ON EMBEDDINGS OF CIRCULAR PLANAR GRAPHS

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Abstract. This paper considers some elementary graph-topological properties of circular planar, critical graphs, with a focus on finding all possible distinct embeddings of a given graph of this type. Methods used are both topological and combinatorial. We prove that the problem of finding all possible topologically distinct embeddings is equivalent to the problem of finding all possible orderings of boundary vertices around the boundary circle. We then show that the existence of a cutvertex is a necessary and sufficient condition for the existence of multiple distinct embeddings, and give a naive algorithm for generating all possible distinct embeddings from a single embedding. Finally, these results are related to a theorem of Perry ([5]).

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1. Introduction

There has been a significant amount of recent research aiming to supplement the study of geometric properties of electrical networks, such as $Y-\Delta$ equivalence, with the study of their graph-topological properties (see, for instance, [8] and [9]). This paper is an attempt to contribute to this effort by investigating planar embeddings of graphs that represent one well studied type of electrical network.

Planar embeddings of graphs have historically received a great deal of attention; relevant examples are papers by MacLane ([3]) and Stallmann ([6]). Most previous results, however, are concerned with the general problem of embedding graphs in the plane, while the present paper addresses the question of planar embeddings subject to a particular type of constraint.

We first introduce some of the basic vocabulary of graph theory.

Definition 1.1. A *graph* G is an ordered pair of sets (V, E), where V is finite and E is a subset of the set $V^{(2)}$ of unordered pairs of V. V is the set of *vertices* and E is the set of *edges*.

If $(u, v) \in E$ for some $u, v \in V$, we say that u and v are *adjacent* and that they are *endvertices* of an edge e in E. If we wish to explicitly identify the endvertices of some $e \in E$, call them u and v, we will write e as uv, or equivalently vu. The *degree* of a vertex v is the number of distinct vertices adjacent to v.

The set of vertices of a particular graph G will be written $V(G)$, and the set of edges of G will be written $E(G)$. If the graph in question is clear from context, we will simply write V and E for these sets.

Given a graph $G = (V, E)$, we will often partition V into two sets ∂V and $int\, V$, so that $\partial V \cup int\, V = V$, $\partial V \cap int\, V = \emptyset$, and $|\partial V| \geq 1$. We will call ∂V the set of *boundary vertices*, or simply *boundary of* G, and $int\, V$ will be the set of *interior vertices*, or simply *interior of* G.

Definition 1.2. G' is a *subgraph* of G if $V(G') \subset V(G), E(G') \subset E(G)$, and the endpoints of all $e' \in E(G')$ are in $V(G')$.

Given a graph G, let V' be a subset of $V(G)$ and let E' be the set of all edges with endvertices in V'. Then the subgraph G' for which $V(G') = V'$ and $E(G') = E'$ is a subgraph of G *induced by* V', written $G' = G(V')$.

Definition 1.3. Let G be a graph, and let p, q be two distinct vertices in V. A *path from* p *to* q in G is a subgraph P of G such that $V(P) = \{p = v_1, v_2, \ldots, v_k = q\}$, with all vertices in $V(P)$ distinct, and $E(P) = \{pv_2, v_2v_3, \ldots, v_{k-1}q\}$. We call p and q the *endpoints* of the path. Two paths P_1 and P_2 are *disjoint* if $V(P_1) \cap V(P_2) = \emptyset$.

If A and B are subsets of vertices, each of order n, such that there exists a path from a_i to b_i for $1 \leq i \leq n$, and furthermore all of these paths are disjoint, then we say that there exists an *n-path from A to B*.

Definition 1.4. A graph G is *connected* if for every two distinct vertices $u, v \in V$ there is a path from u to v, or equivalently, from v to u. A graph that is not connected is *disconnected*.

Let G be a disconnected graph, and take W a nonempty subset of V such that there is a path between any two vertices in W. Then $G(W)$ is a connected subgraph of G. If W is such that adding any vertex to W makes $G(W)$ disconnected, we say that $G(W)$ is a *component* of G. Unless specified otherwise, all graphs discussed in this paper are assumed to be connected.

2. Embeddings and Contiguity Relations

2.1. Embeddings.

Definition 2.1. A *planar embedding* of a graph $G = (V, E)$ is a function σ from G to the plane that assigns to each $v \in V$ a point $\sigma(v)$ in the plane, and to each $e = pq \in E$ a simple, continuous curve $\sigma(e)$ in the plane from $\sigma(p)$ to $\sigma(q)$ (or $\sigma(q)$ to $\sigma(p)$). Furthermore, $\sigma(u) \neq \sigma(v)$ if $u \neq v$ for $u, v \in V$, and if e and f are distinct edges in E, then $\sigma(e)$ and $\sigma(f)$ do not intersect, except possibly at their endpoints. A graph is called *planar* if it has a planar embedding.

Definition 2.2. Let σ and τ be two planar embeddings of a graph G. σ and τ are *topologically identical* if there exists a homeomorphism η of the plane such that either η or η composed with a reflection of the plane maps $\sigma(v)$ to $\tau(v)$ and $\sigma(e)$ to $\tau(e)$, for all $v \in V, e \in E$. If two embeddings are not topologically identical they are *topologically distinct*.

Observe that given a planar embedding, another planar embedding is defined by reflecting the original embedding in the plane. For the purposes of this paper, the embeddings in such a pair will be always considered topologically identical; hence the reference to a reflection of the plane in the definition above.

Remark 2.3. Our definition of a planar embedding is quite general since an edge can be mapped to any simple and continuous curve. It is a fact, however, that every planar embedding is topologically equivalent to a planar embedding in which all of the curves are straight line segments (see, for instance, [1]). It is thus possible to assume without loss of generality that a given planar embedding maps edges to straight line segments.

Of special significance in the theory of electrical networks are planar embeddings whose image lies in a disk, and which map boundary vertices to the boundary of that disk. For the remainder of this paper, $\mathbf{D}$ will denote an open disk in the plane, with $\partial\mathbf{D}$ as its boundary. We will sometimes refer to $\partial\mathbf{D}$ as the *boundary circle*.

Definition 2.4. A *circular planar embedding* of a graph G, where $V = \partial V \cup int\, V$, is a planar embedding σ that sends boundary vertices to points in $\partial\mathbf{D}$ and sends interior vertices to points in $\mathbf{D}$. If e is an edge in E, $\sigma(e)$ is a subset of $\mathbf{D}$ and does not intersect $\partial\mathbf{D}$ except possibly at its endpoints. A graph is called *circular planar* if it has a circular planar embedding.

Definition 2.5. Let σ and τ be two circular planar embeddings of a graph G. σ and τ are *topologically identical* if there exists a homeomorphism η from $\mathbf{D} \cup \partial\mathbf{D}$ to itself such that either η or η composed with a reflection of $\mathbf{D}$ in the plane maps $\partial\mathbf{D}$ to $\partial\mathbf{D}$, and also maps $\sigma(v)$ to $\tau(v)$ and $\sigma(e)$ to $\tau(e)$, for $v \in V, e \in E$. If two embeddings are not topologically identical they are *topologically distinct*.

Since this paper deals strictly with circular planar embeddings, we will henceforth refer to a circular planar embedding simply as an *embedding*. A planar embedding that is not circular will be explicitly identified as such. When a particular embedding σ is clear from context, we will often refer to the image of a vertex v or edge e as simply v and e, respectively, and not $\sigma(v)$ and $\sigma(e)$.

We will generally differentiate between embeddings only up to the conditions of Definition 2.5; that is, we will not differentiate between topologically identical embeddings. Thus it will be useful to refer to equivalence classes of embeddings under homeomorphism (possibly composed with a planar reflection), which are finite in number, rather than to the individual embeddings, of which there are uncountably many. We term these *embedding classes*. When it must be explicitly identified, an embedding class may be represented by any one of its constituents.

2.2. **Boundary Contiguity.**

Remark 2.6. Fix an embedding of some circular planar graph G for which $|\partial V| > 1$. Consider two distinct boundary vertices v and u. We will often reference the fact that v and u partition $\partial\mathbf{D}$ into two arcs $\mathbf{A}_1$ and $\mathbf{A}_2$, so that $\partial\mathbf{D} = \mathbf{A}_1 \cup \mathbf{A}_2 \cup \{v\} \cup \{u\}$.

Definition 2.7. Let G be a circular planar graph with $|\partial V| > 1$, and fix an embedding σ of G. The images of two boundary vertices u and v partition $\partial\mathbf{D}$ into two arcs $\mathbf{A}_1$ and $\mathbf{A}_2$ as in the Remark above. v and u are said to be *contiguous with respect to* σ if either $\mathbf{A}_1$ or $\mathbf{A}_2$ contains the image of no other boundary vertex. We denote contiguity by $v \sim u$.

An embedding of a circular planar graph gives a non-transitive relation $\sim$ on the set of boundary vertices which we will term a *contiguity relation*. A particular contiguity relation will be denoted by Θ (consider Θ to be an ordered subset of V^2 such that $v \sim u$ if and only if $(v, u) \in \Theta$). Now, fix once and for all an orientation around the boundary circle, say clockwise. Further, fix a labeling of the set of boundary vertices $v_1, v_2, \dots, v_n$, where $n = |\partial V|$. Given a contiguity relation, the boundary vertices may be listed in the clockwise order they appear around $\partial\mathbf{D}$; we call this a *circular ordering*. A given contiguity relation gives $2n$ possible circular orderings of the boundary vertices, since the listing can begin with any of the n boundary vertices, and any circular ordering can be reversed to give another circular ordering (corresponding to a reflection in the plane). If it becomes necessary to explicitly exhibit the contiguity relation generated by a particular embedding, we will do so by writing one of the $2n$ possible circular orderings as a sequence $\theta = v_1 v_2 \dots v_n$ (it is clear that any such θ completely determines a contiguity relation).

Remark 2.8. Let G be a circular planar graph with $|\partial V| \geq 4$, and fix an embedding σ of G. Two boundary vertices u and v partition $\partial\mathbf{D}$ into two arcs $\mathbf{A}_1$ and $\mathbf{A}_2$ as in Remark 2.6. Observe that if $v \nsim u$ with respect to the contiguity relation given by this particular embedding, then there must exist at least one boundary vertex with its image in $\mathbf{A}_1$, and there also must exist at least one boundary vertex with its image in $\mathbf{A}_2$. This fact will be often used, though rarely cited.

From its very definition, an embedding of a circular planar graph G uniquely specifies a contiguity relation on the boundary vertices of G. Furthermore, all embeddings in a given embedding class give the same contiguity relation. It is possible, however, that two topologically distinct embeddings will produce the same contiguity relation (see, for example, Fig. 1). That is, two embedding classes correspond to the same contiguity relation. There is a class of circular planar graphs, introduced in the next section, for which the possible contiguity relations on G are in one-to-one correspondence with the embedding classes of G.

2.3. **Critical Graphs.**

Definition 2.9. Let G be a graph, and let p, q be two distinct vertices in V. A *connection from p to q* is a path P from p to q with the additional condition that if $v \in V(P)$ such that $v \neq p$ and $v \neq q$, then $v \in \mathit{int}\, V$.

Let G be a circular planar graph, fix an embedding of G, and consider two subsets of ∂V, $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$, $k > 1$, such that given any two distinct $p_i, p_j \in P$, all $q \in Q$ lie on one of the two arcs into which the images of p_i and p_j partition $\partial \mathbf{D}$. We call P and Q a *k-circular pair*, or simply *circular pair* if the cardinality of P and Q is clear from context. Given a circular pair P and Q, a *k-connection* from P to Q is a set of k disjoint connections, each from a vertex in P to a vertex in Q.

Definition 2.10. Given a graph G, we define two ways of *removing* an edge e from E. If both endvertices of e are in ∂V, we may simply *delete* the edge by removing it from the set E. Otherwise, we *contract* the two endvertices of e; if $e = uv$, remove e from E, remove v from V, and make adjacent to u all vertices that were adjacent to v.

Definition 2.11. A circular planar graph G is *critical* if removing any edge breaks some connection through G.

Though the above definitions require a fixed embedding of G, Proposition 2.13 below demonstrates that criticality is nonetheless a graph-topological property of circular planar graphs; that is, two topologically distinct embeddings of a circular planar graph possess the same k-connections (they are, using terminology from [9], of the same connectivity type). In proving this result (and a few others) we will make use of the following consequence of the Jordan Curve Theorem.

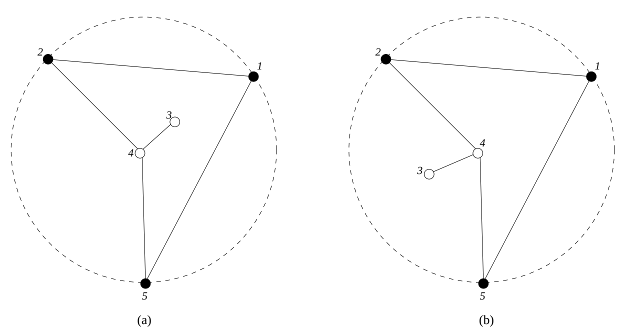

FIGURE 1. A graph for which two distinct embeddings give the same contiguity relation. This graph is not critical.

Lemma 2.12. *Given a circular planar graph G with $|\partial V| \geq 4$, let u_1, u_2, v_1, v_2 be four distinct boundary vertices, and observe that given an embedding σ of G, u_1 and u_2 partition $\partial\mathbf{D}$ into two arcs $\mathbf{A}_1$ and $\mathbf{A}_2$ as in Remark 2.6. Assume that there exist two disjoint paths P_u and P_v, where P_u is a path from u_1 to u_2 and P_v is a path from v_1 to v_2. Then either $v_1, v_2 \in \mathbf{A}_1$ or $v_1, v_2 \in \mathbf{A}_2$.*

Proof. Let the image under σ of a path in G be the union of the images of all edges and vertices that the path contains, so that the images of P_u and P_v, denoted $\sigma(P_u)$ and $\sigma(P_v)$, are continuous simple curves through $\mathbf{D}$ with endpoints in $\partial\mathbf{D}$. By the Jordan Curve Theorem, $\sigma(P_u)$ partitions $\mathbf{D}$ into two disconnected regions, one with boundary $\sigma(P_u) \cup \mathbf{A}_1$ and the other with boundary $\sigma(P_u) \cup \mathbf{A}_2$ (see Fig. 2). Call these region I and region II, respectively.

Assume that v_1 and v_2 are on the boundaries of different regions: $v_1 \in \mathbf{A}_1$, say, and $v_2 \in \mathbf{A}_2$. Then $\sigma(P_v)$ is a path that connects points in region I with points in region II since it must pass through each region. Since the two regions are disconnected they are pathwise disconnected, so a continuous simple curve that begins in one region and ends in the other must intersect the boundary. Consequentially, $\sigma(P_v)$ intersects $\sigma(P_u)$, which implies that the images of P_u and P_v intersect at the image of a vertex (since they cannot intersect at the image of an edge), which implies in turn that they share a vertex. This contradicts our assumption that P_u and P_v are distinct, which implies that v_1 and v_2 cannot be on the boundaries of different regions. That is, either $v_1, v_2 \in \mathbf{A}_1$ or $v_1, v_2 \in \mathbf{A}_2$. □

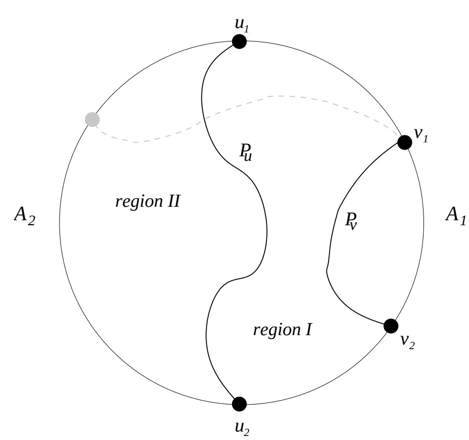

FIGURE 2. The images of u_1 and u_2 partition $\partial\mathbf{D}$. The gray path is forbidden, since a path starting at v_1 may not terminate on $\mathbf{A}_2$; it must terminate on $\mathbf{A}_1$.

Proposition 2.13. *Let σ and τ be two embeddings of a circular planar graph G. Assume that there exists, with respect to σ, a k-connection between a circular pair P and Q. Then there exists a partition of $P \cup Q$ into two sets R and S such that with respect to τ, these sets are a k-circular pair and a k-connection exists between them.*

Proof. The proof is by induction on k. If $k = 1$, then the hypothesis states that there exists a connection between the pair of boundary vertices in question, which is a graph-topological fact and hence must remain true with respect to all embeddings of G. Now, assume that P and Q are an h-circular pair for some $h > 1$, and assume the proposition true for $k < h$. Label the vertices of P and Q so that they appear in the circular order $p_1, p_2, \ldots, p_h, q_h, \ldots, q_1$ around $\partial\mathbf{D}$ with respect to σ. Note that with this notation, the h-connection connects each p_i to q_i.

Consider the sets $P' = \{p_1, \ldots, p_{k-1}\}, Q' = \{q_1, \ldots, q_{k-1}\}$, subsets of P and Q, respectively. P' and Q' are an $(h-1)$-circular pair with an $(h-1)$-connection between them, so by the inductive hypothesis $P' \cup Q'$ can be partitioned into two sets R' and S' such that with respect to τ, these sets are an $(h-1)$-circular pair and an $(h-1)$-connection exists between them.

Now, label the vertices of R' and S' so that they appear in the circular order $r'_1, r'_2, \ldots, r'_{h-1}, s'_{h-1}, \ldots, s'_1$ around $\partial\mathbf{D}$ with respect to τ. It remains to show that τ maps the vertices p_k and q_k unto $\partial\mathbf{D}$ in a way that allows the definition of an h-circular pair. The boundary vertices $r'_1, r'_{h-1}, s'_{h-1}, s'_1$ partition $\partial\mathbf{D}$ into four arcs; denote the arc between vertices x and y as $\mathbf{A}(x, y)$. By the inductive assumption, there exist disjoint connections between the pair of vertices r'_1, s'_1 and the pair of vertices p_k, q_k. By Lemma 2.12, either $p_k, q_k \in \mathbf{A}(r'_1, s'_1)$ or $p_k, q_k \in \mathbf{A}(r'_1, r'_{h-1}) \cup \mathbf{A}(r'_{h-1}, s'_{h-1}) \cup \mathbf{A}(s'_{h-1}, s'_1)$. There also exist disjoint connections between the pair of vertices r'_{h-1}, s'_{h-1} and the pair of vertices p_k, q_k, so again by Lemma 2.12, either $p_k, q_k \in \mathbf{A}(r'_{h-1}, s'_{h-1})$ or $p_k, q_k \in \mathbf{A}(s'_{h-1}, s'_1) \cup \mathbf{A}(s'_1, r'_1) \cup \mathbf{A}(r'_1, r'_{h-1})$.

In other words, it must be the case that either $p_k, q_k \in \mathbf{A}(r'_1, s'_1)$ or $p_k, q_k \in \mathbf{A}(r'_{h-1}, s'_{h-1})$. If the former, then the vertices in $R' \cup S' \cup \{p_k, q_k\}$ appear in the circular order $q_k, r'_1, \ldots, r'_{h-1}, s'_{h-1}, \ldots, s'_1, p_k$ (assuming, without loss of generality, that p_k precedes q_k clockwise around $\partial\mathbf{D}$). Thus $R = R' \cup \{q_k\}$ and $S = S' \cup \{p_k\}$ are an h-circular pair with an h-connection between them. If the latter, then the vertices appear in the circular order $r'_1, \ldots, r'_{h-1}, p_k, q_k, s'_{h-1}, \ldots, s'_1$, so that $R = R' \cup \{p_k\}$ and $S = S' \cup \{q_k\}$ are an h-circular pair with an h-connection between them. In either case the claim is true for h, completing the proof. □

Critical graphs are of great significance in the theory of electrical networks; it is always possible to solve the Dirichlet problem on an electrical network that is represented by a circular planar, critical graph (see [2]). They also possess the following important graph-topological property.

Theorem 2.14. *Suppose G is a circular planar, critical graph. If two embeddings of G give the same contiguity relation on ∂V then they are topologically identical.*

Proof. Fix a contiguity relation Θ, and let σ and τ be embeddings of G that both give Θ. We will show that σ and τ are topologically identical by induction on $|int\, V|$.

If $|int\, V| = 0$, an embedding of G is completely determined by the embedding of ∂V on $\partial\mathbf{D}$; it is easily shown (by an induction on $|\partial V|$, for instance) that for a graph with no interior vertices, two embeddings that give the same contiguity relation must be topologically identical.

Now suppose that $|int\, V| = k$, and assume the claim true for $k-1 \geq 0$. From [2] we know that since G is critical and has at least one interior vertex, there is at least one $e = uv \in E$ that satisfies one of the following two conditions:

(1) u and v are boundary vertices.
(2) u is an interior vertex and v is a boundary vertex adjacent to no vertices but u.

Furthermore, if an edge satisfies one of these conditions and is removed as per Definition 2.10, the resulting graph is still critical. It immediately follows that it is possible to generate a graph G' from G, such that G' is critical and has an edge e that satisfies (2), by deleting a finite number of edges in G that satisfy (1). Now contract e to get a new graph G''; G'' is critical and $|int\, V(G'')| = k-1$. Thus, our strategy will be to establish relationships between embedding classes of G, G', and G'' in order that the inductive hypothesis applied to G'' should imply the claim we are seeking to prove for G.

Let us first consider embeddings of G'. G' is a subgraph of G, so it follows that an embedding of G restricts to an embedding of G'; if e is the edge to be deleted then restrict the embedding to the edges $E(G) - \{e\}$ instead of the entirety of $E(G)$ (all of $V(G)$ is still mapped since $V(G) = V(G')$). This restriction is unique up to homeomorphism in the following sense. Let μ and ν be two embeddings of G that restrict to embeddings of G', call them μ' and ν', respectively.

Assume now that μ' and ν' are topologically identical, that is, there exists a homeomorphism of the plane from μ' to ν' (where we implicitly refer to the image of the graph under the embedding). We claim that μ and ν are topologically identical as well. To see this, first consider some embedding of G and note that restricting it to an embedding of G' creates an empty cell inside $\mathbf{D}'$ (the disk that contains the image of G') that previously contained the image of the deleted edge e and whose boundary contains the vertices that were the endpoints of e, call them t and w. By drawing an arbitrary simple continuous curve from t to w and associating it with the edge e, we create an embedding of G from an embedding of G', one that is homeomorphic to the original embedding of G (the one that begat the embedding of G'). This is illustrated in Fig. 3.

To see why the two embeddings, call them ρ_{old} and ρ_{new}, where ρ_{old} is the original embedding, are homeomorphic, first note that it is a fact that given two disks in the plane, each with a simple continuous curve in its interior, there exists a homeomorphism between the disks that maps one curve to the other. Now, ρ_{old} defines a cell in $\mathbf{D}$ that contains $\rho_{old}(e)$, and ρ_{new} defines a cell in $\mathbf{D}$ that contains $\rho_{new}(e)$. Both these cells are homeomorphic to the disk, so we see that by the above fact there exists a homeomorphism between them. ρ_{old} and ρ_{new} differ only in the interior of the cell containing the image of e, so this homeomorphism of cells is actually a homeomorphism between the embeddings ρ_{old} and ρ_{new}, making them topologically identical.

Returning to μ' and ν', we see that we can extend them as in the discussion above to embeddings of G, call these $\bar{\mu}$ and $\bar{\nu}$, respectively. It is the case that $\bar{\mu}$ is topologically identical to $\bar{\nu}$, because there exists a homeomorphism between the cell of the image of of $\bar{\mu}$ containing e and the cell of the image of of $\bar{\mu}$ containing e; composing this homeomorphism with the homeomorphism between μ' and ν'

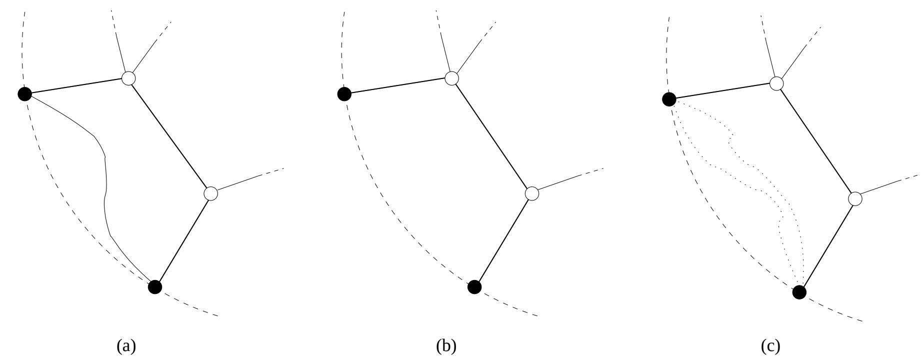

FIGURE 3. (a) An edge between two boundary vertices inside a cell. The cell is bordered by the edges drawn in bold and by an arc of the boundary circle. (b) The edge is removed to give an embedding of a subgraph, which can in turn (c) be extended to an embedding of the original graph with the addition of a curve between the boundary vertices. The embeddings in (a) and (c) are topologically identical.

gives us a homeomorphism between $\bar{\mu}$ and $\bar{\nu}$. But by the discussion above, $\bar{\mu}$ is topologically identical to μ and $\bar{\nu}$ is topologically identical to ν. Thus we see that if μ' and ν' are topologically identical, then μ and ν are topologically identical as well.

Note also that using the observations above it is easy to prove that the contiguity relation on G with respect to some embedding ρ is the same as the contiguity relation on G' with respect to the restriction of ρ. This will be an important element of our proof.

The relationship between embedding classes of G' and embedding classes of G'' is stronger than the one between embedding classes of G and embedding classes of G' above. In particular, there is a bijection between embedding classes of G' and those of G''. To show this, assume that the edge to be contracted in G' is $e = uv$, where u is the interior vertex and v is the boundary vertex. Given an embedding γ of G' and assuming it maps edges to straight line segments, define a subregion $\mathbf{D}'$ of $\mathbf{D}$ as all of $\mathbf{D}$ excepting the interior of the cell, with respect to γ, that contains e (see Fig. 4). By choosing a point d in $\mathbf{D}'$ colinear with e and assuming that $\mathbf{D}'$ is star shaped with respect to this point, it is possible to construct a straight line homotopy that sends the image of u to the image of v, thus producing an embedding of G''.

Intuitively, what happens is the following. The image of u is mapped along the straight line image of e, with its path parametrized so that it is at $\gamma(u)$ when $t = 0$ and at $\gamma(v)$ when $t = 1$. Now, consider any point z that lies on the image under γ of an edge that has u as an endpoint. Draw a line from d to $\partial\mathbf{D}$ that passes through z, and map z along that line, from $t = 0$ to $t = 1$.

This is formalized as follows (henceforth, referencing a vertex or and edge will imply referencing its image under γ). Assume that γ maps e to the parameterized straight line segment $f_v(t) = v(1-t) + ut$, $0 \leq t \leq 1$, from v to u. Now, let W be the set of vertices in $V(G')$ adjacent to u, excluding v, and assume that for $v_i \in W, 1 \leq i \leq |W|$, the edge uv_i is mapped to the parameterized straight line segment $f_i(t) = v_i(1-t) + ut$, $0 \leq t \leq 1$.

First, consider the homotopy that sends e to the point v (that is, sends e to progressively shorter straight lines), given by

$$H_v(s,t) = f_v(t)(1-s) + vs,\ 0 \leq s \leq 1$$

Substituting for $f_v(t)$, this can also be written as

$$H_v(\alpha) = v(1-\alpha) + u\alpha$$

where $\alpha = t(1-s)$.

Now, for $v_i \in W$, this induces a homotopy H_i that sends the image of the edge uv_i to a straight line segment from v_i to the image of u with respect to H_v, given by $H_v(s,1) = u(1-s) + vs$ for each intermediate step in the homotopy H_v. It should be noted that this explains our notation; we refer to homotopy in the singular since the homotopies H_i are all automatically defined once we define H_v. At any rate, these are given by

$$H_i(s,t) = v_i(1-t) + H_v(s,1)t = v_i(1-t) + (u(1-s) + vs)t,\ 0 \leq s \leq 1$$

A quick calculation reveals that for $v_i, v_j \in W$ the assumption that there exist s, t such that $H_i(s,t) = H_j(s,t)$ implies $i = j$. Similarly, for $v_i \in W$ the assumption that there exist s, t such that $H_i(s,t) = H_v(s,t)$ implies $v_i = v$. Thus the line segments in the intermediate steps of our constructed homotopy never intersect, meaning that the image of G' given by some intermediate step of the homotopy is homeomorphic to the image of G' given by any other intermediate step. In particular, the initial ($s = 0$) and final ($s = 1$) images are homeomorphic. This implies that any two embeddings of G'' constructed in this way from an embedding of G' are homeomorphic to each other, thus topologically identical.

This construction gives a well defined function $\mathcal{F}$ from the embedding classes of G' to those of G''. Further, a reverse procedure can be defined almost identically that constructs a homotopy from an embedding of G'' to an embedding of G'; it suffices to reverse the "direction" of the homotopy above. This defines the inverse of $\mathcal{F}$; we confirm that $\mathcal{F}$ is injective as follows. If two topologically identical embeddings of G'' are constructed as above from two embeddings of G', then the topological differences between the two embeddings of G' are restricted to the cells that contain e, the edge that was contracted. But as we saw earlier, any two such cells are homeomorphic, meaning the two embeddings of G' are topologically identical. Conversely, say two topologically identical embeddings of G' are constructed using the "reverse" homotopy from two embeddings of G''. Then, in particular, the subregions $\mathbf{D}'_1$ and $\mathbf{D}'_2$, defined as above, corresponding to each of the two embeddings of G' are homeomorphic. This implies that the original embeddings of G'' were themselves homeomorphic.

Thus $\mathcal{F}$ is bijective (it is clearly surjective since our homotopy is defined for any embedding of G'), so there exists a bijection between embedding classes of G' and those of G''. Note also that the contiguity relation on G' with respect to some embedding γ is the same as the contiguity relation on G'' with respect to the embedding constructed from γ using our homotopy (the proof is by contradiction, using the "reverse" homotopy that creates an embedding of G' from an embedding of G'').

Let us return to σ and τ, the two embeddings of G that share a contiguity relation. From the discussion above, we know σ and τ correspond to embeddings of G', σ' and τ', which in turn correspond to embeddings of G''; call these σ'' and τ''. Recall also that these successive restrictions do not alter the contiguity relation. Now, by our inductive hypothesis, σ'' and τ'' are topologically identical. Since there is a bijection between embedding classes of G' and those of G'', this means that σ' and τ' are also topologically identical. This in turn, as we've observed above, implies that σ and τ are topologically identical, completing the proof. □

Theorem 2.14 tells us that for circular planar, critical graphs, the study of embeddings is equivalent to the study of contiguity relations; results concerning the latter translate into results concerning the former. This is the strategy adopted for the remainder of the paper.

3. Multiple Embeddings

3.1. **Preliminaries.**

We are almost ready to prove a necessary and sufficient condition for the existence of multiple embedding classes for a circular planar, critical graph. Before we do, we must cite without proof a fundamental theorem of graph theory due to Menger.

Definition 3.1. Let G be a graph and consider a subset W of V. We say that W *separates* G if the subgraph $G(V - W)$ has more than one component.

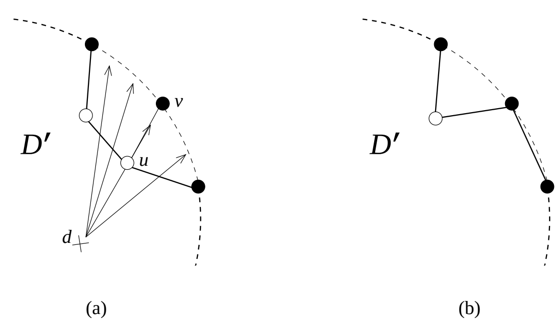

Figure 4. Constructing a homotopy from (a) an embedding of G' to (b) an embedding of G''. Because each intermediate step of the homotopy is a homeomorphism, the procedure can be reversed to construct a homotopy from an embedding of G'' to an embedding of G'.

Definition 3.2. G a graph. A vertex $v \in V$ is called a *cutvertex* if $v \in \partial V$ and $\{v\}$ separates G, or if $v \in int\,V$, $\{v\}$ separates G, and each component of $G(V - \{v\})$ contains fewer than $|V| - 1$ vertices.

This definition, for boundary vertices, is slightly stronger than the canonical one; for instance, if $v \in int\,V$ is a vertex adjacent to a vertex u with degree 1 such that $G(V - \{v\})$ has two components, then v is not a cutvertex because one of the components must be the single vertex u, and so the other contains exactly $|V| - 1$ vertices. If, however, v was adjacent to two vertices of degree 1, then it would indeed be a cutvertex. Compare, for instance, vertices 12 and 13 in Fig. 7.

Remark 3.3. Let G be a circular planar, critical graph with a boundary vertex u of degree 1 adjacent to another vertex v. If $v \in int\,V$, then as we saw this does not imply that G has multiple distinct embedding classes (to wit, using terminology from the proof of Theorem 2.14, contracting u and v would have no effect on the number of distinct embedding classes). If, however, $v \in \partial V$, then the discussion below demonstrates that there exist at least two distinct embedding classes of G (assuming $|\partial V| \geq 4$).

G, u, and $v \in \partial V$ as above. Consider an embedding of G, which gives some circular ordering of ∂V around $\partial \mathbf{D}$. It is easy to see that u and v must be contiguous, since there is an edge connecting them and we stipulated that the images of no two edges intersect. Thus the circular ordering around $\partial \mathbf{D}$ must include as a subsequence either $\dots puvq \dots$ or $\dots pvuq \dots$, where p and q are in ∂V. If the former, then it is possible to give an embedding of G where this subsequence of the circular ordering is replaced by the latter subsequence above. This is done by placing u between v and q on $\partial \mathbf{D}$, and drawing a curve from u to v, which is possible since u and v are on the boundary of a cell in $\mathbf{D}$. A similar statement holds if the latter is the case.

Thus in either case, one embedding immediately gives rise to an alternative contiguity relation, which by Theorem 2.14 corresponds to another, topologically distinct embedding.

Consider a graph G with a cutvertex v. A component of $G(V - \{v\})$ will be called a *v-component* of G. We will denote the set of all cutvertices of a graph G by $\mathcal{C}(G)$.

Definition 3.4. Let G be a graph and consider two subsets A and B of V. Suppose that a subset W of V separates G. If A and B are in different components of $G(V - W)$ we say that *W separates A and B.*

Theorem 3.5 (**Menger, 1927**). *Let G be a graph with A and B two disjoint n-tuples of vertices. Then either G contains an n-path from A to B, or there exists a set of fewer than n vertices that separates A and B.*

3.2. **A Necessary and Sufficient Condition.**

Lemma 3.6. *Suppose G is a circular planar, critical graph, $|\partial V| \geq 4$, such that given any two pairs of boundary vertices (u, v) and (p, q) that appear in any circular ordering of ∂V in the order $uvpq$ (up to rotation), there exists a path from u to v and another disjoint path from p to q. Then G has only one possible contiguity relation, and hence only one embedding class.*

Proof. It suffices to show that given an embedding of G that gives some contiguity relation, any other embedding must also give the same relation, since then Theorem 2.14 would imply that the two embeddings are topologically equivalent. Fix some embedding σ of G that gives a contiguity relation Θ and suppose $u \sim v$ with respect to Θ. Now consider a different embedding τ of G that gives an contiguity relation Θ'. Assume that $u \nsim v$ with respect to Θ'. As noted in Remark 2.8, this implies that there exist boundary vertices p and q such that u, v, p, and q appear in the circular ordering $upvq$ or $uqvp$ around $\partial\mathbf{D}$ with respect to Θ'. We will assume the former without loss of generality.

We will assume that p is the only boundary vertex between u and v with respect to τ; the more general case follows immediately by induction. Consider these four vertices in $\sigma(G)$. Since $u \sim v$ in Θ, we can assume without loss of generality that these vertices appear in the circular order $uvpq$ with respect to Θ. Thus there exists by hypothesis a pair of disjoint paths, one from u to v and another from p to q. These paths are disjoint with respect to any embedding, which in particular means that they are disjoint with respect to τ. Lemma 2.12 now implies that with respect to τ, the four vertices in question must be in one of the following circular orderings around $\partial\mathbf{D}$: $uvpq$ or $uvqp$. Either of these possibilities contradicts our assumption that $u \nsim v$ with respect to Θ', because this assumption implied that the circular ordering with respect to τ is $upvq$. Thus any pair of contiguous vertices in Θ is also contiguous in Θ', meaning $\Theta = \Theta'$. Consequentially, any two embeddings of G are topologically identical by Theorem 2.14. $\square$

Theorem 3.7. *A circular planar, critical graph G has at least two embedding classes if and only if it contains a cutvertex.*

Proof. Assume here that $|\partial V| \geq 4$; otherwise, the result is clear by inspection (see [8]).

If G contains a cutvertex v, then it is easy to exhibit two different contiguity relations on ∂V, hence two distinct embedding classes of G. Note first that if $v \in \partial V$ and v is adjacent to a boundary vertex u of degree 1, then by Remark 3.3 the theorem is true. This allows us to assume that each v-component contains fewer than $|V| - 1$ vertices.

If $v \in \partial V$ and one embedding gives a contiguity relation Θ, we construct Θ' as follows. Choose the $\theta \in \Theta$ that begins with v, so that $\theta = vv_1v_2 \ldots v_{n-1}$. Let v_{i+1} be the first boundary vertex in this sequence such that there exists a path from v_{i+1} to v_{n-1} that does not go through v. Now write $\theta' = vv_iv_{i-1} \ldots v_1v_{i+1} \ldots v_{n-1}$.

If v separates G into subgraphs H and K, then as Fig. 5 illustrates, this is equivalent to reflecting H while fixing K in the plane. This gives Θ' not equal to Θ, hence two distinct embedding classes by Theorem 2.14. Note that we implicitly assumed that not all v-components contain only one vertex. If this is the case, then it is possible to generate, given an embedding, a contiguity relation that corresponds to a topologically distinct embedding by placing v on $\partial\mathbf{D}$ between different v-components, in a manner similar to that of Remark 3.3.

If $v \in int\, V$ the procedure is similar. If $\theta = v_1v_2 \ldots v_n$, let v_i be the first boundary vertex such that a path from v_1 to v_i must go through v, and let v_j be the first boundary vertex such that there exists a path from v_1 to v_j that does not go through v. Now write $\theta' = v_1 \ldots v_{i-1}v_{j-1}v_{j-2} \ldots v_iv_jv_{j+1} \ldots v_n$. It is again easy

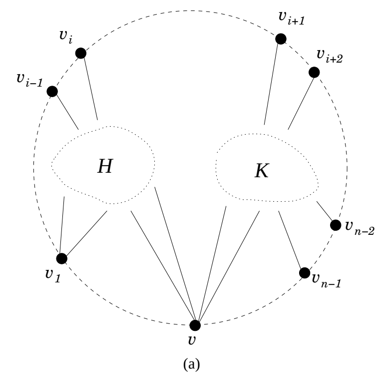

FIGURE 5. (a) An embedding of a graph G that has a cutvertex as a boundary vertex. (b) A distinct embedding produced by reflecting the subgraph H to get a new subgraph H'. Vertices v_{i+2} and v_{n-2} are not necessarily contiguous, and neither are vertices v_1 and v_{i-1}.

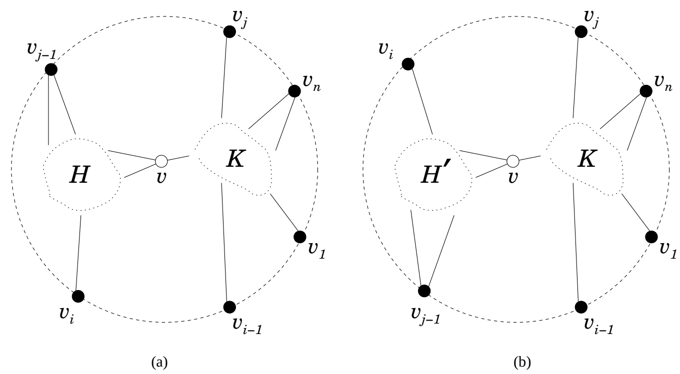

FIGURE 6. (a) An embedding of a graph G that has a cutvertex as an interior vertex, and (b) a distinct embedding produced by reflecting the subgraph H to get a new subgraph H'. Vertices v_1 and v_{i-1} are not necessarily contiguous, and neither are vertices v_i and v_{j-1} nor vertices v_j and v_n.

to check that the contiguity relation given by Θ' as constructed above is distinct from the original Θ. Again, this implies the existence of two embedding classes. An illustration of this procedure is given in Fig. 6.

It remains to prove the other direction; assume now that G has two distinct embeddings. Assume further that the boundary vertex is not of the type discussed in Remark 3.3. G cannot have all 2-paths exist, since that would be equivalent to the assumptions of Lemma 3.6 and would thus imply that all embeddings of G are identical. Thus there exist two disjoint pairs of boundary vertices, $A = \{u_1, u_2\}$ and $B = \{v_1, v_2\}$, that appear in a circular order around $\partial\mathbf{D}$ and have no 2-path between them. By Menger's Theorem, there exists a vertex v that separates A and B, which implies that v is a cutvertex. □

We end this section with a corollary concerning a class of circular planar, critical graphs for which any two embeddings are topologically identical. As defined in [2], a circular planar, critical graph is *well connected* if all possible k-connections exist.

Corollary 3.8. *If a circular planar, critical graph G is well connected, then G has only one embedding class.*

Proof. If $|\partial V| \leq 3$, it is clear by inspection that G has no cutvertices. If $|\partial V| \geq 4$, we know that all 2-connections exist. The existence of a cutvertex would contradict this fact, so by Theorem 3.7 any two embeddings of G are identical. □

4. Enumerating Embedding Classes

4.1. The Combinatorial Representation.

Now that Theorem 3.7 provides a necessary and sufficient condition for the existence of multiple embedding classes, one naturally seeks to enumerate these classes for a given circular planar, critical graph. Though no kind of closed form expression for the number of embeddings can be given at this time, we will prove a result that illustrates a method for enumerating embeddings on an *ad hoc* basis. For this we require the following combinatorial, as opposed to topological, means of representing embedding classes, adapted from [6].

Consider a circular planar graph G, number the set of vertices arbitrarily, and fix a circular orientation, say clockwise. A planar embedding of G can be specified by specifying, for each vertex in $V(G)$, a cyclic permutation given by the clockwise sequence of vertices adjacent to v with respect to this particular embedding. We will often refer to a sequence that represents the permutation, with the understanding that if the vertex in question is adjacent to k distinct vertices, then there are k such possible sequences.

We will call the collection of these permutations for all vertices in G a *combinatorial representation* (or simply *representation*) of the embedding, and denote it by π. Given $v \in V(G)$, the particular permutation that corresponds to v will be denoted $\pi(v)$.

As a consequence of regarding an embedding and its planar reflection to be topologically equivalent, we will not differentiate between a representation π and the representation π' given by reversing all permutations of π (which clearly corresponds to a planar reflection).

It is a fact (see [6]) that an embedding class has a unique combinatorial representation. It is worth noting, however, that according to our definition of an embedding, it is possible for two topologically distinct embeddings to have the same combinatorial representation; in particular, this happens if the graph in question has more than four boundary vertices, and contains a boundary vertex of degree 2 that is adjacent to at least one other boundary vertex (note that such a vertex must be in the boundary because the existence of an interior vertex of degree 2 violates criticality). Note that this vertex is a cutvertex.

An example is the circular planar, critical graph in Figure 7. The embedding in the figure, in which vertex 5 is contiguous to vertex 6 and vertex 7 is contiguous to vertex 8, immediately implies a second embedding in which vertex 5 is contiguous to vertex 7 and vertex 6 is contiguous to vertex 8, while all other contiguity relations are fixed. These two embeddings are topologically distinct, since they correspond to different contiguity relations. Their combinatorial representations, however, are identical.

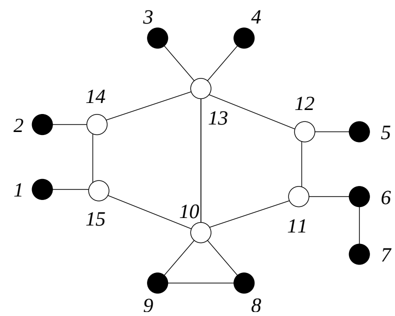

FIGURE 7. An embedding of a circular planar, critical graph.

As a brief aside, note that just as it is possible to formulate a combinatorial definition of a planar embedding, it is possible to formulate a combinatorial definition of a circular planar embedding. In particular, define a subset of vertices as boundary vertices, and demand an embedding of the graph in which the boundary vertices all appear on the boundary of the same face of the graph (it would remain to prove that this is equivalent to the topological definition).

This is actually a special case of a more general type of constraint on the combinatorial representation of a graph discussed in [7]. In that paper, Stallmann defines a data structure that allows for the generation of all planar embedding of a graph subject to a set of such constraints. In particular, this allows one to enumerate all embeddings of a circular planar graph subject to a particular set of constraints corresponding to the requirement that all boundary vertices appear around one face in a certain order.

However, this does not allow for the enumeration of all embeddings of a circular planar graph since there exists, in general, more than one distinct set of constraints for a given graph that expresses the fact that the boundary vertices are on the same face. This is because the boundary vertices may appear in more than one possible order around a face. Thus one would need to first find all possible such sets of constraints. This is not trivial, but might be a simplification of the original problem.

At any rate, we shall not require such a definition for our purposes; the standard combinatorial representation will do fine, though we must be weary of cases in which multiple embeddings correspond to the same representation.

Our enumeration strategy will be based on the isolation of subgraphs of circular planar, critical graphs that have only one embedding class. Recall that $\mathcal{C}(G)$ denotes the set of cutvertices of a graph G. Also, we will from now on say that a cutvertex is adjacent to a particular subgraph if it is adjacent to one or more of the vertices in that subgraph.

Let G be a circular planar, critical graph and let $G' = G(V - \mathcal{C}(G))$, the subgraph induced after removing all cutvertices. Let H be a component of G' that contains k boundary vertices, and let $\mathcal{C}_H$ be the set of cutvertices adjacent to H. We call H and the cutvertices adjacent to it, i.e. the subgraph induced by $V(H) \cup \mathcal{C}_H$, a *rigid component* of G. Note that cutvertices are part of more than one rigid component.

Also note that $k \geq 1$ for any such H because of the following. Assume that $k = 0$ for some H, and that H is adjacent to one cutvertex in G (the general case of an arbitrary number of cutvertices follows by induction). Since all vertices in H are in the interior, contracting them all does not break any connections. This contradicts the assumption that G is critical.

Proposition 4.1. *Let H be a rigid component of a circular planar, critical graph. H has only one embedding class and only one combinatorial representation.*

Proof. Let σ be an embedding of G and consider the subgraph H. For all $v \in \mathcal{C}_H$, if v is in the interior, declare it to be boundary. Call this resulting graph H'. The boundary vertices of H' are the boundary vertices of H and $\mathcal{C}_H$; no elements of $\mathcal{C}_H$ are in interior cells of H', so H' is clearly circular planar. Furthermore, H' is critical; this is obvious if $\mathcal{C}_H \subset \partial V$, as well as otherwise, since in the latter case any connection through G that went through a cutvertex (originally in the interior) and terminated at a boundary vertex not in H now terminates at that cutvertex and is still considered a connection through G, so assuming that H' is not critical would imply that G is not critical.

Now, H' has no cutvertices by definition, so by Theorem 3.7, H' has a unique embedding class. This implies that H' has a unique combinatorial representation, which in turn implies the same for H. Thus H, which has no cutvertices (cutvertices of G cannot be cutvertices of H), must also have only one embedding class, since multiple embedding classes would imply multiple embedding classes of H'. □

Finally, we have the following result due to Stallmann ([6]). The result is true for any vertex, though for our purposes it has been phrased to discuss cutvertices only. Consider a graph G and a vertex v of G. Now consider some permutation α of the vertices adjacent to v, which in turn gives a sequence of vertices around v. We say that this sequence is *embedding-consistent* if there exists a combinatorial representation π of G such that $\pi(v) = \alpha$.

Lemma 4.2 (**Stallmann**). *Let G be a graph, and take a vertex $v \in \mathcal{C}(G)$. A sequence given by a permutation $\pi(v)$ around v is embedding-consistent if and only if the following two conditions hold:*

1. *the subsequence of π corresponding to each v-component that is adjacent to v is embedding-consistent for that v-component;*
2. *π does not contain a subsequence of the form $e_i e_j f_i f_j$, where e_i, f_i belong to one v-component and e_j, f_j belong to another.*

4.2. **An Enumeration Algorithm.**

Proposition 4.1 and Lemma 4.2 outline a thorough, though inefficient, procedure for finding all embeddings of a circular planar graph. We do this by deriving all possible combinatorial representations from one combinatorial representation of the graph, then accounting for multiple embeddings that correspond to the same combinatorial representation. To wit, take a circular planar, critical graph G with an embedding σ and corresponding representation π. For some vertex $u \in V$ that belongs to a rigid component of G and is not a cutvertex, Proposition 4.1 tells us that for any embedding of G giving a representation ρ, $\rho(u)$ is either $\pi(u)$ or the inverse of $\pi(u)$. Thus, we need only look at cutvertices in order to generate all possible combinatorial representations of a graph. We outline an algorithm below. We also describe the algorithm as it acts on circular ordering of G, which is another, and more efficient, way to approach this procedure.

Step 1: Choose some $v \in \mathcal{C}(G)$, and assume that there are k v-components. By Lemma 4.2 the sequence given by the permutation $\pi(v)$ can be partitioned into consecutive subsequences corresponding to each v-component. These can be rearranged to form a sequence of vertices around v that is still embedding-consistent; this can be done in $(k-1)!$ unique ways and satisfy Stallmann's Lemma (there are $k!$ permutations of exactly k items, but each permutation can be written in k ways, so there are $k!/k$ unique permutations).

Note that this is equivalent to partitioning ∂V ($\partial V - \{v\}$ if v is a boundary vertex) into sets of boundary vertices such that two boundary vertices are in the same partition if they are in the same v-component, which in turn divides a circular ordering θ of ∂V, not counting v if it is in the boundary, into k consecutive subsequences (it is easy to see that no circular ordering can contain a subsequence of the form $v_1u_1v_2u_2$, where v_1, v_2 are in one v-component and u_1, u_2 are in another, distinct v-component, and where none of the four vertices are cutvertices). The rearrangement above is thus equivalent to rearranging these subsequences of θ in every possible distinct way to get a new, valid circular ordering (for the subsequent placement of v, if $v \in \partial V$, see Step 3).

Step 2: Each v-component can be drawn in two ways; the way it is already drawn in the given embedding, or "flipped" (see the proof of Theorem 3.7). That is, if the subsequence of $\pi(v)$ corresponding to this v-component is $u_1u_2 \dots u_s$, then it can be replaced with $u_s \dots u_1$, giving a sequence of vertices around v that is still permutation consistent. Considering this step of the algorithm in terms of a circular ordering, this is equivalent to reversing the subsequence of boundary vertices belonging to the v-component in question. Thus there are now $2^k \dot(k-1)!$ ways of arranging v-components around v.

Any new embedding-consistent sequence around v gives a combinatorial representation in the following way; the sequence around v changes in the way specified in the previous steps. If any subsequences of the original sequence around v were reversed, reverse all sequences around vertices in the v-component corresponding to that subsequence.

Step 3: As noted previously, it is possible for distinct embedding to give the same combinatorial representation. This happens only when there exists a cutvertex that is a boundary vertex. This is because if all cutvertices are in the interior, then all boundary vertices belong to rigid components, hence the ordering of rigid components around the cutvertices as specified above uniquely determines a contiguity relation, and hence an embedding by Theorem 2.14.

If v is a cutvertex in ∂V, we must note that it can be placed on $\partial \mathbf{D}$ between any two v-components; that is, in k different ways. In terms of a circular ordering, this is equivalent to the ability to place v, given an arrangement of subsequences of θ as given in Step 1, between any two of these subsequences.

Step 4: Repeat for all $v \in \mathcal{C}(G)$.

Citing Stallmann's Lemma, We have finally accounted for all possible embedding classes of G. It is important to note, however, that in general the representations generated by the algorithm are not distinct; that is, the algorithm will generate a particular representation more than once (see the example below). In particular, if the above algorithm generates some representation, it will also generate its reflection. Because of this an explicit formula for the number of embeddings is not immediate.

Remark 4.3. The algorithm, when applied to a circular ordering of a circular planar, critical graph G, finds all possible contiguity relations on ∂V. This was an open question conceived independently of the effort to find all embedding classes of circular planar, critical graphs, though Theorem 2.14 showed that these two questions are equivalent.

4.3. **An Example.**

As an example, consider the circular planar, critical graph in Figure 7. Here, $\mathcal{C}(G) = \{6, 10, 11, 13\}$, where vertex 6 is a boundary vertex. If we start the algorithm with, say, cutvertex 10, and choose a circular ordering $\theta = 123456789$, we see that the 10-components partition θ into subsequences 1234567 and 89. This gives the following two distinct possible circular orderings of ∂V, each representing an embedding class:

123456789
124356798

If we use cutvertex 6 instead we would partition θ into subsequences 7 and 8912345. Recalling that we may place 6 between these subsequences in two ways when generating new circular orderings, we get these distinct possible circular orderings:

123456789
123457689

Having applied the algorithm to all cutvertices and removed duplicate entries, we are left with the following possible embedding classes, given here by the circular orderings they impose on ∂V:

123456789
124356789
123457689
123456798
124357689
124356798

123457698

This graph provides a good example of a situation in which the same combinatorial representation is given twice by the algorithm. In particular, the 11-components are $G(\{6,7,11\})$ and $G(V-\{6,7\})$. Fixing the latter and reflecting the former gives an embedding the corresponds to the circular ordering 123457689. Now consider vertex 6, with 6-components $G(\{6,7\})$ and $G(V-\{7\})$. Fixing both 6-components and embedding 6 between vertices 5 and 6 instead of between 6 and 8 also gives the same embedding. Both of these approaches would have been taken by the algorithm, so that the embedding giving the circular ordering 123457689 would appear (at least) twice.

5. On a Theorem of Perry

We end this paper by looking at its results and potential applicability in the context of a theorem by Perry, proven in [5]. This theorem and its application to the graph in Fig. 8, discussed below, were the initial motivation for this paper.

Definition 5.1. The *dual graph*, $G_\perp$, of a circular planar graph G is a circular planar graph that is defined as follows. Fix an embedding of G. The graph G together with $\partial\mathbf{D}$ partition the disc into a finite number of disjoint cells. The vertices of the dual graph are defined by placing a vertex in each cell. If one of the edges of a cell is an arc of $\partial\mathbf{D}$, place the vertex on this arc and declare it a boundary vertex. For each edge in the original graph, there is one edge in the dual graph that intersects the original edge and connects the two vertices drawn in the adjacent cells.

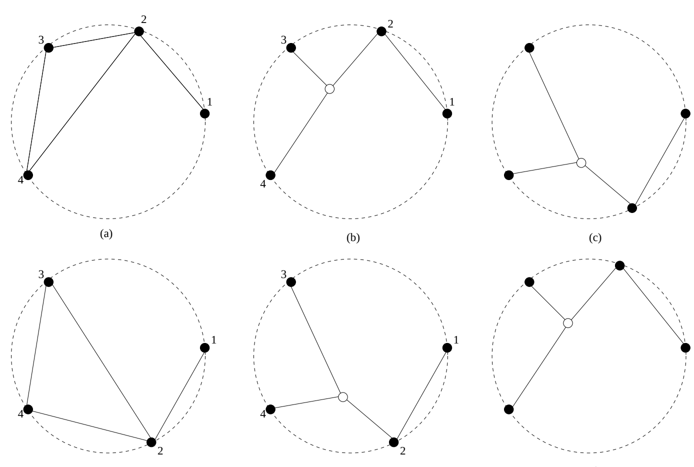

Figure 8. Two different contiguity relations on a circular planar, critical graph. (a) A representative of the embedding class given by one contiguity relation, (b) a graph that is $Y-\Delta$ equivalent to (a), (c) the dual of (a). (d) A representative of the embedding class given by another contiguity relation, (e) a graph that is $Y-\Delta$ equivalent to (d), (f) the dual of (d).

An example is found in Fig. 8. Observe that defining $G_\perp$ simultaneously specifies an embedding of $G_\perp$.

Theorem 5.2 (Perry). *Given a circular planar, critical graph G, the dual graph $G_\perp$ is $Y-\Delta$ equivalent to G if and only if G is well-connected.*

This theorem, as well as previous work on which it is based (e.g. [2]), operates under the implicit assumption of a fixed embedding of the graph G, since otherwise a dual cannot be defined. Consider now the two planar graphs in (a) and (d) of Fig. 8.

The graph in Fig. 8a is not well-connected, hence is not $Y-\Delta$ equivalent to its dual. It is, however, $Y-\Delta$ equivalent to the graph in Fig. 8e, which is the dual of the graph in Fig. 8d. Observe that Fig. 8d is simply a topologically distinct embedding of the graph in Fig. 8a. In this special case, the fact that two topologically distinct embeddings of a graph represent the same connectivity type translates into a somewhat subtle geometric relationship between the two embeddings (more accurately, between the embedding classes they represent). In light of Corollary 3.8, it is easy to see why Theorem 5.2 would ignore this type of relationship. A graph-topological generalization of this theorem would be an interesting consequence of a mathematical framework that classifies electrical networks by their graph-topological, not geometric, properties.

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