On the Nature of Connectivity Types

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Abstract

The purpose of studying connectivity types is to look for a way to classify in a fairly comprehensive manner all critical circular planar n-boundary node graphs. We start by looking at the four-boundary node and five-boundary node graphs. Moreover, by listing all the examples of the two cases, it is possible that we can find bases of connectivity types that imply a set of larger connectivity types. This would allow us to come up with a method of generalizing to n-boundary node critical circular planar graphs.

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1 Introduction

In [1], the topic of connectivity types has been mentioned briefly but has not been studied in detail. By looking at all Y-Δ equivalent well-connected n-boundary node graph and systematically deleting boundary to boundary edges or contracting boundary nodes, we retain critical circular planar graphs. Thus it allows us to list all connected critical circular planar graphs that possess n-boundary nodes in a neat and comprehensive manner. I begin by defininining the terminologies that are employed in this paper.

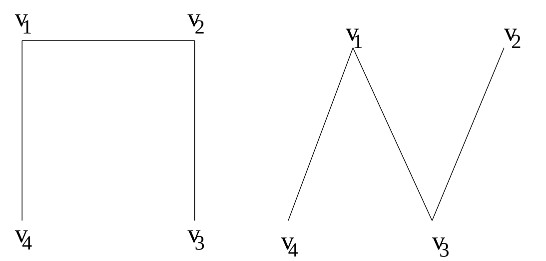

Figure 1: Two Embeddings, Two Different Connectivity Types

Definition 1. Given a graph, $G = (V, E)$, suppose P=$(p_1, ..., p_k)$ and Q=$(q_1, ..., q_k)$ are two sequences of boundary nodes. P and Q are connected through G if there is a permutation τ of the indices 1,...,k, and k disjoint paths α_1, ...,α_k in G, such that for each i, the path α_i starts at p_i, ends at $q_{\tau(i)}$, and passes through no other boundary nodes. To say that the paths α_1, ...,α_k are disjoint means that i different from j implies α_i and α_j have no vertex in common. The set α_1, ...,α_k is called a $k-$*connection* from P to Q. A path which joins one boundary node to another boundary node is a 1-connection.

1.1 Connectivity Types

A *connectivity type* of a certain graph G is understood to be the set of all possible k-connections of G, considering one circular embedding of the graph. For example, the well connected four-boundary node graph has two 2-connections and six 1-connections, and the well connected five boundary node graph has $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ 2-connections and the same number of 1-connections. However, there is a sublety to this. Two graphs can possess the same "counts" of connections, ie., both may have two 2-connections and three 1-connections, but they do not possess the same *connectivity type*, because of different circular pairs. Figure 1 illustrates two graphs which have the same "counts" of connections, but different connectivity types. The two are isomorphic as graphs with boundary nodes.

1.2 Z-Sequences and ζ-Sequences

Definition 2. Suppose G is a critical circular planar graph with n boundary nodes so that $v_1, ..., v_n$ occur in a clockwise order around a circle C, and the rest of G is n the interior of C. The medial graph M(G) has n geodesics each of which intersects C twice, hence in 2n distinct positions. Label them $t_1, ..., t_{2n}$, so that $t_1 < v_1 < t_2 < t_3 < ... < t_{2n-1} < v_n < t_{2n} < t_1$ is in a circular order around C. We obtain a $z-$*sequence* by labeling the geodesics in a circular order, starting with g_1, which begins at t_1, and continuing with g_2, until we have labeled all n geodesics, but only counting each geodesic once.

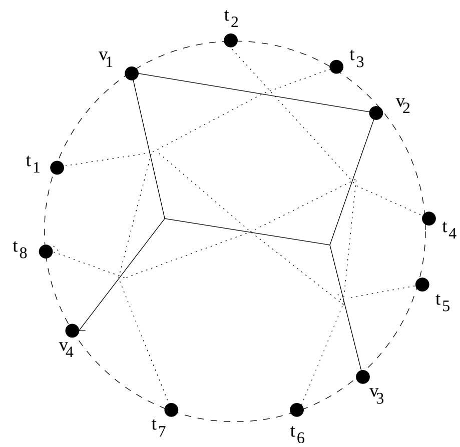

Figure 2: Well Connected 4-Boundary Node Graph G and M(G)

Observation 1. The well connected n-boundary node graph has a $z-sequence$ of 1,2,...n,1,2,...n.

Definition 3. Suppose we renumber the $z - sequences$ in a manner where we start from g_1, and we count the number of intersecting geodesics. This is our first entry in the new sequence. We continue in a clockwise manner with g_2, until we get back to g_1. Our renumbered sequence has 2n terms, the same as before with the $z - sequence$. We refer to this new sequence as a ζ-*sequence*. [2]

Observation 2. The well-connected n-boundary node graph has a ζ-*sequence* of n-1,n-1,...n-1 for 2n terms.

Observation 3. The first n terms in a ζ-*sequence* is necessary and sufficient to determine the ζ-*sequence* and hence the $z - sequence$.

For practical purposes, from here on out let us refer of a unique $z-sequence$ or ζ-*sequence* by fixing an embedding of the boundary nodes. Figure 2 illustrates the well-connected four boundary node graph and its medial graph, and Figure 3 illustrates its $z - sequence$ superimposed on its medial graph.

Observation 4. Two critical circular planar graphs have different $z-sequences$ if and only if their ζ-*sequences* is different, that is, the map from $z - sequences$ to ζ-*sequences* is bijective.

Theorem 1. Given a critical circular planar graph G, a graph is Y-Δ equivalent to G if and only if their $z - sequences$ are the same.

Proof. The proof is straighforward. A Y-Δ transformation of a particular critical circular planar graph corresponds to motions in the geodesics, but its

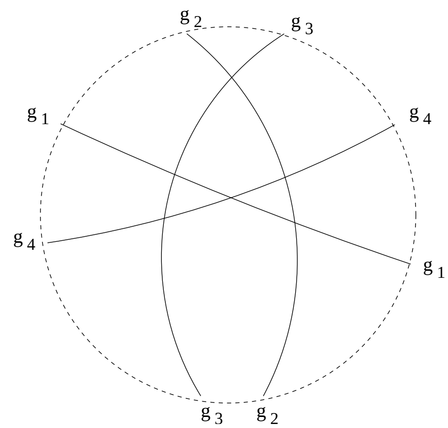

Figure 3: Z-Sequence of the Well Connected 4-Boundary Node Graph G

$z-sequence$ is still the same, since motions preserve ordering of geodesics at the boundary nodes, and conversely, two graphs that have the same $z - sequence$ have the same medial graphs, but this implies that two graphs are Y-Δ equivalent.

□

Theorem 2. Two critical circular planar graphs are Y-Δ equivalent if and only if they possess the same connectivity type.

Proof. Suppose a graph G is transformed into a graph G' by a Y-Δ transformation. Then suppose we have a Y with p, q, and r being boundary nodes, and w the interior node. A Y-Δ transformation leaves us with p, q, and r as boundary nodes, and no interior nodes. Suppose α and β are disjoint paths in G where α passes through a boundary node, say p, and β passes through edges rw and wq. This implies that there are corresponding paths α' and β' in G', where $\alpha = \alpha$' and β is the same as β' except the two aforementioned edges are replaced by a single edge. Hence it follows that if α is any connection $P \leftrightarrow Q$ through G, there is a corresponding connection $P \leftrightarrow Q$ through G'. Hence a Y-Δ transformation implies a bijection from G onto G', and the connectivity type stays the same. Likewise, if two graphs have the same connectivity type, and they are critical, their medial graphs are lens-free. Hence the equivalence class M under motions of geodesics may be found using the cut-point lemma, according to Proposition 9.2 of [1]. But motions of geodesics in a medial graph preserve Y-Δ equivalence. Hence this proves our theorem.

□

Lemma 1. Two graphs have the same $z - sequences$ and ζ-*sequences* if and only if their connectivity types are the same.

Type1	$Z - Sequence$	Type2	$Z - Sequence$	Type3	$Z - Sequence$
$(4;2)$	12342134	$(4,1;2,3)$ $(4;3)$	12342314	$(1;2)$	12343214
$(4;2)$	12342134	$(1;3)$	12342143	$(4,1;2,3)$ $(3;4)$	12342413
$(3,4;1,2)$	12341324	$(4;3)$	12341423	$(4;2)$	12342413
$(3,4;1,2)$	12341324	$(1,2;3,4)$	12314324	$(4;2),(4;3)$ $(2;3)$	12324314
$(1;3)$	12341243	$(1,2;3,4)$ $(1;4)$	12134234	$(4;2)$	12132434
$(2,3;4,1)$	12341324	$(4;2),(4;3)$	12342314	$(1;4)$	12343214

Table 1: All Z-Sequences Of 4-Boundary Node Graphs (Read Horizontally)

Our question hence can be rephrased like this: Can we locate all connectivity types by permuting the $z - sequence$ or ζ-*sequence*? Specifically, is it possible to find an algorithm that allows us to relate a unique connectivity type to a unique sequence, and hence a unique recoverable graph? We examine the four boundary node case. In Figure 4 we have all possible four-boundary node critical circular planar graphs up to an isomorphism, or according to [4], all possible circular embeddings of the four-boundary node critical circular planar graph, and in Table 1 we have the $z - sequences$ of all four-boundary node critical circular planar graphs. I have conveniently labeled the table as "types," by which I imply the number of edges broken. For example, Type 1 includes all four-boundary node critical circular planar graphs with 1 edge having been deleted off of the well connected 4-boundary node graph through boundary edge deletion or boundary edge contraction. I list them by their sets of connections broken at each type, and across each row is a sequence of edge deletions.

1.3 Counting the Z-sequences

According to [2], it is possible by a set of motions of geodesics to arrive at all possible permutations of $z - sequences$, and by [3] there is an algorithm for the cardinality of $z - sequences$ for a specific n-boundary node critical circular planar graph. The number of $z - sequences$ for any particular n-boundary node case is

$$a(n) = \prod_{k=1}^{n} (2k - 1)$$

and it can be proved using induction. This motivates the next conjecture.

Conjecture 1. By permuting the $z - sequence$ of any n-boundary node critical circular planar graph, it is possible to find all connectivity types of a specific

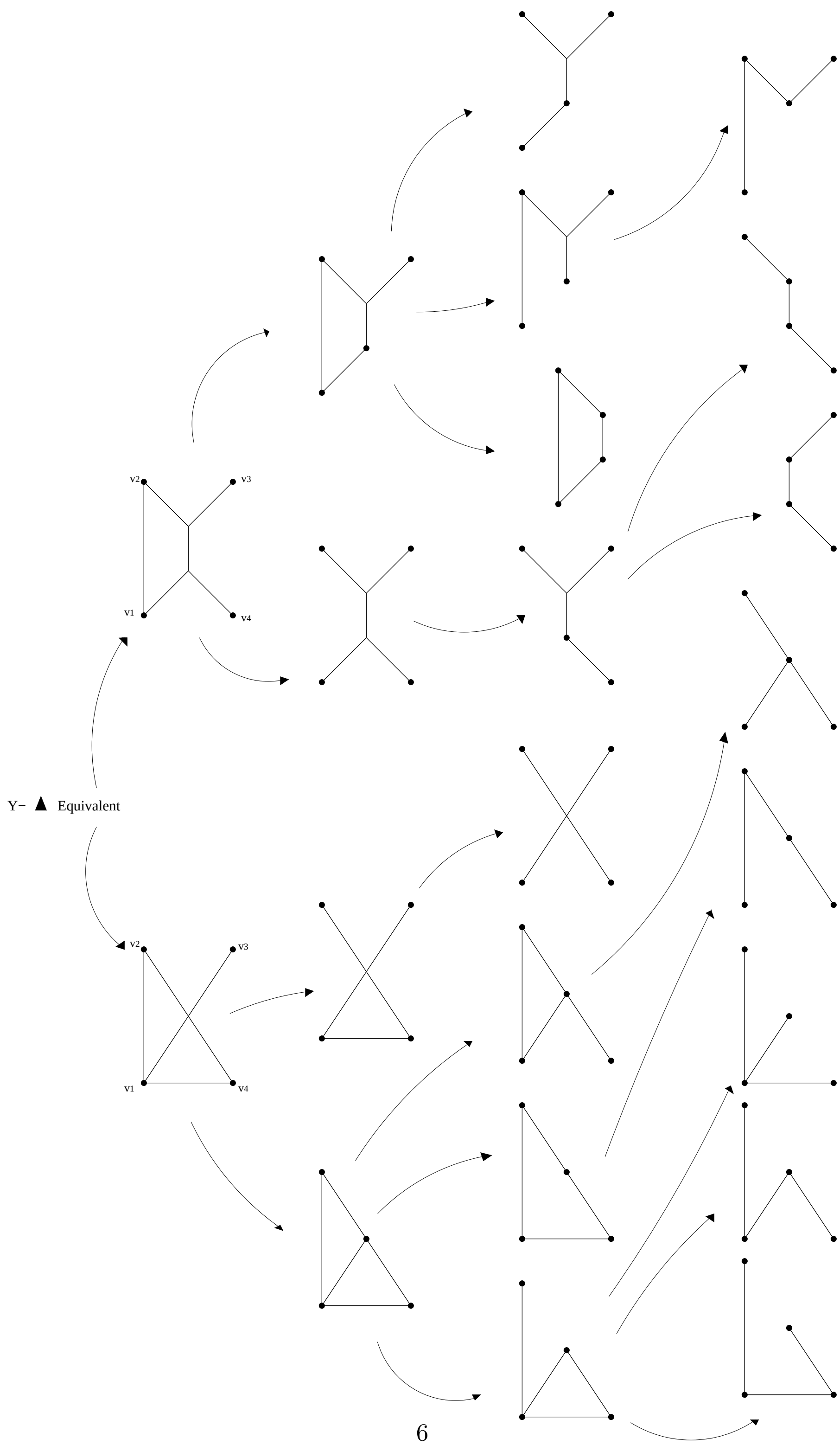

Figure 4: All 4-Boundary Node Critical Circular Planar Graphs

n-boundary node graph. That is, it is possible to find an algorithm using $z-$ *sequences* to classify connectivity types.

2 Basis of Connections

However, there is a more intuitive way of looking at sets of connectivity types. Suppose we know that breaking a set of edges or contracting a set of boundary nodes, or even both, gives us a set of broken connections. Is there a way we can find a core group of broken connections that imply all other broken connections, specifically in the n-boundary node critical circular planar case? In other terms, can we find a basis of equations that spans a larger set of equations? And if so, which connections make up the basis, and are these connections always needed for other bases? This motivates the next group of terminologies.

Theorem 3. Suppose Γ = (G,γ) is a circular planar resistor network and (P,Q)=($p_1, ..., p_k$;$q_1, ..., q_k$) is a circular pair of sequences of boundary nodes.
a) If (P;Q) are not connected through G, then detΛ(P;Q) = 0
b) If (P;Q) are connected through G, then $(-1)^k$ $\cdot det\Lambda$(P;Q) > 0.

The proof is given in [1]. We can readily deduce from this that Λ(P;Q) is non-singular if and only if P and Q are connected through G, and in particular, any well connected n-boundary node graph has the property that Λ(P;Q) is non-singular.

Definition 4. An *algebraic variety* is a generalization to n dimensions of algebraic curves. An algebraic variety V is defined as the set of points in the reals satisfying a system of polynomial equations $f_i(x_1, ..., x_n) = 0$ for i = 1, 2 ,... formally written as V $\subset$ R^n. According to the Hilbert Basis Theorem, a finite number of equations suffices.

We can think of R^n as the space of entries in the upper diagonal of the response matrix, with n entries, corresponding to n edge conductivities. The set V is the set of all edge conductivities remaining after a finite number of boundary edge deletions or boundary node contractions, where dim V = m, and the set of edge conductivities that have been broken by these finite number of operations is codim V = n-m.

2.1 Complete Intersection

Definition 5. If the minimal number of equations used to span an ideal defining the algebraic variety is equal to the codimension of V, then the space of response matrices forms a *complete intersection*.

The idea is that there is a basis of equations which spans a wider range of equations. In the case of critical circular planar graphs, by deleting three edges, and hence reducing the dimension of V by 3, it is possible that doing this operation generates five equations. For example, we can have two broken

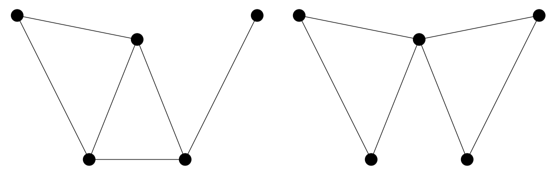

Figure 5: Same Counts, Two Different Graphs

2-connections, which imply that the determinants of the two circular pairs are 0, and three broken 1-connections, which imply that the three 1 by 1 determinants are 0. A complete intersection implies there exists a basis of three homogeneous equations, $f_1 = f_2 = f_3 = 0$ that spans the five other homogeneous conditions, $g_1 = g_2 = g_3 = g_4 = g_5 = 0$.

Remark 1. Two four boundary node critical circular planar graphs having the same "counts" of connections imply that either the two graphs are Y-Δ equivalent or they are isomorphic as graphs. (different embeddings of the same graph)

This idea leads to the somewhat naive conjecture that all critical circular planar graphs having the same "counts" of connections imply either Y-Δ equivalence or graph isomorphisms, which would imply that by "counts" of connections we can determine a graph. However, for the five boundary node case, having the same "counts" of connections does not imply Y-Δ equivalence or graph isomorphisms. Figure 5 illustrates a counterexample. Both graphs have six 1-connections and five 2-connections, but they are different graphs. Hence, it is generally not true that for any two n-boundary node critical circular planar graph, if their "counts" of connections are the same, then the two graphs are Y-Δ equivalent or isomorphic as graphs.

Theorem 4. For all four-boundary node critical circular planar graphs, the set V^m, where $6 \leq m \leq 9$, is a complete intersection.

Proof. The proof goes by way of exhaustion. I have exhausted all four boundary node critical circular planar graphs and found a basis for each one of them. However, I will give an example. Let us take the Y-Δ equivalent graph of the symmetric well connected four-boundary node graph. By contracting boundary node 3, we break the (2;4) connection, and following that we delete the edge between nodes 2 and 3, hence breaking the (2;3) and (1,2;3,4) connections. Finally we delete the edge between nodes 1 and 3, and in doing so break the (1;3) connection. Refer back to Figure 4 for the sequence of graphs.

From the three edge deletions, we get $\lambda_{24} = \lambda_{23} = \lambda_{13} = 0$ and $\det\Lambda(1,2;3,4) = 0$. Our goal is to try to represent these four equations in terms of three. Suppose we assume $\lambda_{24} = \lambda_{23} = \lambda_{13} = 0$. Hence we are trying to find the missing component, $\det\Lambda(1,2;3,4) = 0$. Looking at the response matrix, we get

$$\begin{bmatrix} \Sigma & & 0 & \\ & \Sigma & 0 & 0 \\ 0 & 0 & \Sigma & \\ & 0 & & \Sigma \end{bmatrix}$$

Looking at this we can readily determine that $\det\Lambda(1,2;3,4) = 0$, either from λ_{24} and λ_{23}, or λ_{23} and λ_{13}. This implies a complete intersection.

□

The point of basis of connections now brings up a very good question. If we look back at the example of complete intersection, how would we know which connections form a basis, that is, is there a way we can generalize it so that we not only have a basis, but we know which basis it is. Hence we want to come up with an algorithm that would describe this basis.

2.2 Some Nice Arguments for a Complete Intersection

Here is one argument for a basis for a five-boundary node graph. We will look at the Tower of Hanoi network, one of the well connected 5-boundary node graphs, and number the boundary nodes in a clockwise order starting from the top. First we contract boundary node 4, then delete the edge between boundary nodes 4 and 5, then. As a result, we have three 2-connections broken, the (2,3;5,1), (3,4;5,1), and (2,4;5,1) connections. Then suppose we draw a closed curve with boundary nodes 1 through 5 in a clockwise manner. By the Jordan Curve Theorem, we find that the (3,4;5,1) and (2,4;5,1) connections establish the (2,3;5,1) connection through contradiction. If (2,3;5,1) connection were not broken, then implicitly the (2,4;5,1) connection would not be broken, by the Theorem. Figure 6 gives the appropriate graph. However, this argument only establishes a complete intersection and fails to give us an algorithm.

Here is another argument for a basis for the same five-boundary node graph that will produce this result we want. The (2,3;5,1), (3,4;5,1), and (2,4;5,1) connections are still broken, and now we want to find an algorithm that gives a specific basis of two 2-connections. The following matrix represents rows 2, 3, 4 and columns 5, 1.

$$\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

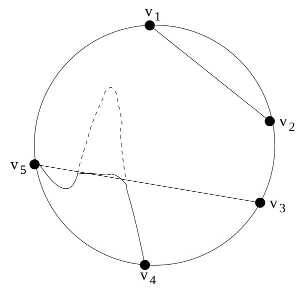

Figure 6: (2,3;5,1) connection

Looking at this, we can determine that

(2,3;5,1) = ae-bd = D_1, (3,4;5,1) = bf-ce = D_2, and (2,4;5,1) = af-cd = D_3

Then necessarily

fD_1+dD_2 =$e(af - cd)$ = eD_3

Hence we have the following formula represented as linear combinations of 1-connections and 2-connections:

fD_1+dD_2-eD_3 = 0, where e,f,d $\neq 0$

From this we can readily see that any two combinations of two broken 2-connections would imply that the third 2-connection is also broken. Hence this algebraic argument is more thorough than the geometric argument in the previous case, and leaves us wondering whether the argument of a basis extends further than a complete intersection.

Conjecture 2. For any n-boundary node critical circular planar graphs, the space of response matrices is a complete intersection.

A good way to start off would be to look at all five boundary node critical circular planar graphs, and for n > 5, look at the six term identity. From it, we can readily see that sets of broken 1 connections do not necessarily mean sets of broken 2 or higher connections.

2.3 Determinant-Connection Formula

Another way to start is by looking at the determinant-connection formula, as presented by Lemma 3.12 of [1]. For two disjoint sequences of boundary nodes, the equation follows:

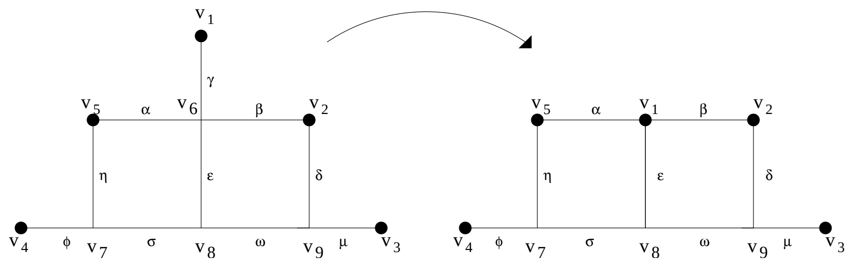

Figure 7: Contracting Node 1 Breaks a 2-Connection

$$\det \Lambda(P;Q) \cdot \det K(I;I) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \left\{ \sum_{\substack{\alpha \\ \tau_\alpha = \tau}} \prod_{e \in E_\alpha} \gamma(e) \cdot D_\alpha \right\} \tag{1}$$

Using this, let us take the Tower of Hanoi graph with the usual circular ordering and contract boundary node 1. Doing this breaks the (2,3;4,5) connection. We proceed to label each edge conductivity. Refer to Figure 7 for the graphs. By the determinant-connection formula for the Tower of Hanoi graph is

detΛ(2,3;4,5)*K = $\alpha\beta\phi\sigma\omega\mu$

but by contracting boundary node 1, we break the (2,3;4,5) connection.The argument for this is by contracting node 1, $\gamma \to \infty$. But γ is implicitly in the row sum for interior node 6, which is implicitly in the K factor on the left hand side. Hence as $\gamma \to \infty$, K $\to \infty$, so by dividing this factor to the right hand side, we can see that

detΛ(2,3;4,5) $\to$ 0.

It remains to be seen if there is something to be discovered in the form of a linear subspace, so that there is the possibility of finding an ideal that spans all connections. However, at this point, it is nice to have so many arguments for a basis that gives us a complete intersection.

2.4 Future Research

The question of whether there is a basis of connections for any n-boundary node critical circular planar graph is still to be answered, and so is the classifying connectivity types with our two aforementioned sequences. It would also be interesting to find analogues in a dual graph.

2.5 Acknowledgements

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