UTILIZING EIGENVALUES AND EIGENVECTORS

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Abstract. In studying most matrices, examining the eigenvalue problem yields a great wealth of information. Likewise, in the inverse problem for electrical networks, the eigenvalues and eigenvectors contain data which characterizes the entire system. Using this idea, it is possible to make sense of concepts that appear in the continuous counterpart.

1. Introduction

The paper revolves around the comcept of solving inverse problems with eigenvalues and eigenvectors. As in any calculations involving matrices, examining the eigenvalue problem brings forth elegant and strightforward results, which reveals valuable information from another perpective.

In order to understand the process of recovering stars with the eigenvalue problem, it is first necessary to understand some of the properties of the response and Kirchoff Matrices.

1.1. Properties of Symmetric Matrices.

Observe that both the response and the Kirchhoff matrices are symmetric. Consequently, the two matrices have important properties illustrated in the following theorem:

Theorem 1.1. *Suppose we have a symmetric matrix M. Then M has the properties such that:*
1. All eigenvalues are real.
2. Eigenvectors corresponding to different eigenvalues are orthogonal.
3. The eigenvectors span a linearly-independent complete set, or form a basis.

1.2. Other Theorems.

In order to proceed with the eigenvalue method in the simplest manner, it is necessary to state the following theorems.

Theorem 1.2. *Given an orthonormal matrix N, the inverse matrix* $N^{-1} = N^T$.

Proof. Let N be represented in the form of vectors, or

$$N = [n_1, n_2, \cdots, n_k].$$

Then the tranpose of N can be written as

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$$N^T = \begin{bmatrix} n_1^T \\ n_2^T \\ \vdots \\ n_1^T \end{bmatrix}.$$

Multiplying the two yields

$$N^T N = \begin{bmatrix} n_1^T \\ n_2^T \\ \vdots \\ n_1^T \end{bmatrix} [n_1, n_2, \cdots, n_k].$$

Since $n_i^T n_j = \begin{cases} 1, & if\ i = j \\ 0, & if\ i \neq j \end{cases}$, we can conclude that $N^T N = I$.

□

In order to use Theorem(1.2), an orthogonal set must be given. Thus the following theorem ensures that orthogonality condition is satisfied.

Theorem 1.3. *Given a symmetric matrix M, it is possible to produce the eigenvectors in such a way as to obtain an orthonormal basis.*

Proof. For eigenvectors corresponding to different eigenvalues, Theorem(1.1) ensures that we obtain an orthogonal set. From statement three of Theorem (1.1), the multiplicity number of each eigenvalue equals the number of linearly independent eigenvectors. Thus applying the Gram Schmidt Orthogonalization process to the set, we obtain an orthogonal basis which can then be normalized.

□

2. The Three Boundary Node Network

With the theorems in the previous section, it is now possible to represent the response matrix in a form utilizing the eigenvalues and eigenvectors. Further, we will examine when the response matrix is valid.

2.1. **A Representation of the Response Matrix.**

In recovering the response matrix, we will make use the following eigenvalue problem:

$$\Lambda v = \mu v \tag{1}$$

where Λ is the response matrix, μ is an eigenvalue, and v is a normalized eigenvector. Since Λ has the property such that the row sum is zero, we immediately observe that one of the solution to (1) is $\mu = \mu_1 = 0$ with the corresponding eigenvector $v = v_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$. Now, let us assign the other eigenvalues and eigenvectors as $\mu = \mu_2$ with $v_2 = (a_1, a_2, a_3)$ and μ_3 with (b_1, b_2, b_3) such that the two vectors are normalized.

Let P and D be a matrices such that $P = [v_1, v_2, v_3]$ while D be a diagonal matrix with entries μ_1, μ_2, μ_3 in the respective order. Thus, the eigenvalue problem can be reformulated in the matrix form

$$\Lambda P = PD \tag{2}$$

Notice that P is an orthonormal matrix, possessing the property such that $P^{-1} = P^T$. Utilizing the orthonormality, we see that the response matrix can be written as

$$\Lambda = PDP^T \tag{3}$$

or

$$\Lambda = \begin{bmatrix} \frac{1}{\sqrt{3}} & a_1 & b_1 \\ \frac{1}{\sqrt{3}} & a_2 & b_2 \\ \frac{1}{\sqrt{3}} & a_3 & b_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

Computing the matrix multiplication term by term, we reach the following formula

$$\Lambda = \begin{bmatrix} a_1^2\mu_2 + b_1^2\mu_3 & a_1a_2\mu_2 + b_1b_2\mu_3 & a_1a_2\mu_2 + b_1b_3\mu_3 \\ a_1a_2\mu_2 + b_1b_2\mu_3 & a_2^2\mu_2 + b_2^2\mu_3 & a_2a_3\mu_2 + b_2b_3\mu_3 \\ a_1a_2\mu_2 + b_1b_3\mu_3 & a_2a_3\mu_2 + b_2b_3\mu_3 & a_2^2\mu_2 + b_2^2\mu_3 \end{bmatrix}. \tag{4}$$

2.2. **The Validity of the Response Matrix.** In order to confirm that (4) is indeed a valid representation, it is necessary to provide conditions to ensure the properties held by a response matrix. Thus, we will now present two ways to reach the conditional statements. The first approach will utilize the response matrix of form (4) to obtain three conditions with eight parameters. In contrast, the second method will use a slightly modified version of (4) to simply the three conditions using three parameters.

2.2.1. *The Three Conditions with Eight Parameters.* Observing that one of the eigenvalue is $\mu = 0$, it can be readily be seen that the row sums of (4) are zero. Thus let us proceed to check the sign of the diasgonal entries.
Since the eigenvalues are non-negative numbers, it follows that the diagonal terms are also positive. To ensure that the off-diagonals are negative, we find the following relations:

$$\begin{aligned} a_1a_2\mu_2 + b_1b_2\mu_3 &\leq 0 \\ a_1a_3\mu_2 + b_1b_3\mu_3 &\leq 0 \\ a_2a_3\mu_2 + b_2b_3\mu_3 &\leq 0 \end{aligned} \tag{5}$$

To simply the three relations , let us first examine a property of the eigenvectors. Recalling Theorem(1.3), we see that the vectors form an orthogonal set. Note that one of the eigenvector is $v = v_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$, which directly implies that the sum of the components of each eigenvector is equal to zero.

$$a_1 + a_2 + a_3 = 0 \tag{6}$$

$$b_1 + b_2 + b_3 = 0$$

Since the eigenvectors form an orthonormal set, we see that

$$a_1^2 + a_2^2 + a_3^2 = 1$$

Completing the square and using (6) yields

$$(a_1 + a_2 + a_3)^2 - 2(a_1a_2 + a_2a_3 + a_3a_1) = -2(a_1a_2 + a_2a_3 + a_3a_1) = 1$$

Applying the same procedure to v_3 and multiplying each of the equation with its respective eigenvalue, we arrive at

$$\mu_2(a_1a_2 + a_2a_3 + a_3a_1) = \frac{-1}{2}\mu_2$$
$$\mu_3(b_1b_2 + b_2b_3 + b_3b_1) = \frac{-1}{2}\mu_3$$

Adding the two equations, we obtain the following formula:

$$X + Y + Z = \frac{-1}{2}(\mu_2 + \mu_3) \tag{7}$$

where $X = a_1a_2\mu_2 + b_1b_2\mu_3$, $Y = a_2a_3\mu_2 + b_2b_3\mu_3$, $Z = a_1a_3\mu_2 + b_1b_3\mu_3$.
Recall that the row sums of the response matrix are zeroes. Thus, we see that

$$\begin{aligned} Z + X &= -(a_1^2\mu_2 + b_1^2\mu_3) \\ X + Y &= -(a_2^2\mu_2 + b_2^2\mu_3) \\ Y + Z &= -(a_3^2\mu_2 + b_3^2\mu_3) \end{aligned} \tag{8}$$

We see that X, Y, and Z must be negative for the response matrix to be valid. Now, taking the difference of (7) with (8), we arrive at a condition to ensure a valid Λ.

$$\mu_2(a_i^2 - \frac{1}{2}) + \mu_3(b_i^2 - \frac{1}{2}) \leq 0 \tag{9}$$

where i = 1, 2, 3.

2.2.2. *Reduction to Three Parameters.* Since the eigenvectors of the response matrix form an orthogonal set and have component sum equal to zero, it is natural to search for the existence of a condition using fewer variables in checking the validity of Λ. Before we proceed, we must first modifiy the eigenvectors to gain a representation in which the vectors are described by three parameters.
We see that setting the last components, or a_3 and b_3 as one will give us no loss of generality. By the property of component sums equaling to zeroes and setting a_1 as $x > 0$, we see that a_2 is $-(1 + x)$. Similarily, b_2 can be written as $-(1 + b_1)$. From the orthogonality property, we arrive at an equation relating the eigenvectors, where $b_1 = -\frac{2+x}{(1+2x)}$. Thus the eigenvectors, in the orthonormal form, are

$$a = \nu_1 \begin{bmatrix} x \\ -(1+x) \\ 1 \end{bmatrix}, b = \nu_2 \begin{bmatrix} b_1 \\ -(1+b_1) \\ 1 \end{bmatrix} \tag{10}$$

where $\nu_1 = \frac{1}{2(x^2+x+1)}$ and $\nu_2 = \frac{(2+x)^2}{6(x^2+x+1)}$ are normalization constants.
Using the eigenvectors (10) and following the same steps as in section 2.2.1, we see the following relations:

$$-\frac{1}{2})(\mu_2 + \mu_3) + \frac{\mu_2 a_i^2}{\nu_1} + \frac{\mu_3 b_i^2}{\nu_2} \leq 0 \tag{11}$$

for $i = 1, 2, 3$.

Using the single parameter x to represent the values of the eigenvectors, it is possible to derive the following sets of equations.

$$\frac{1}{2(x^2+x+1}[(-1)\mu_2(x+1) + (-1)\frac{\mu_3}{3}(2x^2 - x - 1)] \leq 0 \tag{12}$$

$$\frac{1}{2(x^2+x+1}[\mu_2(x) - \frac{\mu_3}{3}(2x^2 + 5x + 2)] \leq 0$$

$$\frac{1}{2(x^2+x+1}[(-1)\mu_2(x^2+x) + \frac{\mu_3}{3}(x^2 + x - 2)] \leq 0$$

2.2.3. *Symmetric Network.* From the representation in equation (12), it is possible to observe an interesting property. Taking a look at a symmetric network, or graphs with $\mu_2 = \mu_3$, we see that the substitution μ_2 with μ_3 for each of the expression in (12) yields

$$\frac{-1}{3}\mu_3 \leq 0 \tag{13}$$

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which is satisfied for all cases involving valid eigenvalues. Thus for a three boundary node network with symmetric conductivities, the eigenvector and eigenvalue pair can be constructed with equation (10) to ensure a valid response matrix for all cases.

2.2.4. *The Single Condition.* Examining equation (12), observe that the factor $\frac{1}{2(x^2+x+1)}$ can be cancelled out since $x \geq 0$. Thus we will examine the simplified version of (12).

$$[(-1)\mu_2(x+1) + (-1)\frac{\mu_3}{3}(2x^2 - x - 1)] \leq 0 \tag{14}$$

$$[\mu_2(x) - \frac{\mu_3}{3}(2x^2 + 5x + 2)] \leq 0$$

$$[(-1)\mu_2(x^2 + x) + \frac{\mu_3}{3}(x^2 + x - 2)] \leq 0$$

Now let us find a relation to relate the two eigenvalues μ_2 and μ_3 by shifting one of the terms in each of the the inequalities to the other side. With simple manipulations to obtain a relation with μ_2 being represented by the parameters $x > 0$ and μ_3, we find that

$$\mu_2 \geq \frac{\mu_3}{3}[\frac{-2x^2 + x + 1}{x + 1}] \tag{15}$$

$$\mu_2 \leq \frac{\mu_3}{3}[\frac{(2x + 1)(x + 2)}{x}] \tag{16}$$

$$\mu_2 \geq \frac{\mu_3}{3}[\frac{(x - 1)(x + 2)}{(x)(x + 1)}] \tag{17}$$

.

Observe that (16) gives the upper bound for μ_2 while the inequalities (15) and (17) give the lower bounds. For later purposes, we will introduce the following notation: $A = \frac{\mu_3}{3}[\frac{-2x^2+x+1}{x+1}]$, $B = \frac{\mu_3}{3}[\frac{(2x+1)(x+2)}{x}]$, and $C = \frac{\mu_3}{3}[\frac{(x-1)(x+2)}{(x)(x+1)}]$. Now, let us compare (15) and (17).

Suppose $A \geq C$. Then, by some algebra, we find that $(2x^3 + 3x^2 + 3x - 2) \leq 0$. Similarily, if we consider $A \leq C$, we obtain $(2x^3 + 3x^2 + 3x - 2) \geq 0$.

Examination of the polynomial $(2x^3 + 3x^2 + 3x - 2)$ yields that there exists only one real root, which by numerical calculation is given by $x = x_c = .42944454\cdots$. Thus, the *Critical Point* for the system occurs at $x = x_c$.

Summing up the results, we arrive at the following condition that will ensure a valid response matrix.

$$\Lambda \ valid \ <=> \begin{cases} \frac{\mu_3}{3}[\frac{(x-1)(x+2)}{(x)(x+1)}] \leq \mu_2 \leq \frac{\mu_3}{3}[\frac{(2x+1)(x+2)}{x}], & if\ x \leq x_c \\ \frac{\mu_3}{3}[\frac{-2x^2+x+1}{x+1}] \leq \mu_2 \leq \frac{\mu_3}{3}[\frac{(2x+1)(x+2)}{x}], & if\ x \geq x_c \end{cases} \tag{18}$$

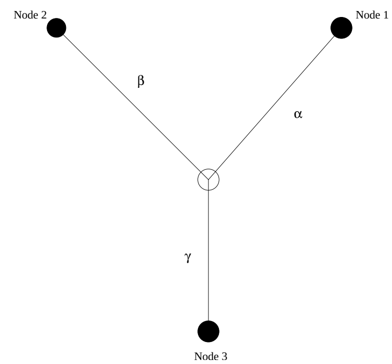

FIGURE 1. Three Node Star Graph

2.3. **Recovering the Conductivities for Y-Graph.** With the response matrix in the form of (4), it is possible to find the conductivities of the Y-Graph using only the values contained in the eigenvalues and eigenvectors.
To recover the conductivities of the network, the following relations found in [1] can be used

$$\alpha = -\lambda_{1,1} + \lambda_{1,2}(\lambda_{2,3})^{-1}\lambda_{3,1} \tag{19}$$

$$\beta = -\lambda_{2,2} + \lambda_{2,3}(\lambda_{3,1})^{-1}\lambda_{1,2}$$

$$\delta = -\lambda_{3,3} + \lambda_{3,1}(\lambda_{1,2})^{-1}\lambda_{2,3} -$$

Through explicit calculations, (19) reduces to the following which gives a symmetric representation in terms of eigenvalues and eigenvectors.

$$\alpha = -\frac{\mu_2\mu_3}{a_2a_3\mu_2 + b_2b_3\mu_3}[[a_1^2(b_2b_3) + (a_2a_3)b_1^2] - [a_1b_1(a_2b_3 + a_3b_2)]] \tag{20}$$

$$\beta = -\frac{\mu_2\mu_3}{a_1a_3\mu_2 + b_1b_3\mu_3}[[a_2^2(b_1b_3) + (a_1a_3)b_2^2] - [a_2b_2(a_1b_3 + a_3b_1)]]$$

$$\delta = -\frac{\mu_2\mu_3}{a_1a_2\mu_2 + b_1b_2\mu_3}[[a_3^2(b_1b_2) + (a_1a_2)b_3^2] - [a_3b_3(a_1b_2 + a_2b_1)]]$$

3. THE GENERAL N BOUNDARY NODE NETWORK

In the previous sections, we introduced the readers to the case with three boundary nodes. Now, let us generalize the results to a N boundary node system.

3.1. **The Response Matrix.** Following the steps in section 2.1, we see that the Response Matrix can be represented as $\Lambda = PDP^{-1}$. To see how Λ is constructed by the entries of μ and $\hat{v}$, we will introduce the following notation:

$$\hat{v_1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \hat{v_2} = \begin{bmatrix} v_{2,1} \\ v_{2,2} \\ \vdots \\ v_{2,N} \end{bmatrix}, \ \cdots, \ \hat{v_N} = \begin{bmatrix} v_{N,1} \\ v_{N,2} \\ \vdots \\ v_{N,N} \end{bmatrix},$$

with corresponding eigenvalues $\mu_1, \ \mu_2, \cdots,$ and μ_N. Then the entries of the Response Matrix can be written as follows:

$$\lambda_{i,j} = \begin{cases} \sum_{k=2}^{N} \mu_k \ v_{k,i}^2, & \text{if } i = j \\ \\ \sum_{k=2}^{N} \mu_k \ v_{k,i} \ v_{k,j}, & \text{if } i = j. \end{cases} \tag{21}$$

3.2. **Vaildity of the Response Matrix.** To ensure that the NxN matrix is a valid response, let us take a look at the row sums. Since each of the vector component sum is zero, or $\sum_{k=1}^{N} v_{i,k} = 0$, the row sums of the matrix equal to zero, which can be shown by explicit computation.

The second condition requires that the diagonal entries are non-negative and the off-diagonal values to be non-positive. Since the first case of (21) involves only the combinations of the eigenvalues and the squares of the eigenvector components, the diagonal terms are indeed non-negative for all choices of v's and μ's.

Note the normal property $\sum_{k=1}^{N} v_{i,k}^2 = 1$. By completing the squares and multiplying the corresponding μ for each $i = 2, 3 \cdots, N$, we obtain the following inequalities:

$$\frac{-1}{2}\left(\sum_{k=2}^{N} \mu_k\right) + \sum_{k=2}^{N} \mu_k v_{k,i}^2 \leq 0 \tag{22}$$

for $i = 1, 2, \cdots, N$.

4. The Green's Function

Like in the continuous case, an analogue to the Green's function can be understood for the discrete problems. In Karen Perry's paper [3], the Green's function is presented in the following manner:

$$G = C^{-1} \tag{23}$$

where $K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$.

Now let us proceed with finding a spectral, or the eigenvalue/eigenvector representation for the Green's function given in (23).

4.1. **The Kirchhoff Matrix and the M-Interior/N-Boundary Node Network.**

Since the Green's function is associated with the interior nodes of the network, it is natural to work with the Kirchhoff matrix instead of the response matrix. However, given that the eigenvalues and the eigenvectors given are those associated with the Kirchhoff matrix, the same arguement given in section 3 can be given. Thus let us introduce the following eigenvalues and the eigenvectors.

$$\hat{w}_i = \begin{bmatrix} \vec{U_i} \\ \vec{I_i} \end{bmatrix} \text{ with } \phi_i \tag{24}$$

where $\vec{U_i}$ is a Nx1 vector that correspond to the potentials on the boundary nodes and $\vec{I_i}$ a Mx1 vector incorporating the voltages at the interior nodes. Here, we note that w_i is a normalized.

Using the pair given in (24), we see that the Kirchhoff matrix is represented as

$$K = \begin{bmatrix} \sum_{i=2}^{N+M} \phi_i \vec{U_i} \vec{U_i}^T & \sum_{i=2}^{N+M} \phi_i \vec{U_i} \vec{I_i}^T \\ \sum_{i=2}^{N+M} \phi_i \vec{I_i} \vec{U_i}^T & \sum_{i=2}^{N+M} \phi_i \vec{I_i} \vec{I_i}^T \end{bmatrix} \tag{25}$$

4.2. **The First Spectral Representation of the Green's Function.**

With the Kirchhoff matrix in the form of (25), we can now procceed to show the first spectral analogue of the Green's function for a M-interior node network. By (25), we see that

$$C = \sum_{i=2}^{N+M} \phi_i \vec{I_i} \vec{I_i}^T \tag{26}$$

Thus the Green's function is given by

$$G = [\sum_{i=2}^{N+M} \phi_i \vec{I_i} \vec{I_i}^T]^{-1} \tag{27}$$

.

4.3. **The Second Spectral Representation of the Green's Function.** To find the second representation of the Green's Function, we will use the property such that the boundary voltages are zeroes. Thus, the only relevent information is contained in the interior nodes.

Let $\vec{z_i}$ and ψ_i for $i = 1, 2, \cdots, M$ be the eigenvector, eigenvalue pair for matrix C. Then C can be written as

$$C = \sum_{i=2}^{M} \psi_i \vec{z_i} \vec{z_i}^T \tag{28}$$

Thus, we see that the Green's function is represented as

$$G = [\sum_{i=2}^{M} \psi_i \vec{z_i} \vec{z_i}^T]^{-1} \tag{29}$$

5. Conclusion

In examining the roles of the eigenvector/eigenvalue pairs in the response matrix in, it is possible to understand exactly how powerful and useful the eigenvalue problem is. Since the cases studied can be applied to any network with the specified number of nodes, it can be concluded that the eigenvalues and the eigenvectors contain the entire information about a network.

6. Future Work

For future studies, he following two problems may yield interesting results:

1.To find a single condition for the generalized N node network to satisfy the characterization of the response matrix.
2.To find the relationship between the eigenvalues and eigenvectors of (27) and (29).

References

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[2] Curtis, Edward and Morrow, James, *Inverse Problems for Electrival Networks*, Worl Scientific, Singapore, 2000.
[3] Perry, Karen, "Discrete Complex Analysis".