MOBIUS STRIPS, PINWHEELS, AND OTHER TWO-TO-ONE GENERALIZATIONS OF N-GON-IN-N-GON GRAPHS

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Abstract. This paper considers a few new two-to-one graphs, and shows how they are similar to the well studied n-gon-in-n-gon graphs. Then it argues that a large class of graphs must necessarily be two-to-one.

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1. The Mobius Graph

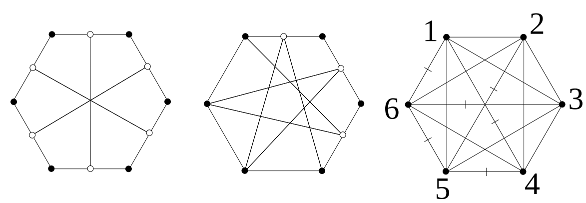

Figure 1. $\bigstar - K$ Steps

Consider the leftmost graph in figure 1, where dots represent boundary nodes and open circles represent interior nodes (for the remainder of this paper, any intersection of lines not indicated by a circle or a dot is not a node) . Then three $Y - \Delta$ transformations convert it to the second graph. Finally, three $\bigstar - K$ transformations result in the final graph. Now, the three doubled outer edges can easily be recovered using quadrilateral relations. The three "diameters", however, are more difficult. We can use quadrilateral relations (see figure 2) to solve for the double edges in much the same way as for the triangle-in-triangle case.

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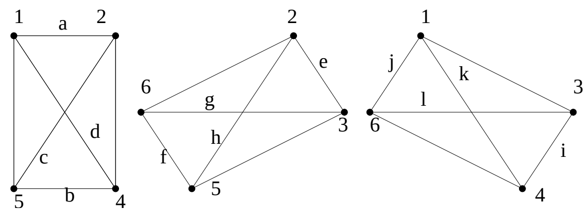

FIGURE 2. The Quadrilaterals

From quadrilateral relations, we have

$$ab = cd \qquad ef = gh \qquad ij = kl$$

Furthermore, from the entries in the response matrix we know

$$c + h = \lambda_{25} \qquad g + l = \lambda_{36} \qquad k + d = \lambda_{14}$$

Thus, we can solve for any of the unknowns, c, d, g, h, k, or l. For example,

$$c = \cfrac{ab}{\lambda_{14} - \cfrac{ij}{\lambda_{36} - \cfrac{ef}{\lambda_{25} - c}}}$$

This is a quadratic equation in c, suggesting that we have a two to one graph. Near the end of this paper, this graph, which we shall call $Mobius_3$ (the names of graphs will be justified later in the paper), will be shown to be at most two-to-one.

Remark 1.1. The terminology which I will use in some of the definitions is not particularly clear, as the word "edge", traditionally, has multiple meanings. When I am referring to an edge as part of a graph, I will simply say "edge". However, when I am referring to an edge as a edge of an n-gon, I will explicitly say "edge of the n-gon".

Definition 1.2. We define (visually) the the $Mobius_n$ to be the graph such that the $2n$ boundary nodes can be placed on the vertices of a 2n-gon and the $2n$ interior vertices can be placed on the midpoints of the edges of the 2n-gon. Furthermore, the edges in the $Mobius_n$ graph are those connecting each boundary node to the two interior nodes closest to it in the plane, as well as the edges connecting the interior node on the midpoint of a given edge of the 2n-gon to the interior node on the midpoint of the opposite edge of the 2n-gon.

2. A More Familiar Context

One may ask how the $Mobius_3$ graph relates to other two-to-one graphs, specifically the triangle-in-triangle graph. Instead of considering this relation explicitly, we will first examine another graph to serve as a bridge between the two, the graph shown second from the left in figure 3. This graph is obtained by "separating" each of the interior nodes in the triangle-in-triangle graph radially outward from the center. Once again, we give a visual definition of the graph.

Definition 2.1. We define the $Hexcyl_n$ graph to be as follows. First draw two concentric regular n-gons in the plane, such that the vertices of those n-gons lie on n radial lines. Then, place a boundary node at each vertex of those two n-gons, and an interior node at the midpoint of each edge of each n-gon. The edges in the $Hexcyl_n$ graph are precisely those that connect each boundary node to each of the two interior nodes on the edges of the n-gon adjacent to the vertex where the boundary node is placed. The rest of the edges connect each pair of interior nodes lying on the same radial line. See figure 3 for an example of a $Hexcyl_3$ graph.

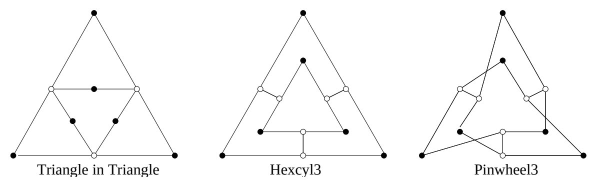

FIGURE 3. Some Generalizations of the Triangle in Triangle

To better understand the $Mobius_3$ graph, we attempt draw it so that, visually, it is most analagous to the triangle in triangle. At first glance, it appears that the best way of doing so is to draw the $Mobius_3$ graph as the graph on the right of figure 3; certainly, this drawing maintains the rotational symmetry of the triangle-in-triangle as well as the hexagonal embedding of the $Mobius_3$ graph.

Definition 2.2. The $Pinwheel_n$ graph is defined, visually, by the following algorithm. Place two concentric regular n-gons in the plane so that the vertices of those n-gons lie on n radial lines. Place the interior and boundary nodes as in the $Hexcyl_n$ case. Choose a boundary node on the inner n-gon, and call it bi_1. Label the rest boundary nodes on the inner n-gon in circular order around the vertices of the inner n-gon. Call the interior node lying on the edge of the inner n-gon connecting bi_k and bi_{k+1} ii_k. Likewise, define bo_k to be the boundary node lying on the same radial line as bi_k, and io_k to be the interior node lying on the same radial line as ii_k. Then the edges of $Pinwheel_n$ connect bo_k to io_k, io_k to ii_k, io_k to bi_{k+1}, bi_k to ii_k, and ii_k to bo_{k+1}. For an example of a $Pinwheel_3$ graph, see the graph on the right in figure 3.

Remark 2.3. The $Pinwheel_n$ graph is similar to the $Hexcyl_n$ graph in that they can both be obtained from the triangle in triangle by separating nodes, except that the separation is "twisted" in the $Pinwheel_n$ graph instead of radial, as in the $Hexcyl_n$ graph.

In the $n = 3$ case, $Pinwheel_3$ is isomorphic to $Mobius_3$, but, when we consider higher n, we find that the $Pinwheel_n$ graph is not always isomorphic to $Mobius_n$. In the next section, we find when the two graphs are equivalent, and, when they are not, say how we should draw the $Mobius_n$ graph so that, visually, it looks the most simliar to an n-gon-in-n-gon graph.

3. THE $n = 4$ CASE

When $n = 4$, figure 4 shows the equivalence of $Pinwheel_4$ and $Hexcyl_4$ (dotted lines represent the boundary/edge of the cylinder, dashed lines represent edges on the hidden surface).

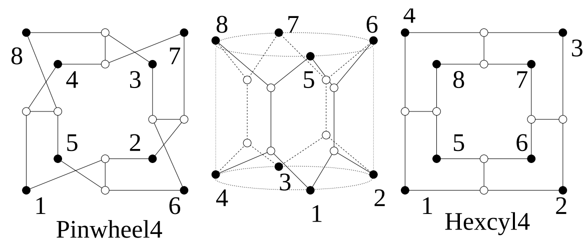

FIGURE 4. Different Embeddings Show Equivalence

In this way we see that $Hexcyl_n$ can be embedded as the union of n hexagons on the graph (HEXagon on CYLinder gives the name Hexcyl). For instance, one hexagon in the figure is that consisting of boundaring nodes one and six, and the edges and interior nodes making up the two paths between them. How, then, do we embed the $Mobius_n$ graph like an n-gon-in-n-gon? Is $Mobius_4$ isomorphic to $Hexcyl_4$? The answer, as the following argument shows, is no.

Consider the drawing of the $Mobius_4$ graph shown on the left in figure 5. To show that $Mobius_4$ and $Hexcyl_4$ are not isomorphic, we show that there can be no isomorphism taking boundary node k in $Mobius_4$ to boundary node h_k in $Hexcyl_4$ that satisfies the requirements of isomorphisms. So, choose any point h_1 on $Hexcyl_4$. By symmetry, we do not lose any generality mapping boundary node 1 from $Mobius_4$ to h_1. We can also assume, without loss of generality, that that point is on the outer square of $Hexcyl_4$. Now, 1 is two edges away from exactly two other boundary nodes, node 2 and node 8. So, we can say that if we had an isomorphism from $Mobius_4$ to $Hexcyl_4$, we would have that h_2 and h_8 were the boundary node of the outer square closest to the node at h_1. But then, extending this argument, we see that 3 would have to map to h_3 such that the shortest path from h_2 to h_3 had a length of two edges. There are only two boundary nodes in $Hexcyl_4$ that satisfy this property, and one of them is h_1. So h_3 must be the other (if not, then we have no bijection, contradicting our assertion that we have an isomorphism), which is located at the point opposite from h_1 on the outer square. But by a similar argument, h_7 must be that same point! Thus we have no bijection, so there is no isomorphism, so there is a contradiction. So $Mobius_4$ is not isomorphic to $Hexcyl_4$.

The preceeding argument can easily be generalized to higher n, giving us the following result.

Proposition 3.1. *$Mobius_n$ is not isomorphic to $Hexcyl_n$ for any n.*

As we can see, the $Mobius$ graph is equivalent to the $Hexcyl$ graph, except with one "side" twisted. So when are $Mobius$ and $Pinwheel$ graphs equivalent? The result is summed up in the following theorem.

Theorem 3.2. *The $Pinwheel_n$ graph is isomorphic as a graph with boundary to the $Mobius_n$ graph iff n is odd,and is isomorphic as a graph with boundary to $Hexcyl_n$ iff n is even.*

Proof. First we consider the case when n is odd. Then we have boundary nodes lying at the vertices of two concentric n-gons. Number them as shown on the left in figure 6, so that the outer boundary nodes are labeled with odd numbers and the

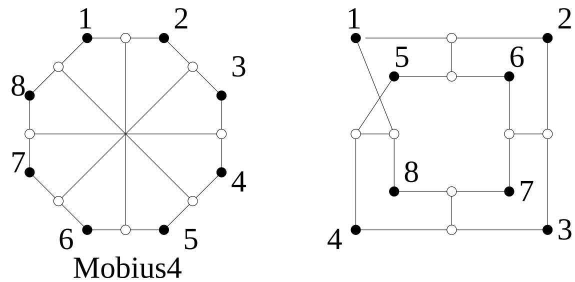

FIGURE 5. Alternative Embedding of the Mobius Graph

inner boundary nodes are labeled with even ones. Furthermore, label the boundary nodes around each polygon so that the order is $1, n+2, 3, n+4, ...2n-1, n$ for the exterior polygon and $n+1, 2, n+3, 4, ...n-1, 2n$ for the interior polygon with $n+1$ and n lying on the same radial line. Then if we embed the $Mobius_n$ graph in a 2n-gon with boundary nodes in circular order, we see that the two graphs are isomorphic.

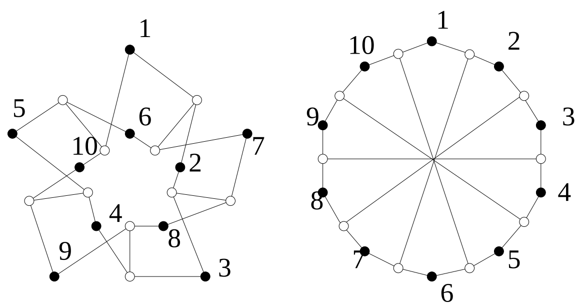

FIGURE 6. Reordering of Boundary Nodes: In the Case n is Odd

Finally, we prove the case when n is even. Label the boundary nodes as before, except with the outer boundary node sequence being $1, n+1, 3, n+3, ...n-1, 2n-1$ and the inner one being $2, n+2, 4, n+4, ...n, 2n$. We then have a drawing of $Pinwheel_n$. Next, place two conentric n-gons in the plane. Place boundary nodes at the vertices of the two n-gons, and interior nodes at the midtpoints of the two $n-gons$. Place an edge between boundary node and each interior node at the midpoint of each side adjacent to each boundary node. Also, place an edge between each pair of interior nodes lying on the same radial ray. Label the boundary nodes on the outer n-gon $1, 2, ...n$ and the boundary nodes on the inner n-gon $n+1, n+2, ...2n$. In addition, we require that 1 and $n+1$ lie on the same radial ray. Then we see that the two graphs are isomorphic (see figure 7).

We have so far shown that $Pinwheel_n$ is isomorphic to $Mobius_n$ when n is odd and isomorphic to $Hexcyl_n$ when n is even. To see that these are not just if statements, but iff statements, we simply refer to Proposition 3.1. $\square$

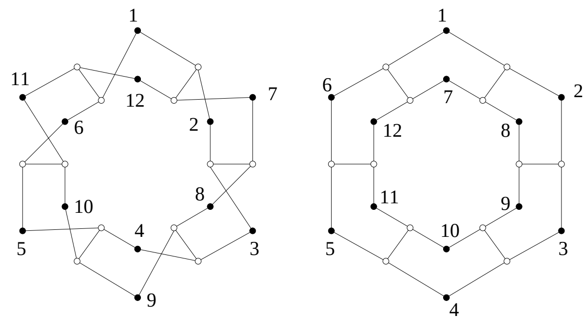

FIGURE 7. Reordering of Boundary Nodes: In the Case n is Even

4. RECOVERABILITY

What are the recoverability properties of these graphs? Are they also two-to-one? To answer this question, we first look in detail at the four-boundary-node subgraphs of which they are comprised.

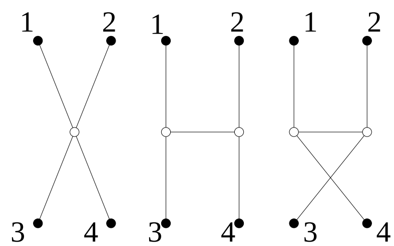

FIGURE 8. Subgraphs of type I, II, and III

Definition 4.1. We will say that a graph is a *generalized* $n - gon - in - n - gon$ if it is a cycle $C_1C_2C_3...C_nC_1$ of n subgraphs of type I, II, or III such that nodes 1 and 2 of C_i are identified with nodes 3 and 4 of C_{i+1}, respectively.

Remark 4.2. Each C_i in a generalized n-gon-in-n-gon graph need not be the same. Figure 8 shows an example of a generalized n-gon-in-n-gon graph with this property. Note the asymmetries of the graph.

With this in mind, we are ready to state a general recoverability result.

Theorem 4.3. *All generalized n-gon-in-n-gon graphs are at most two-to-one.*

Proof. Consider the subgraphs of type I, II, and III. The ★ - K transformation applied to those subgraphs result in the graphs shown in figure 8.

The double edges in the $\bigstar - K'd$ graphs can be found using quadrilateral relations on the edges 14, 13, 24, and 23 since none of those edges overlap with the edges from other subgraphs. Thus, the problem of solving for recoverability in this case simply reduces to the original n-gon-in-n-gon case. For a more rigorous proof, refer to the discussion of the counting principle in [**?**, $FrenchPan$].

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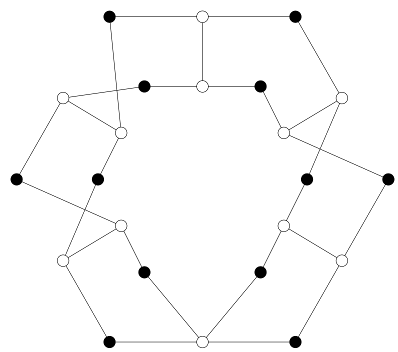

FIGURE 9. A Generalized Hexagon in Hexagon

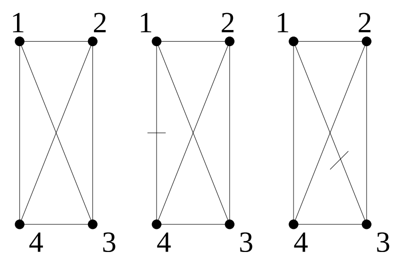

FIGURE 10. The $\bigstar - K$ Transformation Applied to the Three Subgraphs

From this we see that the $Pinwheel$, $Hexcyl$, and $Mobius$ graphs are all at most 2-1.

5. Conclusion

From the re-embedding of the $Mobius_3$ graph from a hexagon to a twisted triangle-in-triangle, we see that what originally looked like a completely new type of two-to-one graph is quite similar to one that is well known. The difference between the two simply reduces to splitting interior nodes. However, by mixing and matching different types of splits, we find a large class of two-to-one graphs with significant asymmetries, something much different from the normal n-gon-in-n-gon case

6. Further Research

Ideally, this idea should be extendable to "stacked" n-gons-in-n-gons—for example, if we have three concentric triangles, for a total of nine boundary nodes, what are the subgraphs for which we can generalize recoverability properties? What might those recoverability properties be? The question of splitting interior nodes is very interesting because, in the three triangle case as an example, it is possible to separate nodes to get two-to-one, four-to-one, and infinity-to-one graphs. What if we take k concentric n-gons-in-n-gons and identify the boundary (and possibly interior) nodes of the outermost layer with the innermost layer. How about identifying different nodes? Should each boundary node only be connected to two interior nodes? What characterizes the response matrices of these graphs? For further reading on n-to-1 networks, see references.

References

[1] Ernie Esser. On Solving the Inverse Conductivity Problem for Annular Networks. 2000. http://www.math.washington.edu/ reu/papers/2000/esser/esser.pdf

[2] Tracy Lovejoy. Applications of the Star-K Tool. 2003. http://www.math.washington.edu/ reu/papers/2003/lovejoy/stark.pdf

[3] Jennifer French and Jerry Pan. 2^n To 1 Graphs. 2004. http://www.math.washington.edu/ reu/papers/2004/jenny/jennjerr.pdf

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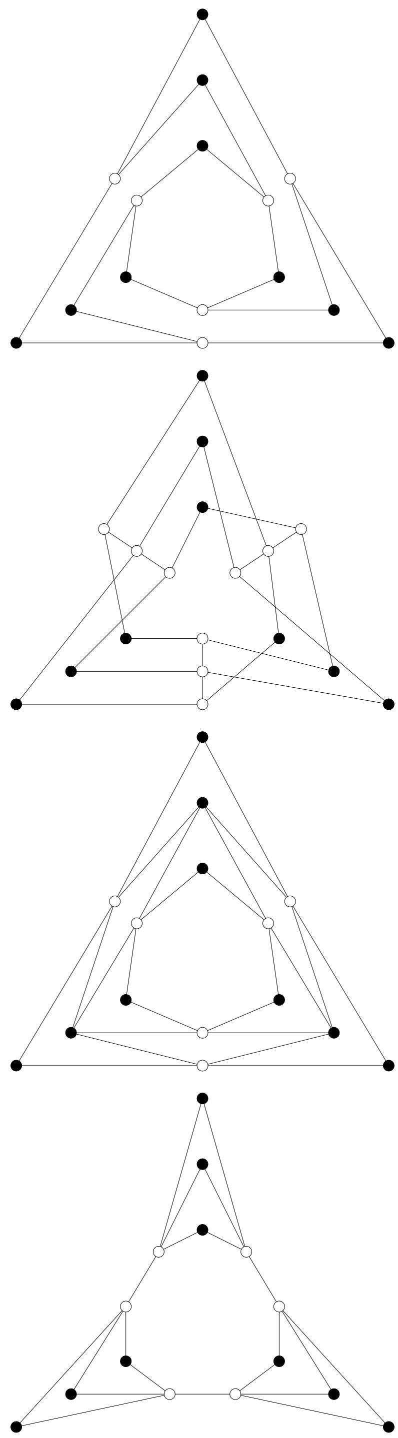

FIGURE 11. Recoverable, 2-1, 4-1, and ∞-1 Graphs. Note that their rotational symmetry is not necessarily required—perhaps there is some way to mix and match subgraphs as in the generalized n-gon-in-n-gon case.