On the Nature of Connectivity Types

Miao Xu

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Abstract

The purpose of studying connectivity types is to look for a way to classify in a fairly comprehensive manner all critical circular planar n-boundary node graphs. We start by looking at the four-boundary node and five-boundary node graphs. Moreover, by listing out all the examples of these two cases, it is possible that we can find bases of connectivity types that imply a set of larger connectivity types. This would allow us to come up with a method of generalizing n-boundary node critical circular planar graphs.

1 Introduction

In [1], the topic of connectivity types has been mentioned briefly but has not been studied in detail. By looking at all Y-Δ equivalent well-connected n-boundary node graph and systematically deleting boundary to boundary edges or contracting boundary nodes, we retain critical circular planar graphs. Thus it allows us to list all connected critical circular planar graphs that possess n-boundary nodes in a neat and comprehensive manner. I begin by defininig the sets of terminologies that are employed in this paper.

Definition 1. Given a graph, $G = (V, E)$, suppose P=$(p_1, ..., p_k)$ and Q=$(q_1, ..., q_k)$ are two sequences of boundary nodes. P and Q are connected through G if there is a permutation τ of the indices 1,...,k, and k disjoint paths α_1, ...,α_k in G, such that for each i, the path α_i starts at p_i, ends at $q_{\tau(i)}$, and passes through no other boundary nodes. To say that the paths α_1, ...,α_k are disjoint means that i different from j implies α_i and α_j have no vertex in common. The set α_1, ...,α_k is called a $k-connection$ from P to Q. A path which joins one boundary node to another boundary node is a 1-connection.

A *connectivity type* of a certain graph G is understood to be the set of all possible k-connections of G. For example, a well-connected graph with four boundary nodes has two 2-connections and six 1-connections, and a well-connected graph with five boundary nodes has $\binom{5}{2}$ 2-connections and the same number of 1-connections. Hence we classify the well-connected five boundary node

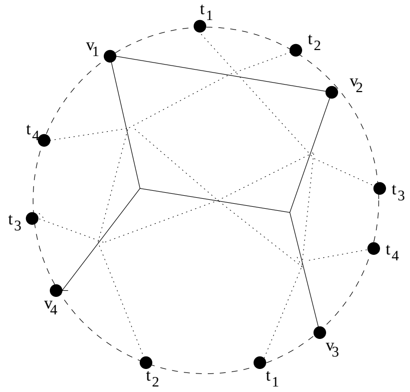

Figure 1: Graph and its Medial Graph

graph as having "ten 2-connections and ten 1-connections." However, there is a sublety to this. Two graphs can possess the same "counts" of connections, ie., both may have two 2-connections and a 1-connection, but they do not possess the same *connectivity type*. However, it is readily evident that these two graphs are isomorphic.

Definition 2. Suppose G is a critical circular planar graph with n boundary nodes so that $v_1, ..., v_n$ occur in a circular order around a circle C, and the rest of G is in the interior of C. The medial graph M(G) has n geodesics each of which intersects C twice, hence in 2n distinct positions. Label them $t_1, ..., t_{2n}$, so that $v_1 < t_1 < t_2 < v_2 < ... < v_n < t_{2n-1} < t_{2n} < v_1$ is in a circular order around C. We obtain a $z - sequence$ by labeling the geodesics in a circular order, starting with g_1, which begins at t_1, and continuing with g_2, until we get have labeled all n geodesics in a circular order, but only counting each geodesic once.

For practical purposes, from here on out let us refer of a unique $z - sequence$ by fixing an embedding of the boundary nodes. Figure 1 illustrates the well-connected four boundary node graph and its medial graph, and Figure 2 illustrates its $z - sequence$.

Theorem 1. Given a critical circular planar graph G, a graph is Y-Δ equivalent to G if and only if their $z - sequences$ are the same.

Proof. The proof is straighforward. a Y-Δ transformation of a particular critical circular planar graph corresponds to motions in the geodesics, but its $z - sequence$ is still the same, and conversely, two graphs that have the same $z - sequence$ have the same medial graphs, but this implies that two graphs are Y-Δ equivalent. □

We know that two graphs are Y-Δ equivalent if and only if they possess the same connectivity types. Hence we deduce the following lemma.

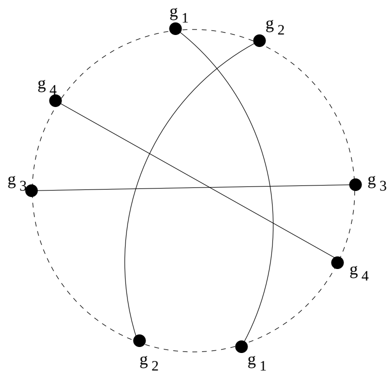

Figure 2: $z - sequence$ of the well connected 4 boundary node graph

Lemma 1. Two graphs have the same $z - sequences$ if and only if their connectivity types are the same.

This allows us to classify graphs and their connectivity types by $z{-}sequences$. Our question can be rephrased like this: Can we locate all connectivity types by a set of unlacings of the $z - sequence$? Specifically, is it possible to find a connectivity type that corresponds to a unique $z - sequence$? We examine the four boundary node case. In Figure 3 we have all possible four boundary node critical circular planar graphs up to an isomorphism.

According to [2], it is possible by a set of motions of geodesics to arrive at all possible combinations of $z - sequences$. This motivates the next conjecture.

Conjecture 1. By unlacing the $z - sequence$ of any n-boundary node critical circular planar graph, it is possible to find all connectivity types of a specific n-boundary node graph.

However, there is a more intuitive way of looking at the sets of connectivity types. Suppose we know that breaking a set of edges or contracting a set of boundary nodes, or even both, gives us a set of broken connections. Is there a way we can find a core group of broken connections that imply all other broken connections, specifically in the n-boundary node critical circular planar case? In other terms, can we find a basis of equations that imply a larger set of equations? This motivates the next group of terminologies.

Definition 3. An *algebraic variety* is a generalization to n dimensions of algebraic curves. An algebraic variety V is defined as the set of points in the reals satisfying a system of polynomial equations $f_i(x_1, ..., x_n) = 0$ for $i = 1, 2$,... formally written as $\mathrm{V} \subset \mathrm{R}^n$. According to the Hilbert Basis Theorem, a finite number of equations suffices.

We can think ot R^n as the space of entries in the upper diagonal of the response matrix, with n entries, corresponding to n edge conductivities. The

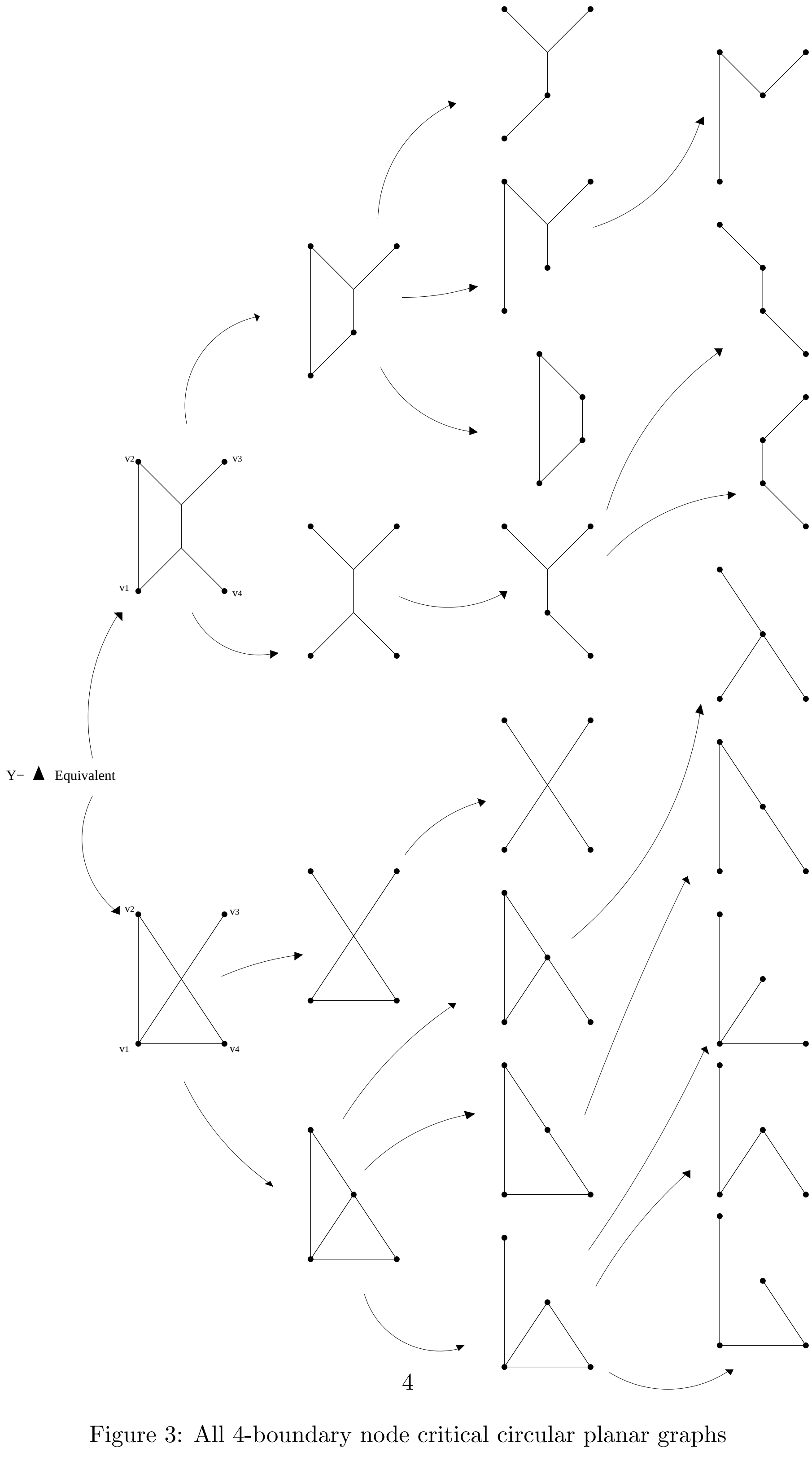

Figure 3: All 4-boundary node critical circular planar graphs

set V is the set of all edge conductivities remaining, where V = dim (m). The codimension, which is n-m, implies the set of edge conductivities that have been broken.

Definition 4. If the number of equations in an algebraic variety is equal to the codimension of V, then the space of response matrices forms a *complete intersection*.

The idea is that there is a basis of equations which spans a wider range of equations. In our graph case, by deleting three edges, we sometimes come up with five equations, ie, two determinantal conditions, which imply broken 2-connections or three λ conditions, which imply broken 1- connections. A complete intersection means that there exists a basis of three equations that represents these five conditions.

Theorem 2. For all four-boundary node critical circular planar graphs, there always exists a complete intersection.

Proof. The proof goes by way of exhaustion. I have exhausted all four boundary node critical circular planar graphs and found a basis for each one of them. (I will give one example of this in class)

□

Conjecture 2. For any two n-boundary node critical circular planar graph, if their "counts" of connections are the same, then either the two graphs are Y-Δ equivalent or are graph isomorphisms.

We know that if two graphs have the same connectivity types, then they are Y-Δ equivalent. However, if this conjecture holds true, then by looking at "counts" of connections we can determine a basis without having to look at all examples of graphs which have the same counts.

Conjecture 3. For any n-boundary node critical circular planar graph, there exists a complete intersection.

A good way to start looking at this problem is with the six-term identity. We can readily see that sets of broken 1-connections do not necessarily imply sets of broken 2 or higher connections.

References

[1] Curtis, Edward B., and James A. Morrow, *Inverse Problems for Electrical Networks*, World Scientific, Singapore, 2000.

[2] Gorodezky, Igor. Various Informal Talks, 2004.