RECOVERABILITY OF SUBGRAPHS

J.T. RUSSELL

Abstract. We present a number of flexible definitions for subgraphs of a graph-with-boundary and prove several straightforward recovery results. Our main goal is to clearly present some vocabulary and fundamental results for dealing with subgraphs.

Let $G = (V, V_B, E)$ denote a graph with boundary throughout this paper.

Definition 1. Let $G^+ = G \wedge \{v_1, v_2, ...v_k\}$ denote the graph where the vertices v_1, v_2, ..., v_k are promoted to the boundary, as in [1]. We say G^+ is a *promotion* of G.

Definition 2. A *subgraph* of G is a triple $(V', V'_B E')$ such that V' is a subset of V, V'_B is any subset of V' that contains $V'_B \cap V'$, and E' is a relation on V' such that if $p \sim_{E'} q$ then $p \sim_E q$. That is, a subgraph is a set of vertices and some collection of the edges from G that join vertices in V', with additional boundary vertices chosen arbitrarily.

We say a subgraph G_1 *contains* a subgraph G_2 if G_2 is a subgraph of G_1; i.e. $V_1 \supset V_2$, $V_{1B} \supset V_{2B}$, and $E_1 \supset E_2$.

Definition 3. A *simple subgraph* of G is a subgraph $(V', V'_B E')$ such that $V'_B = V' \cap V_B$.

Definition 4. A *full subgraph* of G is a subgraph $(V', V'_B E')$ such that E' is simply the restriction of E to V'. That is, it contains all edges from G that link vertices in V'.

Definition 5. A subgraph G' of G is *interior-connected* if every pair of vertices in G' is connected through the interior of G'. I.e., for every p, q in S there exists a path joining p and q that goes through no boundary vertices except possibly p or q themselves.

Definition 6. An *interior-connected-component* (ICC) G' of G is a maximal interior-connected simple subgraph. That is, G' is an interior-connected simple subgraph of G such that if H is an interior-connected simple subgraph that contains G' then $H = G'$.

Each interior-connected component is a full subgraph. Every interior node i of G is contained in a unique interior-connected component, which we will denote $ICC_G(i)$. The same is not true for every boundary node: a boundary node may belong to several ICC's. The intersection of two ICC's contains only boundary nodes and boundary-boundary edges.

Date: July 23, 2003.

It has come to my attention that some of the definitions and results in this paper can be found incidentally in various forms in a number of papers, including [3], [4], and [5]. These results were produced independently in this paper, but more importantly they are assembled together in what I hope is a useful and easily accessible form.

Definition 7. A *Simon-subgraph* of G is a full subgraph $G' = (V', V'_B, E')$ such that V'_B is a union of $V' \cap V_B$ with the set of all vertices in V' that are adjacent to some vertex outside of V'.

Note that a Simon-subgraph is fully determined by its set of vertices.

Proposition 8. *A Simon-subgraph $G' = (V', V'_B, E')$ of G is a simple subgraph of $G \wedge V'_B$.*

Definition 9. Given a subgraph $G' = (V', V'_B, E')$ in G, its *complement* is the simple subgraph $H = (W, W_B, F)$ such that F includes every edge not in E', and W includes all vertices not in V' and all vertices that are endpoints of edges in F.

Note that a subgraph and its complement are *not disjoint*. Rather, their intersection consists of all vertices that are adjacent to points inside and outside of V'. We will refer to these points as the *topological boundary* of the subgraph, as distinct from the *electrical boundary*, which is V'_B. The *Simon boundary* of a subgraph is the union of its topological boundary with the boundary inherited from G; i.e., the Simon boundary is the set of all vertices in the topological boundary or in $V_B \cap V'$. Thus a Simon-subgraph is precisely a full subgraph whose electrical boundary coincides with its Simon boundary.

The intersection of a subgraph with its complement contains no edges.

Definition 10. An *isolated subgraph* G' of G is a simple subgraph such that no interior node of G' is connected through the interior to any interior node of its complement. Equivalently, an isolated subgraph is one whose electrical boundary contains its topological boundary.

Proposition 11. *Any ICC of G or any union of ICC's of G is an isolated subgraph of G.*

Proposition 12. *Every Simon-subgraph (V', V'_B, E') of G is an isolated subgraph of $G \wedge V'_B$.*

Proposition 13. *Let G' be an isolated subgraph of G such that every boundary node is connected-through-the-interior to an interior node. Then G' is a union of ICC's of G. In particular,*

$$G' = \bigcup_{i \in \operatorname{int} G'} ICC_G(i) \tag{1}$$

Proof. Every interior vertex in G' is contained in its own ICC, and since each boundary vertex is connected-through-the-interior to an interior node, it is contained in the ICC of that node. So G' is contained in the union. On the other hand, if v is a vertex not in G', then any path from an interior node of G' to v must go through a boundary node. So v is not contained in any of the ICC's. Thus the union is contained in G'; hence equality. $\square$

Lemma 14. *Suppose that $G = G_1 \cup G_2$ and $G_1 \cap G_2$ consists only of boundary nodes (and no edges). Then if G is recoverable, G_1 and G_2 are recoverable.*

Proof. Let Λ_1 and Λ_2 be the respective response matrices for G_1 and G_2. Then we can compute Λ, the response matrix for G, in terms of Λ_1 and Λ_2. If A designates the vertices contained only in G_1, B the vertices in the intersection, and C the vertices only in G_2, then

$$\Lambda = \begin{bmatrix} (\Lambda_1)_{AA} & (\Lambda_1)_{AB} & 0 \\ (\Lambda_1)_{BA} & (\Lambda_1)_{BB} + (\Lambda_2)_{BB} & (\Lambda_2)_{BC} \\ 0 & (\Lambda_2)_{CB} & (\Lambda_2)_{CC} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{2}$$

Suppose we are given Λ_1. Then we can assign arbitrary positive conductivities to the edges in G_2 and compute the corresponding response matrix Λ_2. Next, add together Λ_1 and Λ_2 to obtain Λ for the combined graph. Since G is recoverable, we can recover all of the conductivities in G from Λ—in particular, we recover all of the conductivities in G_1. $\square$

By applying Lemma 14 repeatedly, we also obtain

Corollary 15. *If $G = G_1 \cup G_2 \cup ... \cup G_n$ and for each i and j the intersection $G_i \cap G_j$ contains only boundary nodes, then each G_i is recoverable.*

Since (by definition) the intersection of an isolated subgraph with its complement contains only boundary nodes, we can apply Lemma 14 to obtain

Corollary 16. *Any isolated subgraph of a recoverable graph is recoverable.*

Thus by Proposition 11 we obtain

Corollary 17. *Any ICC (or any union of ICC's) of a recoverable graph is recoverable.*

And finally,

Corollary 18. *Any Simon-subgraph of a recoverable graph is recoverable.*

Proof. Let G' be a Simon-subgraph of G, and G be recoverable. By Proposition 12, G' is an isolated subgraph of a promotion of G. By [1], any promotion of a recoverable graph is recoverable. Thus G' is an isolated subgraph of a recoverable graph, and by Corollary 16, G' is recoverable. $\square$

References

[1] Johnston, Andrew. "Modification of the Boundary Set for Solving the Inverse Problem"; 2003.
[2] Simon Pai's talk, 2003.
[3] Card, Ryan. "Using Network Amalagamation and Separation to Solve the Inverse Problem"; 2000.
[4] Ingerman, David. "Theory of Equivalent Networks and Some of Its Applications"; 1992.
[5] Carlson, Thomas. "A Natural Representation for a Class of Circular Planar Graphs"; 1999.

Stanford University / University of Washington REU
E-mail address: `jefe@stanford.edu`