THE SMALLEST RECOVERABLE FLOWER

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Abstract. This paper gives an example of recoverable flower and argues that it is the smallest one in existence. A big thanks to Tracy Lovejoy, who supplied many of the arguments in this paper.

1. Introduction

A flower is defined to be a graph with no boundary spikes and no boundary to boundary connections. This implies

(1) The valence of every boundary node in a flower is at least two.

What, then, is the fewest number of boundary nodes a recoverable flower can have?

2. Non-existence of Small Recoverable Flowers

We know that since there are $n(n-1)/2$ independent entries in a response matrix for a graph with n boundary nodes, we have

(2) No recoverable graph can possibly have more than $n(n-1)/2$ edges.

Otherwise, we would have too few equations for the number of unknowns. Thus, by (1) and (2), there are no recoverable flowers with fewer than five boundary nodes.

Here we condsider the five boundary node case by examining the number of interior nodes. We assume all graphs are connected.

2.1. **0 Interior Nodes.** There cannot be zero interior nodes, otherwise all edges would be boundary to boundary connections and we would not have a flower.

2.2. **1 Interior Node.** There cannot be one interior node, because each edge would have to connect to that node. Then we would have five pairs of parallel connections, and the graph would not be recoverable.

2.3. **2 Interior Nodes.** The graph is shown in figure 1.

One way to see that this graph is not recoverable is to place conductances of one on all edges in the upper half of the graph and place conductances of two on all edges on the lower half of the graph. Then by symmetry, that electrical network must have the same response as one with the conductances reflected across the horizontal axis.

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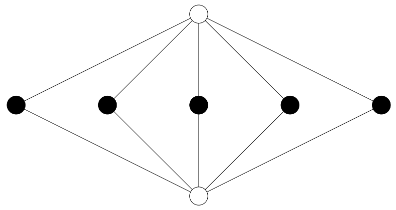

FIGURE 1. Not Recoverable, By Symmetry

Remark 2.1. Notice the arguments for zero, one, or two interior nodes do not depend on the fact that the flower had five boundary nodes. Thus, we have

(3) Any recoverable flower must have at least three interior nodes

2.4. **3 Interior Nodes.** Combining (1), the fact that there are no boundary to boundary connections, and that we have five boundary nodes, we know that there must be at least ten edges. Furthermore, we cannot have more than $5(5-1)/2 = 10$ edges by (2). So since $10 \leq \#$ of edges ≤ 10,

(4) A recoverable flower with five boundary nodes must have exactly ten edges.

It follows that there can be no interior to interior connections in a recoverable flower with five boundary nodes, and that each boundary node has valence of exactly two. Later, we will see that this is vacuously true, but for the time being, it is a useful fact. Now, we notice that since an interior node of valence one is a pendant, and an interior node of valence two is a series connection, we have

(5) Interior nodes must have valence of at least three in a recoverable graph.

Now, considering the three interior node specifically, we can combine (4) and (5) to see that two interior vertices, call them v_{i1} and v_{i2} have valence three, while the remaining interior vertice, v_{i3} has valence four. v_{i1} and v_{i3} are both adjacent to a boundary vertice, v_{b1}, because otherwise we would either have at least one pendant or series connection, or a disconnected graph. Furthermore, v_{i2} and v_{i3} are both adjacent to another boundary vertice, v_{b2}, otherwise we would, once again, have a pendant or a series connection. To see a visualization of what we know must be true, so far, for a recoverable five boundary node flower, see figure 2. Continuing, we note that v_{i3} is adjacent to a boundary node, v_{b3}, which is adjacent to another interior node (if this were not the case, we would have a pendant at v_{i3}. WLOG, assume that interior node is v_{i1}. Then v_{i1} is adjacent to a boundary node v_{b4}, which is also adjacent to v_{i2}, otherwise there would be a series at v_{i2}. This determines the position of the final boundary node, v_{b5}, which is necessarily adjacent to v_{i2} and v_{i3}. Then if a five boundary node three interior node graph is to be recoverable, it must be the graph shown on the left of figure 3. But that graph is $Y-\Delta$ equivalent to the graph on the right of figure 3. That graph contains a series connection and hence is not recoverable. So we know that there are no five boundary node three interior node recoverable flowers.

2.5. **4 Interior Nodes.** From the discussion following (4), we know that there are no interior to interior connections in the five boundary node case. This, combined with (5), implies that a five boundary node four interior node flower must have at least twelve edges. But from (4), we know it must have exactly ten, giving a

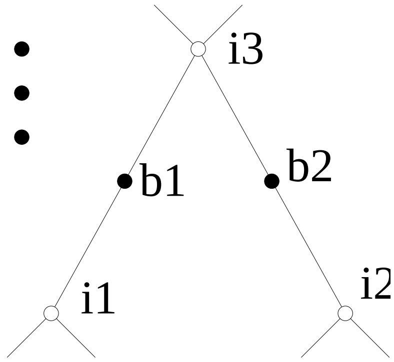

FIGURE 2. Where Do We Put the Remaining Boundary Nodes?

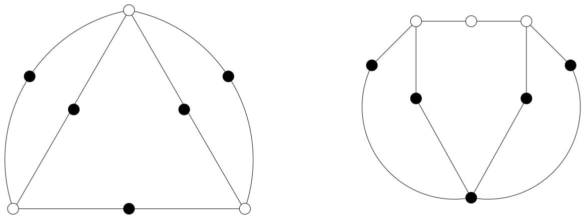

FIGURE 3. A Graph and its $Y-\Delta$ equivalent

contradiction. Thus, there are no five boundary node four interior node recoverable flowers.

2.6. ≥ 5 **Interior Nodes.** The discussion from the four interior node case easily generalizes to any higher number of interior nodes. So we can conclude that there are no five boundary node recoverable graphs.

Remark 2.2. It should be noted that some of the above arguments can be generalized to any number of boundary and interior nodes. Let i be the number of interior nodes, and let b be the number of boundary nodes. Since each interior node must have valence of at least three, we know that there are at least $3i$ edges adjacent to interior nodes (note that interior to interior connections are counted twice*, once for each end, whereas boundary to interior connections are only counted once). Now, all edges adjacent to interior nodes must be accounted for. Otherwise we would have an interior node an interior node with valence less than three, which would give us a non-recoverable graph. These restrictions give rise to two inequalities.

For a recoverable flower with no interior to interior connections, we find, we note that there are a minimum of $3i$ adjacent to interior nodes. Since we have a maximum of $b(b-1)/2$ edges adjacent to those interior nodes (since we are considering the case where all edges connect an interior node to a boundary node) we have $3i - b(b-1)/2 \leq 0$, with the inequality instead of equality because we could have more than three edges adjacent to a given interior node. This can be rewritten as

$$i \leq \frac{b(b-1)}{6} \tag{6}$$

For a recoverable flower with minimal boundary to interior connections (specifically, each boundary node has valence two), we can derive a similar inequality. Once again, we have a minimum of $3i$ edges, remembering to double count when necessary*. Since each boundary node is adjacent to exactly two edges, boundary to interior connections account for $2b$ of those edges. We know, from (2), that there are at most $b(b-1)/2 - 2b$ edges remaining, which, by assumption, are all interior to interior connections. As noted*, interior to interior connections must be counted twice, so we have that $3i - 2b - 2(b(b-1)/2 - 2b) \leq 0$, or equivalently,

$$i \leq \frac{b(b-1) - 2b}{3} \tag{7}$$

Whenever $b > 5$ (i.e. for all recoverable flowers), the second inequality is less restrictive than the first. Furthermore, it is the most general in that it applies to every recoverable flower. Note that equality can only hold in (7) if exactly two edges are adjacent to each boundary node (though this is not a sufficient condition). If more edges are adjacent to boundary nodes, it is possible to put a better bound on the maximum number of interior nodes we can have and retain recoverability. This is due to the fact that inerior to interior connections are double counted, and boundary to interior connections are not.

3. Six Boundary Nodes

Theorem 3.1. *The smallest number of boundary nodes a recoverable flower can have is six. Furthermore, it may have exactly six.*

Proof. The proof that six is a lower bound is above. The proof that six is the greatest lower bound is the graph on the left of figure 4, which is recoverable because it is $Y - \Delta$ equivalent to a graph (shown in the middle and on the right) composed entirely of boundary to boundary edges and which contains no parallel connections. □

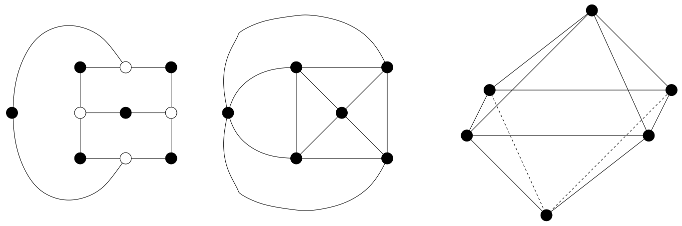

Figure 4. The Smallest Recoverable Flower, and its $Y - \Delta$ Equivalent

Remark 3.2. The graph on the right of figure 4 shows the embedding on the octahedron of a graph that is $Y - \Delta$ equivalent to the six boundary node recoverable flower, with the nodes at vertices, and the edges of the graph lying on the edges of the octahedron. Perhaps there are interesting properties of graphs that are $Y - \Delta$ equivalent to ones that can be embedded on Platonic Solids. Note that one would have to be careful with their approach—one way to generalize the graph on the left would be to consider a twenty boundary node, five interior node flower such that there was a two edge path from each interior node to every other interior node, with the vertex separating those edges being a boundary vertex. The choice of twenty boundary nodes in the five interior node case is due to the fact that 5 choose $2 = 20$ (in the four interior node case, we had 4 choose $2 = 6$, which was the number of boundary nodes). Then, if we consider an dodecahedron, we could, in similar fashion to the octahedral case, place boundary nodes of the $-K'd$ graph on the vertices of the surface—but then, none of the edges of the graph would lie on the edges of the dodecahedron! Perhaps it would be better to place the boundary nodes at the centers of the faces of an icosahedron, and find a way to draw the graph on that surface.

Remark 3.3. When we say "smallest recoverable flower" we mean the flower with the smallest number of boundary nodes, as well as smallest number of edges. However, it may be possible to have a six-node recoverable flower with fewer interior nodes than the example given. If it existed, however, it would have to have three interior nodes. It would be interesting to give an example of such a graph, or to show that it cannot exist. The only way I can think of to show non-existence would be a proof by cases (of which there would be many—between fifty and one hundred or so) to show that each such flower is non-recoverable. Hopefully, there is a more elegant way to prove this result. Furthermore, I do not know if there are any other six node recoverable flowers at all—though we do know from Remark 2.1 that it would have to have three, four, five, or six interior nodes.

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