complete bipartite graphs as its components such that each component is connected to the other components only by boundary-to-boundary edges (or they are not connected at all), then the techniques developed above can be applied piecewise.

2.4 Harmonic Homomorphisms are Open Maps

It was assumed as part of the definition of harmonic homomorphism that interior vertices get mapped to interior vertices. To verify that harmonic homomorphisms are open maps, the only other condition that we must check is that neighborhoods in the domain graph get sent surjectively to neighborhoods in the codomain (i.e. open sets get mapped onto open sets). More specifically,

Theorem: *Let $\Gamma = (G, \gamma)$ and $\Gamma' = (G', \gamma')$ be electrical networks such that $\phi : G \to G'$ is a harmonic homomorphism. Let $p \in \mathrm{int}V(G)$ and let $p' = \phi(p)$. Then for all $q' \in N(p')$, there exists $q \in N(p)$ such that $\phi(q) = q'$.*

Proof: Assume, by contradiction, that there is some q' in the neighborhood of p' that is not the image of any vertices in the neighborhood of p. Let u be the germ of a harmonic function defined on p' and all vertices adjacent to it. Then since ϕ is a harmonic homomorphism, we know that ϕ pulls back u to the germ of a harmonic function whose domain of definition contains p and all vertices adjacent to it. By letting $u(q') \to \infty$, we know from the averaging principle that $u(p') \to \infty$. However, since we assumed that q' is not the image of any neighbor of p, and by the definition of homomorphism we know that no neighbors of p get sent to p', we see that $u(p) \to \infty$ whereas u evaluated at all neighbors of p remains finite. This is indeed a contradiction of the averaging principle, thus the theorem is proven. $\square$

Remark: If we were to allow adjacent vertices to have the same image, as in the definition of horizontal conformality, it would be possible to find counter-examples to this result. For example, imagine two copies of the graph K_2, say $G = (g_1, g_2; g_1g_2)$ and $H = (h_1, h_2; h_1h_2)$, each consisting of two adjacent interior vertices. We can send both g_1 and g_2 to h_1, so that G does *not* map surjectively onto the neighborhood of h_1. However, all harmonic functions on H are constant and pull back to constant, and hence harmonic, functions on G.

The proof of openness in the case of Riemannian manifolds follows a similar construction, although it is more complicated as the contradiction derived is in terms of a Green's function on the image neighborhood and a violation of Harnack's inequality. See [1] for the details.

2.5 Composition of Harmonic Homomorphisms

An easy result [1] in the case of Riemannian Manifolds is that the composition of two harmonic morphisms is itself a harmonic morphism. We quickly derive the analogous result for harmonic homomorphisms on graphs:

Theorem: *Let $\phi : A \to B$ and $\psi : B \to C$ be harmonic homomorphisms on electrical networks $\Gamma_A = (A, \gamma_A), \Gamma_B(B, \gamma_B)$, and $\Gamma_C = (C, \gamma_C)$. Then the composition $\psi \circ \phi : A \to C$ is a harmonic homomorphism.*

Proof: There are three properties to check to verify that the map $\psi \circ \phi$ is a harmonic homomorphism:

1. We know that ϕ maps interior vertices of A to interior vertices of B, and then ψ takes these interior vertices of B to interior vertices of C, thus $\psi \circ \phi$ maps interior vertices to interior vertices. The same argument applies to boundary vertices.

2. If $x \sim y$ in A, then ϕ preserves this adjacency so that $\phi(x) \sim \phi(y)$ in B and $\phi(x) \neq \phi(y)$. Since ψ is also a homomorphism, we know that $\psi(\phi(x)) \sim \psi(\phi(y))$ and $\psi(\phi(x)) \neq \psi(\phi(y))$, thus $\psi \circ \phi$ is a homomorphism as well.

3. Finally, the germ of a harmonic function at an interior vertex p of C gets pulled back by ψ to the germ of a harmonic function at each of the interior vertices in the inverse image $\psi^{-1}(p)$. Then ϕ pulls back each of these germs to the germs of harmonic functions on A. Thus $\psi \circ \phi$ pulls back the germs of harmonic functions on C to the germs of harmonic functions on A.

Since these three properties are verified, we see that $\psi \circ \phi$ is indeed a harmonic homomorphism. $\square$

In order to construct more general harmonic homomorphisms, it would be nice to generalize this composition result in the following manner:

Conjecture: Suppose we are given three graphs A, B, and C with the following property: we can choose the conductivities on A and B so that there is a harmonic homomorphism from A to B, or we can *independently* choose the conductivities on B and C so that there is a harmonic homomorphism from B to C. Then it is possible to choose the conductivities on A and C so that there is a harmonic homomorphism from A to C.

We refer to this statement as a conjecture since a proof has thus far eluded this author. The plausibility of such a result is justified as follows. In all the examples of harmonic homomorphisms found in this paper, the conductivities of the target graph were arbitrary and determined the conductivities of the domain graph. If this occurrence holds in general — that is, whenever there is a harmonic homomorphism from G to H, we can choose arbitrary conductivities for H and have those values determine conductivities on G that will preserve the harmonicity of the map — then we can have arbitrary conductivities on C, take the appropriate conductivities on B that are determined from ψ, and then have these conductivities on B determine conductivities A so that ϕ is a harmonic homomorphism. Then it is simply a matter of applying the previous result.

3 Analogues From the Continuous Case

We now return to the continuous case of complex analysis and Riemannian manifolds to produce some further examples of harmonic homomorphisms.

3.1 Isometries and Automorphisms

An *isometry* is a bijective map that preserves distance. It is mentioned in [1] that an isometry of a Riemannian manifold is a harmonic morphism. We would like to find a discrete analogue to this fact.

Consider the case that a harmonic homomorphism is actually a graph automorphism. It is clear that the identity map is a harmonic homomorphism, since all given conditions exactly match all necessary conditions. However, the only other maps in the automorphism group of a graph that are harmonic are ones that have a symmetry in the conductivities exactly corresponding to the symmetry in the graph that produced the automorphism. For example, the quadrilateral graph in **Figure 6** has an automorphism group that is isomorphic to the dihedral group of order 8. If $\alpha = \gamma$, then there is a harmonic homomorphism sending vertex 1 to 4, 4 to 1, 2 to 3, and 3 to 2. If $\beta = \delta$ as well, then

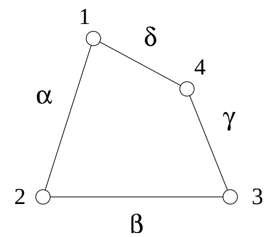

Figure 6: A graph with automorphism group isomorphic to D_4

there is also a harmonic homomorphism corresponding to the map defined by $1 \leftrightarrow 2$ and $4 \leftrightarrow 3$. Finally, if $\alpha = \beta = \gamma = \delta$, then there will be eight possible harmonic homomorphisms, one for each of the four rotations and four reflections in the automorphism group of this graph.

If we return to the interpretation of an electrical network as a metric graph, we see that these harmonic automorphisms are precisely the bijective graph maps that preserve distance between vertices — and thus graph isometries.

3.2 The Reflection Principle

We have already shown that analytic functions are harmonic morphisms. We now show that another result from complex analysis, known as *Schwarz reflection*, carries over to harmonic homomorphisms.

Under appropriate circumstances, an analytic function on a subset of the upper half plane, say U^+, may be uniquely extended, via complex conjugation, to an analytic function on $U^+ \cup I \cup U^-$, where U^- is the reflection of U^+ across the real axis and I is the interval that joins U^+ and U^-. More generally, this reflection may occur over any analytic curve, not just the real axis. We can derive a similar result in terms of electrical networks and harmonic homomorphisms.

Let us begin by investigating the meaning of a *reflection* in the case of a graph. First assume that the axis of reflection is a straight line that possibly intersects G, but does not penetrate the convex hull of $V(G)$. There are a few cases to consider:

- If the axis of reflection does not intersect any edges or vertices of G, then reflection produces an isomorphic copy of G, denoted $\overline{G}$, sitting in the same plane as G but sharing no edges or vertices with G.
- If the axis intersects a nonempty set of vertices $\{v_1, \ldots, v_n\} \in V(G)$, but no edges of G, then we can imagine producing an isomorphic copy of G off in the distance that is geometrically inverted about this axis and then sliding it closer to G until the corresponding vertices $\{\overline{v_1}, \ldots, \overline{v_n}\} \in V(\overline{G})$ are overlayed with their conjugate counterparts.
- If the preceding case occurs except that the axis of reflection covers at least one edge (and therefore its endpoints), then we repeat the above

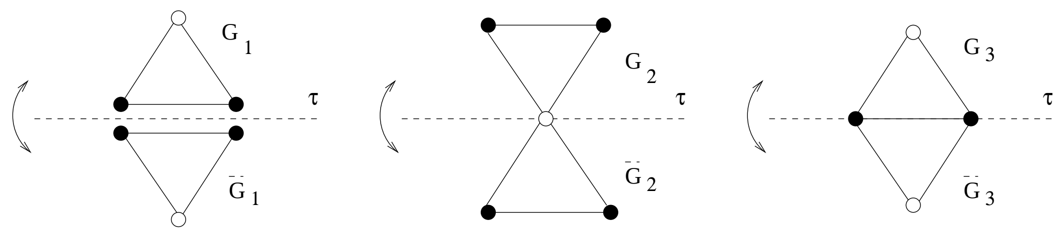

Figure 7: Various ways in which to reflect a graph

process except that we only keep one copy of the edge in question, rather than doubling the edge with its conjugate.

One can see the effect of these reflections in **Figure 7**.

By avoiding the use of a geometric object (line, curve, etc.), we can produce a useful generalization of this reflection process:

Definition: Let $G = (V, E)$ be a graph embedded in the plane (not necessarily planar). Let $V_R \subset V$ be a proper subset of vertices of G, and let $E_R \subset E$ be the set of all edges in G that have both endpoints in V_R (this is typically referred to as an *induced subgraph*). Produce an isomorphic copy of G, denoted by $\overline{G} = (\overline{V}, \overline{E})$. The *reflection of G induced by V_R* is the graph $G \cup \overline{G}$ obtained by identifying each $v \in V_R$ with its conjugate $\overline{v} \in \overline{V}$, and similarly for edges. We refer to (V_R, E_R) as the *reflecting set* for $G \cup \overline{G}$. The conductivity of each edge in $G \cup \overline{G}$ is defined to be the value of the edge it came from if it is not in the reflecting set, and twice this value if it is in this set (since two edges were merged to one in the identification process).

This definition reduces to the previous (more intuitive) case when the reflecting set is chosen to be all edges and vertices that lie on a line drawn through the plane in which G is embedded that does not penetrate the convex hull of the vertices of G. This general definition of reflection is more versatile, but it also leads to some unnatural looking operations. In **Figure 8** we see how the same graph may be reflected geometrically given one embedding, but the more abstract definition of reflection is needed if we are given another embedding.

These concepts allow us to formulate the discrete analogue of Schwarz reflection for harmonic homomorphisms:

Theorem: *Given graphs G and H, a harmonic homomorphisms ϕ from G to H, and a non-empty reflecting set (V_R, E_R) for G, we may extend ϕ to a harmonic homomorphism ϕ' from $G \cup \overline{G}$ to H.*

Proof: As you might expect, we define ϕ' to be equal to ϕ on all vertices of G, and $\phi'(\overline{v}) = \phi(v)$ for all vertices in $\overline{G}$ that are not in G. Let $p \in \text{int}V(G \cup \overline{G})$. Consider the following cases:

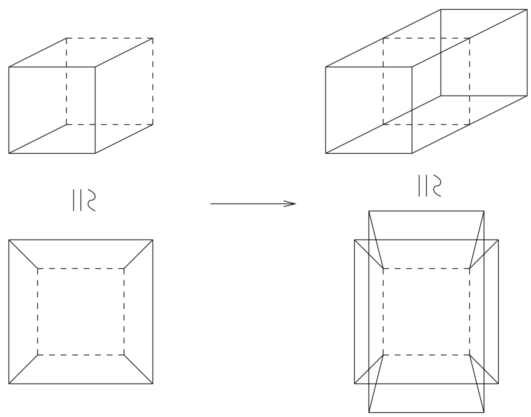

Figure 8: A reflection of the cube with two different embeddings

- If $p \in V(G), p \notin V_R$, then no edge incident to p is in E_R, nor is there any edge joining p to any vertices of $\overline{G}$. Thus the process of reflecting G to produce $G \cup \overline{G}$ cannot destroy the harmonicity of any pulled back function on G given by the original map ϕ.

- If $p \in V(\overline{G}), p \notin V_R$, then no edge incident to p is in the conjugate of E_R, nor is there an edge joining p to G. By the definition of ϕ', the values of all pulled-back vertex functions at neighbors of p are equal to the corresponding values in the conjugate neighborhood, thus this case reduces exactly to the case described above.

- Now suppose p is in V_R, but that p has no incident edges in E_R. By the definition of ϕ, we know that for all harmonic functions u on H,

  $$\sum_{q \in V(G): pq \in E(G)} \gamma_{pq}(u(\phi(p)) - u(\phi(q))) = 0$$

  By the symmetry of reflection, we also know that the same statement is true if we replace G by $\overline{G}$. Thus,

  $$\sum_{q \in V(G \cup \overline{G}): pq \in E(G \cup \overline{G})} \gamma_{pq}(u(\phi(p)) - u(\phi(q))) = 0$$

- Finally, consider the case in which there are one or more edges incident to p that lie in E_R, say $\{e_1, \ldots, e_n\}$. By the definition of reflecting set, we know that $p \in V_R$ as well. Let $\gamma_1, \ldots, \gamma_n$ denote the conductivities of these edges. These conductivities are by definition twice the value they were in the original graph G. For each edge e_i in this set, consider half of its conductivity coming from an edge in G, and the other half coming

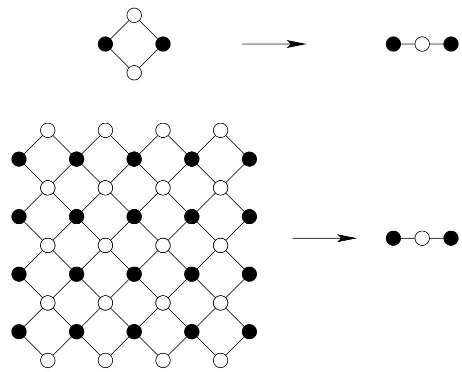

Figure 9: Repeated applications of the reflection principle

from an edge in $\overline{G}$. When we write the sum of the currents flowing out of p in the graph $G \cup \overline{G}$, we simply have the sum of the currents out of p in G plus the sum of the currents out of p in $\overline{G}$, which is zero since ϕ' restricts to a harmonic homomorphism on both G and $\overline{G}$.

Since the reflecting set was chosen to be an induced subgraph, we will never have an edge joining G to $\overline{G}$ that is not already in either of the two graphs alone, thus we are sure that adjacent vertices never have the same image. Also, adjacency in each graph separately is preserved, thus ϕ is indeed a homomorphism. $\square$

We can now construct a much larger class of harmonic homomorphisms by repeated applications of the reflection principle, as is demonstrated in **Figure 9**. By taking the simple example of the harmonic homomorphism that identifies two interior vertices described earlier, we can extend this map to a larger graph by reflecting twice in both the vertical and horizontal directions.

3.3 Graph Products, Projections, are Related Maps

There are two examples of harmonic morphisms on Riemannian manifolds that we would like to discretize, both found on page 107 of [1]:

1. Let M and N be Riemannian manifolds, and denote their product manifold by $M \times N$. Then the natural projections $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are both harmonic morphisms.

2. Let M, N, and P be smooth Riemannian manifolds, and let $\phi : M \times N \to P$ be a smooth map. If each of the partial maps $\phi_y : M \to P$ and $\phi_x : N \to P$ defined by $\phi_y(x) = \phi_x(y) = \phi(x, y)$ are harmonic morphisms for all $(x, y) \in M \times N$, then ϕ itself is a harmonic morphism.

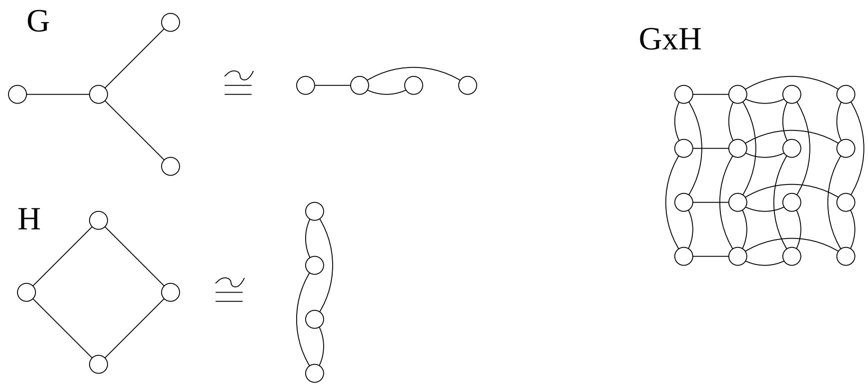

Figure 10: The cartesian product of two graphs

We must begin by discussing the concept of a *graph product*. Given two graphs G and H, the product graph $G \times H$ has a vertex for every pair of vertices $(g, h) \in V(G) \times V(H)$. The difficulty is in deciding how the edges in the component graphs determine the edges in the product graph. We use the following definition of graph product:

Definition: Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, the *cartesian product* $G \times H$ is the graph with vertex set $V_G \times V_H$. If $g_1, g_2 \in V_G$ and $h_1, h_2 \in V_H$, then in $G \times H$ we have $(g_1, h_1) \sim (g_2, h_2)$ if and only if one of these two possibilities occurs:

$$g_1 \sim g_2 \text{ and } h_1 = h_2 \textit{ or } g_1 = g_2 \text{ and } h_1 \sim h_2$$

Remark: Traditionally the cartesian product of two graphs G and H is denoted by $G \square H$, but we use $G \times H$ to reinforce the analogy between product graphs and product manifolds.

When discussing product graphs it is useful to have the following notation:

Definition: Given a vertex $p = (g, h)$ in a product graph $G \times H$, we refer to the set of all vertices (g, h') such that $h \sim h'$ as the *vertical neighborhood* of p, denoted $N_V(p)$, and the set of all vertices (g', h) such that $g \sim g'$ as the *horizontal neighborhood* of p, denoted $N_H(p)$. If a vertex q is in the vertical neighborhood of p, we say that p and q are *vertical neighbors*, and we similarly define *horizontal neighbors*.

In order to draw the product of two graphs, it is often helpful to align the vertices of the first component graph in a horizontal line and those of the second component graph in a vertical line. This allows us to place the vertices of the product graph in a grid, as is illustrated in **Figure 10**.

There are a couple problems when we try to define the projection maps from a product graph to either of its component graphs. First, it is not clear how to designate the boundary and interior in the product graph. Even if we were to pick some convention, say $(g, h) \in \partial(V_G \times V_H)$ if and only if $g \in \partial V_G$ or $h \in \partial V_H$, in most cases we would not be able to project the product graph onto its component graphs because of the restriction that interior maps to interior and boundary maps to boundary. A further difficulty is that the definition of homomorphism we have adopted does not allow adjacent vertices to get mapped to the same image vertex.

Note: Because of these two complications, we restrict our discussion for the remainder of this section to harmonic morphisms defined as in [3] and [4] — that is, we no longer consider graphs with boundary, and the graph homomorphism condition is replaced by the following:

$$\text{If } x \sim y \text{ then } \phi(x) \sim \phi(y) \textit{ or } \phi(x) = \phi(y)$$

which we refer to as the *weak homomorphism condition.*

We still need to decide how the conductivities on the edges of a product graph are determined. Since we chose the cartesian product for our definition of graph product, this turns out to be quite simple:

- If $g_1 \sim g_2$ and $h_1 = h_2$, then let $\gamma[(g_1, h_1)(g_2, h_2)] = \gamma(g_1 g_2)$
- If $g_1 = g_2$ and $h_1 \sim h_2$, then let $\gamma[(g_1, h_1)(g_2, h_2)] = \gamma(h_1 h_2)$

We can now state the discretization of the fact that projections from product manifolds are harmonic morphisms:

Theorem: *Let $\Gamma_G = (G, \gamma_G)$ and $\Gamma_H = (H, \gamma_H)$ be two electrical networks, and denote their product network by $\Gamma_{G\times H} = (G \times H, \gamma_{G\times H})$. Then the maps $\pi_1 : G \times H \to G$ and $\pi_2 : G \times H \to H$ defined by $\pi_1(g, h) = g$ and $\pi_2(g, h) = h$, respectively, are both harmonic morphisms.*

Proof: Clearly it suffices to show that one of the projections, say π_1, is a harmonic morphism, since the cartesian product is symmetric. Pick a vertex $p' \in V(H)$ and let u be an arbitrary germ of a harmonic function at p'. We will show that $u \circ \pi_1$ is the germ of a harmonic function at an arbitrary vertex p in the inverse image $\pi_1^{-1}(p')$. Since all vertical neighbors of p are also in $\pi_1^{-1}(p')$, we know that for all $q \in N_V(p)$,

$$(u \circ \pi_1)(q) = (u \circ \pi_1)(p) = u(p')$$

thus

$$\sum_{q\in N_V(p)} \gamma_{G\times H}(pq)(u(\pi_1(p)) - u(\pi_1(q))) = \sum_{q\in N_V(p)} \gamma_{G\times H}(pq)(u(p') - u(p')) = 0$$

Now for the horizontal neighbors of p we have the that

$$\sum_{q\in N_H(p)} \gamma_{G\times H}(pq)(u(\pi_1(p)) - u(\pi_1(q))) = \sum_{q'\in N(p')} \gamma_G(p'q')(u(p') - u(q')) = 0$$

since $u \circ \pi_1$ essentially restricts to u on horizontal slices of $\Gamma_{G\times H}$. We also know

$$\sum_{q\in N(p)} = \sum_{q\in N_V(p)} + \sum_{q\in N_H(p)}$$

so that

$$\sum_{q\in N(p)} \gamma_{G\times H}(pq)(u(\pi_1(p)) - u(\pi_1(q))) = 0$$

and thus u pulls back to the germ of a harmonic function at p. $\square$

Let us now investigate the discretization of the second class of harmonic morphisms on Riemannian manifolds listed at the beginning of this section. We first need to define partial maps in the case of graphs:

Definition: Let $G \times H$ be a product graph and let $\phi : G \times H \to L$ be a map from this product to a third graph L. For every vertex $h \in V(H)$, there is an isomorphic copy of G sitting in $G \times H$ which is the subgraph induced by the set of vertices contained in a horizontal slice of the product graph:

$$G \cong \mathrm{Ind}\{(g,h) \in V(G\times H) \mid g \in V(G)\}$$

Given a fixed $h \in V(H)$, the *partial map* ϕ_h is the map from this induced subgraph to L given by $\phi_h(g) = \phi(g,h)$. The partial map $\phi_g : H \to L$ is defined similarly.

We can now state and prove the result that a map on a product graph will be a harmonic morphism whenever both partial maps are harmonic morphisms for every vertex in the graph:

Theorem: *Let $\Gamma_G = (G, \gamma_G), \Gamma_H = (H, \gamma_H)$, and $\Gamma_L = (L, \gamma_L)$ be electrical networks, and let $\phi : G \times H \to L$ be a graph map. If the partial maps ϕ_g and ϕ_h are harmonic morphisms for all $g \in V(G)$ and $h \in V(H)$, then ϕ itself is a harmonic morphism.*

Proof: Denote the product network by $\Gamma_{G\times H} = (G \times H, \gamma_{G\times H})$. Choose $p' \in V(L)$ and $p = (g,h) \in \phi^{-1}(p') \subset V(G\times H)$ to be arbitrary. As in the proof of the previous theorem, we use the fact that the sum of the current out of p is equal to the sum of the current out in the horizontal direction plus the sum of the current out in the vertical direction. Let u be the germ of a harmonic function at p', so that

$$\sum_{q'\in N(p')} \gamma_L(p'q')(u(p') - u(q')) = 0$$

Since ϕ_h is a harmonic morphism, we know that

$$\sum_{q\in N_H(p)} \gamma_{G\times H}(pq)(u(\phi(p)) - u(\phi(q))) = 0$$

and similarly

$$\sum_{q \in N_V(p)} \gamma_{G\times H}(pq)(u(\phi(p)) - u(\phi(q))) = 0$$

because ϕ_g is a harmonic morphism. As mentioned above, we also know that

$$\sum_{q \in N(p)} = \sum_{q \in N_H(p)} + \sum_{q \in N_V(p)}$$

Thus

$$\sum_{q \in N(p)} \gamma_{G\times H}(pq)(u(\phi(p)) - u(\phi(q))) = 0$$

Therefore $u \circ \phi$ is the germ of a harmonic function a p. $\square$

The following corollary is an application of this theorem:

Corollary: *Let Γ_G, Γ_H, and Γ_L be electrical networks. If there are harmonic morphisms $\phi : G \to L$ and $\psi : H \to L$, then there is a harmonic morphism $\omega : G \times H \to L$.*

Proof: It is easiest to describe ω as the composition of three simple maps. First, let $\alpha : G \times H \to G \times L$ be the map obtained by applying ψ to each of the vertical copies of H. Then let $\alpha' : G \times L \to L \times L$ be the map obtained by applying ϕ to each horizontal copy of G. Finally, let $\alpha'' : L \times L \to L$ be the projection map onto a horizontal copy of L.

To see that all the partial maps of $\omega = \alpha'' \circ \alpha' \circ \alpha$ are harmonic morphisms, we first note that the partial maps ω_g are harmonic morphisms for all $g \in V(G)$ because the projection map α'' sends all vertices in a vertical slice down to one vertex, thus all harmonic functions on L trivially pull back to constant functions on the vertical slices of $G \times H$. Then for the partial maps ω_h, we see that there is no dependence on h, since all horizontal slices of $G \times H$ map onto L in the same manner. Moreover, for each $h \in V(H)$ the map $\omega_h : G \to L$ is identical to the map $\phi : G \to L$, thus all partial maps are indeed harmonic morphism. $\square$

All the maps involved in the above proof are pictured in **Figure 11**.

4 Miscellaneous Remarks and Future Research

The aim of this paper is to explore the topic of harmonic morphisms on graphs in an example-oriented fashion. Now that the groundwork has been laid, it would be nice to prove some more general results about such maps and develop some related material. We list below a few of the possible topics of future research.

4.1 A Geometric Representation of Harmonic Functions

Suppose we are interested in studying a harmonic homomorphism $\phi : G \to H$. Given an interior vertex $p \in V(H)$, let u be an arbitrary germ of a harmonic

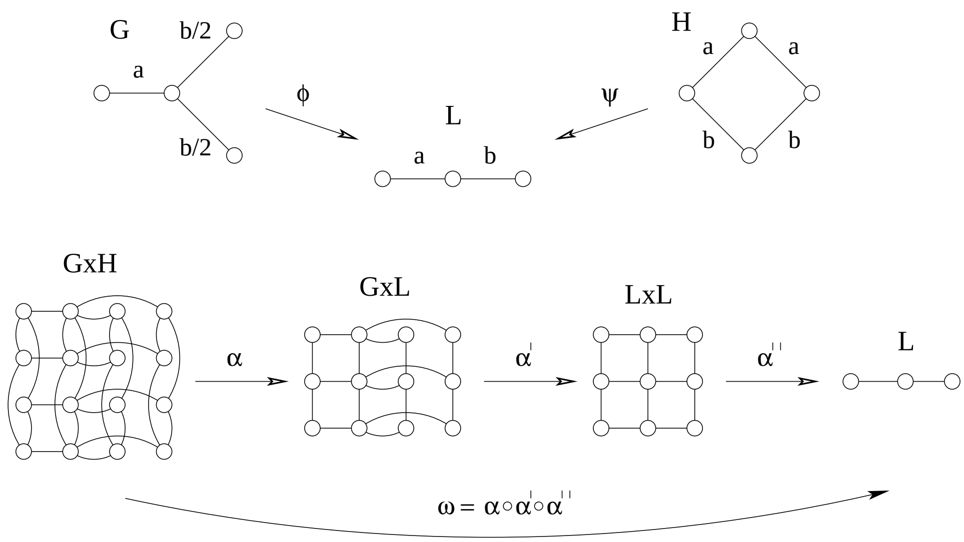

Figure 11: A harmonic homomorphism ω from $G \times H$ to L

function at p. Since harmonicity is entirely determined by the averaging principle, we can assume without loss of generality that $u(p) = 0$. A physical argument for this is that absolute potentials never matter, it is only their differences that are important. With this simplification, Kirchhoff's current rule becomes

$$\sum_{q \in N(p)} \gamma_{pq}(-u(q)) = 0$$

where the negative sign is clearly irrelevent.

Now suppose the *valency* of p is n (that is, p has n neighbors). We can number these n neighbors, say $q_1, \ldots, q_n$, in some consistent fashion: for example, start due north of p and travel in a clockwise direction. Then we can write the conductivities on all edges incident to p as a vector with n entries, say $\Gamma = (\gamma_{pq_1}, \gamma_{pq_2}, \ldots, \gamma_{pq_n})$. Furthermore, we can write the value of u at these n vertices as another vector $U = (u(q_1), u(q_2), \ldots, u(q_n))$. With this notation, we see that the condition of u satisfying Kirchhoff's rule can be written as $\Gamma \cdot U = 0$. In other words,

Let u be the germ of a function. Then u is harmonic if and only if the vectors U and Γ described above are orthogonal in the standard Euclidean sense.

Because ϕ is a harmonic homomorphism, we know that any vertex p' in the inverse image of p has valency of at least n (by the openness of harmonic homomorphisms) and that each neighbor of p' gets mapped to a vertex in the neighborhood of p (since ϕ is a graph homomorphism). Thus the pulled back values of u at $N(p')$ will all be entries in U. These values induce an ordering

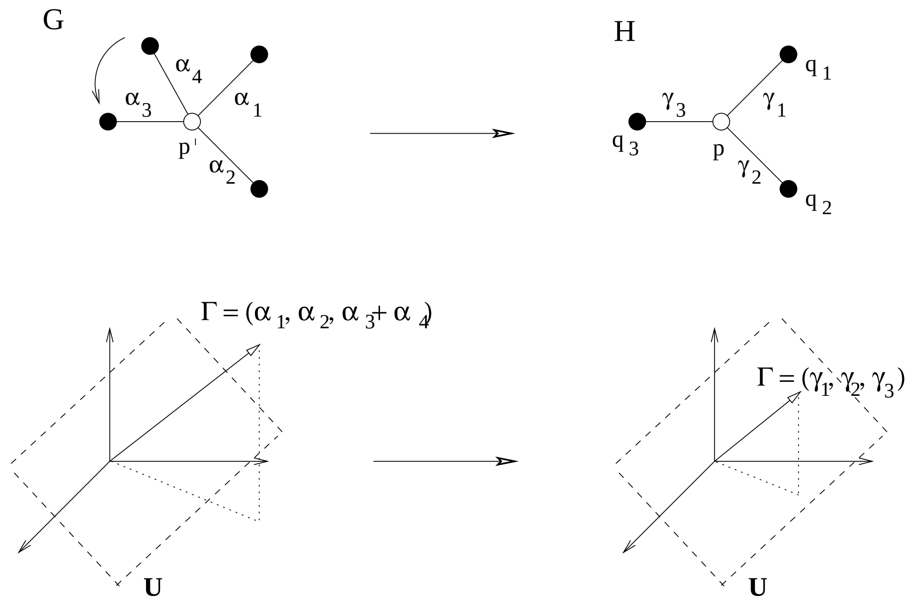

Figure 12: A vector representation of harmonicity

on the vertices in the neighborhood of p'. However, because the valency of p' may be greater than n, there could be multiple neighbors of p' that share the same value of $u \circ \phi$. If this is the case, then the vector of conductivities $\Gamma' = (\gamma'_1, \ldots, \gamma'_n)$ at p' is obtained as follows: each entry γ'_i is the sum of the conductivities on all edges from p' to a vertex q' that gets mapped to q_i. Because ϕ pulls back germs of harmonic functions to germs of harmonic functions, we see that $\Gamma' \perp U$. Thus harmonic homomorphisms preserve orthogonality in some sense. We can summarize this process as follows:

Given an n-star H, we choose an ordering on its vertices and represent its conductivites according to a unique vector in the positive orthant of $\mathbb{R}^n$. The space $\mathbf{U}$ of all germs of harmonic functions that take the value zero at the center vertex p of H forms an $n-1$ dimensional vector space orthogonal to the vector of conductivities. If we have a homomorphism ϕ from a graph G to H, then for each interior vertex p' in the inverse image of p, we can form the conductivity vector for $N(p')$ described above. We know that ϕ is a harmonic homomorphism if and only if this conductivity vector is orthogonal to the space $\mathbf{U}$. An example of this geometric interpretation is illustrated in **Figure 12**.

It seems quite plausible that this representation would be useful for investigating certain properties of harmonic homomorphisms.

4.2 Germs of Functions and Harmonic Continuation

It is interesting to note that there is a distinction between the set of maps that pull back germs of harmonic functions to germs of harmonic functions and those that pull back globally defined harmonic functions to globally defined harmonic functions. This is because not every germ of a harmonic function can be extended to a harmonic function on the entire graph. If a vertex has valency n, then every vector in $\mathbb{R}^n$ corresponds to exactly one germ of a harmonic function at this vertex, since we are free to choose any real number for each of the n neighbors and then the averaging principle determines the value at the center vertex. However, if a graph has strictly less than n boundary vertices, say $m < n$, then it will be impossible to have each of these germs extend to a globally defined harmonic function — since these global functions are in bijective correspondence with $\mathbb{R}^m$.

In the trivial case, the only globally defined harmonic functions on a graph with one boundary vertex will necessarily be constant functions, so any vertex with valency two or more will have a whole family of germs of harmonic functions that do no extend globally.

It is an interesting problem to determine exactly when the germ of a harmonic function can be extended to a globally defined harmonic function in general. It is possible that this has something to do with the existence of disjoint paths to the boundary from each vertex in the neighborhood of the locally defined function, but there has really been no work on this topic yet.

4.3 Characterization of Harmonic Homomorphisms

As is commonly the case in mathematics, the characterization of harmonic homomorphisms seems to be quite difficult compared to proving basic properties about such maps and producing examples. One approach is the following:

Since a harmonic homomorphism is a vertex map from one graph with conductivities to another graph with conductivities, we can interpret characterization as a problem with four given quantities and one unknown such that we are interested in determining what values of the unknown will ensure that the map is a harmonic homomorphism. For example, we could be given two graphs, one of which with fixed conductivities, and a vertex map between them. The question would be to determine what conductivities on the other graph would guarantee that the map is a harmonic homomorphism. Or we could be given two graphs, both with fixed conductivities, and the problem is to determine what vertex maps are harmonic homomorphisms.

Perhaps the most difficult problem in this paper is to develop a suitable interpretation of harmonic homomorphisms on resistor networks in terms of electrical properties. Also, there is likely to be much we can learn about two electrical networks based on the existence of a harmonic homomorphism from one to the other. However, at the time this paper is written there are no known applications to potential theory and inverse problems arising from this work.

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