$\bigstar$ AND $\mathcal{K}$ SOLVE THE INVERSE PROBLEM

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Abstract. We present an algorithm for recovering conductivities on arbitrary networks. This algorithm finds the full solution set of possible conductivities on any given graph that produce a particular response matrix. This set consists of rational functions of entries of the response matrix, and possibly some parameters under a system of polynomial constraints.

Contents

1. Graphs With Boundary and Multiplicity

A graph with boundary and multiplicity (or a *multi-graph*) is a triple $G = (V, V_B, M)$ where V is a set of nodes, V_B is any subset of V, designated as the boundary nodes, and M is the *edge-multiplicity* function, defined on pairs of vertices such that $M(i,j)$ is the number of edges joining nodes i and j. M is always a non-negative integer, and $M(i,j) = M(j,i)$.

We say i is adjacent to j, or $i \sim j$, if $M(i,j)$ is positive, and we say they are connected by a single edge (or a *singleton*) when $M(i,j) = 1$. If $M(i,j) = m$ is greater than one, we say that (i,j) are joined by a *cable of multiplicity m*. If $M(i,j)$ is zero or one for all pairs (i,j), we say that G is *simple.*

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Thanks to Sam Coskey and Tracy Lovejoy for figures, Simon Pai for the neat organization of the algebra in Theorem 3.1, Dr.Jim Morrow for numerous valuable discussions, and Nicolas Addington for many of the ideas in this paper. And props to everybody at the University of Washington REU and the NSF—go team! It has also since come to my attention that earlier work in [7] explored a similar idea to the main observation of this paper.

2. The Forward and Inverse Problems

A *network* is a pair $\Gamma = (G, \gamma)$ where G is a graph and γ is a positive conductivity function defined on all cables in G. If G is a simple graph, then γ is defined on G's edges. If G is not simple, then γ is assigns a conductivity to each *cable* of G, not to individual edges. The *Kirchhoff matrix* [1] for Γ, denoted K is defined such that

$$K_{ij} = \begin{cases} \gamma_{ij} & i \sim j \\ -\sum_{k \neq i} K_{ik} & i = j \\ 0 & i \not\sim j \text{ and } i \neq j \end{cases} \tag{1}$$

K is symmetric and negative semi-definite [1]. Suppose that we write

$$K = \begin{array}{cc} & \begin{array}{cc} \partial & \text{int} \end{array} \\ \begin{array}{c} \partial \\ \text{int} \end{array} & \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \end{array} \tag{2}$$

Then the *response matrix* of Γ is defined in terms of the Kirchhoff matrix as the Schur complement

$$\Lambda = K/K(I;I) = A - BC^{-1}B^\top \tag{3}$$

We will refer to the (easy) problem of computing Λ for a given network as the forward problem.

Λ is itself a Kirchhoff matrix; in particular, it is the Kirchhoff matrix of a complete graph with conductivities given by its off-diagonal entries.

Note that we can solve the forward problem by successively interiorizing nodes. Consider a connected network Γ with N nodes and n interior nodes i_1, i_2, ... i_n. Consider an identical network where all nodes are considered to be boundary nodes. Denote this by Γ_0. The response matrix Λ_0 of Γ_0 is precisely equal to the Kirchhoff matrix of Γ. Now consider one node to be an interior node, and denote the new network by Γ_1. The response matrix Λ_1 of Γ_1 is equal to the Schur complement $\Lambda_0/\Lambda_0(i_n; i_n)$. We then continue to interiorize the nodes, one by one. At each stage the response matrix is a Schur complement of the previous response matrix. The original network Γ is equal to Γ_n, and so we conclude that

$$\Lambda = \Lambda_n = K/K(i_n; i_n)/K(i_n - 1; i_n - 1)/\cdots/K(i_1; i_1) \tag{4}$$

Taking a Schur complement is the same as doing Gaussian elimination, so this result shows that in this case we can perform Gaussian elimination in any order. This perspective is also useful because it has a nice geometric interpretation, which we will discuss later in this paper.

Suppose that Λ and G are given. We say that a network Γ is *recoverable* when Λ determines the conductivity γ uniquely, and we say a graph G is recoverable when $\Gamma = (G, \gamma)$ is recoverable for any positive γ. The inverse problem is to (1) determine when Γ is recoverable, and (2a) compute γ in terms of Λ and G when Γ is recoverable, or (2b) compute the set of all γ that yield Λ on G when Γ is not recoverable.

[1] Note that the sign convention used in this paper is the opposite of that used in many works, such as [1]

3. ★ AND $\mathcal{K}$

An *n-star*, denoted $\bigstar_n$, is a graph with boundary that has n boundary nodes and one interior node, where each boundary node is connected by a single edge to the interior node and there are no other edges. The *complete graph* on n vertices, denoted $\mathcal{K}_n$, is a graph with n boundary nodes, no interior nodes, and a single edge connecting every pair of nodes (Figure 1).

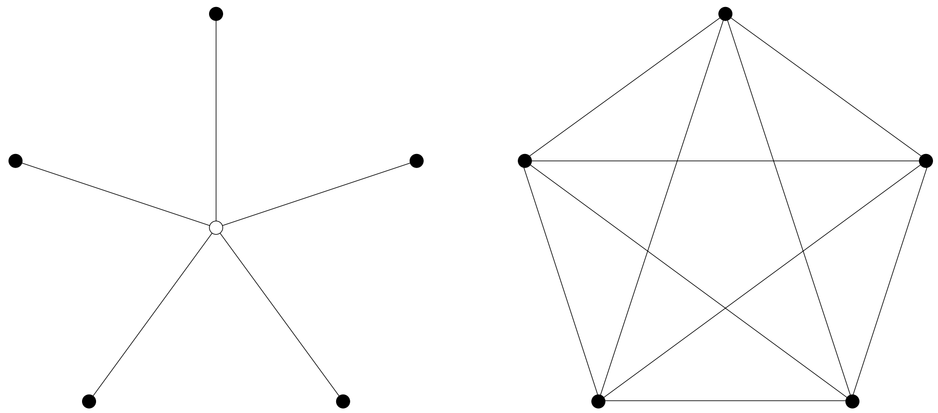

FIGURE 1. $\bigstar_5$ and $\mathcal{K}_5$

The response matrix of a complete graph is easy: λ_{ij} is equal to the conductivity on the edge joining i and j. Conversely, any response matrix can be thought of as coming from a complete graph with conductivies equal to the entries. Thus every network Γ is response-equivalent to the complete graph whose conductivities are equal to the entries in Γ's response matrix. In particular, every n-star is response-equivalent to a complete graph, and there is a simple relationship between the conductivities of the two networks.

Consider networks on $\bigstar_n$ and $\mathcal{K}_n$. Denote the n conductivities on $\bigstar_n$ by γ_1, γ_2, ..., γ_n, and the $n(n-1)/2$ conductivities on $\mathcal{K}_n$ by λ_{ij}. Then we can transform the star to the $\mathcal{K}$ that corresponds to the star's response matrix, which we calculate by interiorizing the central node. This yields the formula

$$\lambda_{ij} = \frac{\gamma_i \gamma_j}{\sum_{k=1}^{n} \gamma_k} \tag{5}$$

We refer to this as the $\bigstar - \mathcal{K}$ transformation.

Clearly not every complete graph (i.e., not every response matrix) corresponds to a star, however. In fact, we will show that a complete graph comes from a star if and only if the following condition is satisfied:

$$\lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{jl} \text{ for all } i \neq j \neq k \neq l \tag{6}$$

This condition is the same as the determinantal condition corresponding to the fact that a star has no two-connections. It can also be interpreted geometrically. Consider the quadrilateral $ijkl$ in the complete graph. The determinantal condition says that the conductivities on each pair of opposite edges have the same product. Hence we call Equation 6 the quadrilateral condition (Figure 2).

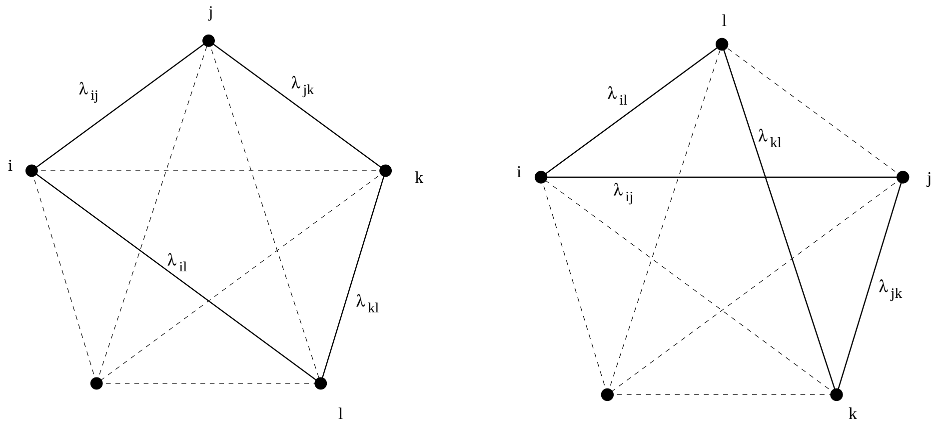

FIGURE 2. The quadrilateral condition.

The quadrilateral condition is equivalent to the fact that $\frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}}$ $(i \neq j \neq k)$ is the same quantity regardless of how j and k are chosen. This fact can also be interpreted geometrically. Fix a vertex i in $\mathcal{K}_n$. Then for all triangles with a vertex at i, the product of the two adjacent sides over the opposite side is always the same number. We call this the triangle condition (Figure 3).

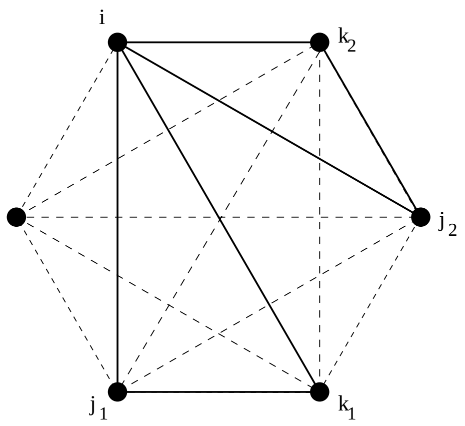

FIGURE 3. The triangle condition.

Suppose that a network on $\mathcal{K}_n$ with conductivities λ_{ij} satisfies (6). Then we can calculate the corresponding conductivities on $\bigstar_n$ by this formula:

$$\gamma_i = -\lambda_{ii} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} = \sum_{i \neq j} \lambda_{ij} + \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} \tag{7}$$

where $i \neq j \neq k$ and λ_{ii} is the ith entry in the diagonal of the response matrix. Because of the triangle condition, this expression is unambiguous. Formula 7 is the $\mathcal{K} - \bigstar$ transformation. We will prove in the following lemma that $\mathcal{K} - \bigstar$ is actually the inverse of $\bigstar - \mathcal{K}$.

Theorem 3.1. *A network on a complete graph is response-equivalent to a (unique) star if and only if its conductivities satisfy Equation 6.*

Proof. The preceding discussion shows that any complete graph that is response-equivalent to a star must satisfy the quadrilateral condition. On the other hand, if (6) is satisfied, then so is the triangle condition, and we can write without ambiguity

$$\alpha_i = \sqrt{\frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}}} \tag{8}$$

and so

$$\alpha_i\alpha_j = \sqrt{\frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}}}\sqrt{\frac{\lambda_{ji}\lambda_{jk}}{\lambda_{ik}}} = \lambda_{ij} \tag{9}$$

Then define γ according to Equation 7:

$$\gamma_i = \frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} + \sum_{j\neq i}\lambda_{ij} = \alpha_i^2 + \sum_{j\neq i}\alpha_i\alpha_j = \alpha_i\sum_j \alpha_j \tag{10}$$

We need to check that $(\bigstar_n, \gamma)$ is response-equivalent to $(\mathcal{K}_n, \lambda)$—that is, we need to check that if we perform the $\bigstar - \mathcal{K}$ tranformation, on γ, we get λ. This we see because

$$\frac{\gamma_i\gamma_j}{\sum_k \gamma_k} = \frac{(\alpha_i\sum_k \alpha_k)(\alpha_j\sum_k\alpha_k)}{\sum_k(\alpha_k\sum_l \alpha_l)} = \frac{\alpha_i\alpha_j(\sum_k\alpha_k)^2}{(\sum_k\alpha_k)^2} = \alpha_i\alpha_j = \lambda_{ij} \tag{11}$$

□

Corollary 3.2. For star networks, the range of the map $L : \gamma \mapsto \Lambda_\gamma$ is the set

$$V = \{\Lambda : \lambda_{ij}\lambda_{kl} = \lambda_{ik}\lambda_{jl} \text{ for } i \neq j \neq k \neq l, \lambda_{ij} > 0 \text{ for } i \neq j, \text{ and } \sum_j \lambda_{ij} = 0 \text{ for each } i\}$$

and L is a diffeomorphism onto V (the foregoing theorem demonstrates a continuous inverse).

This is an independent (and much more direct) proof of a special case of Theorem 3.1 in [4], which is in turn an extension of an argument in §4.6 of [1].

Remark 3.3. Because of the triangle condition we can also write Equation 7 in a more symmetric form:

$$\gamma_i = \sum_{i\neq j}\lambda_{ij} + \frac{2}{(n-1)(n-2)}\sum_{i\neq j\neq k}\frac{\lambda_{ij}\lambda_{ik}}{\lambda_{jk}} \tag{12}$$

We can also interpret this formula geometrically. For each triangle with a vertex at i, calculate the product of the sides adjacent to i over the opposite side. Take the average of this value over all such triangles. The conductivity γ_i is equal to this average plus the sum of the conductivities of the edges in $\mathcal{K}_n$ that meet at i.

Performing a $\bigstar - \mathcal{K}$ transformation on any network on $\bigstar_n$ will produce a conductivity function λ that satisfies (6). Moreover, every conductivity function satisfying (6) corresponds to a unique star-network, whose conductivity function is determined by Equation (7). Thus the conductivity functions λ on $\mathcal{K}_n$ that satisfy Equation (6) are in 1-1 correspondence with networks on $\bigstar_n$.

4. Recovering a Star

Suppose we have a network on $\mathcal{K}_n$ where some of the conductivities are known and some are unknown. What conductivities can be assigned to the unknown edges in order to satisfy all of the star conditions?

Consider a graph $\mathcal{K}_n$. Let E be the set of edges in $\mathcal{K}_n$, E_0 be the set of edges with known conductivities and λ a conductivity function defined on E_0. We will call λ a *partial conductivity function.* The values of λ determine a unique star-network if and only if there exists a unique extension $\overline{\lambda}$ defined on E that satisfies condition 6. The problem, then, is to find $\overline{\lambda}$.

We can consider this algebraic problem from a graph theoretic perspective. Each of the equations corresponds to the quadrilateral in $\mathcal{K}_n$ with vertices i, j, k, l. If an unknown edge is part of a quadrilateral $ijkl$ in $\mathcal{K}_n$ where the three other edges are known, we can immediately deduce the only possible value of the fourth edge from (6), namely

$$\overline{\lambda}_{ij} = \frac{\lambda_{jk}\lambda_{il}}{\lambda_{kl}} \tag{13}$$

We call this "closing a quadrilateral".

Note: if an edge can be calculated by closing two different quadrilaterals, and the two formulae give different results, then there is no possible extension of λ.

Once we have determined the value of λ_{ij} by this method, we can treat it as one of the knowns and search for another quadrilateral. Continuing in this way, we may be able to determine the conductivities of all edges in E—or then again, we may not. Let $\overline{E_0}$ denote the set of all edges whose conductivities may be determined by closing quadrilaterals.

False Theorem 4.1. *Suppose λ is a partial conductivity function on $\mathcal{K}_n$ such that there exists an extension $\overline{\lambda}$ such that $(\mathcal{K}_n, \overline{\lambda})$ comes from a star network $(\bigstar_n, \gamma)$. Then the network $(\bigstar_n, \gamma)$ is unique if and only if $\overline{E_0} = E$.*

In fact, this can fail. Consider a partial conductivity function λ defined on $\mathcal{K}_6$ such that λ_{13}, λ_{24}, λ_{35}, λ_{46}, λ_{51}, and λ_{62} are known and the rest unknown (Figure 4). Then, since the known edges form two disjoint triangles, it's clear that no quadrilaterals can be formed. Nonetheless, it turns out that the full conductivity function $\overline{\lambda}$ can be determined uniquely.

One way to see why this is so is to rearrange the nodes so the known edges form two concentric triangles as in Figure 4. If we assign a parameter t to one of the unknown edges, the parameter propagates by closing quadrilaterals in both directions. Then the two propagations meet in the opposite quadrilateral. Applying Equation 6 to this quadrilateral fixes t uniquely.

Remark 4.2. In most cases quadrilateral closure will be enough to determine whether a unique extension exists. If quadrilateral closure is insufficient to fix λ on all edges, then we can assign a parameter to an unknown edge and close quadrilaterals again. We proceed, assigning parameters and closing quadrilaterals until we find a value of $\overline{\lambda}$ for every edge.

The problem that the counterexample above points out is that we may introduce false parameters by this method—that is, we may write a parametric formula for an edge which is actually known uniquely due to restrictions on the parameter. This is a hassle, but spurious solutions will be eliminated if we simply check for

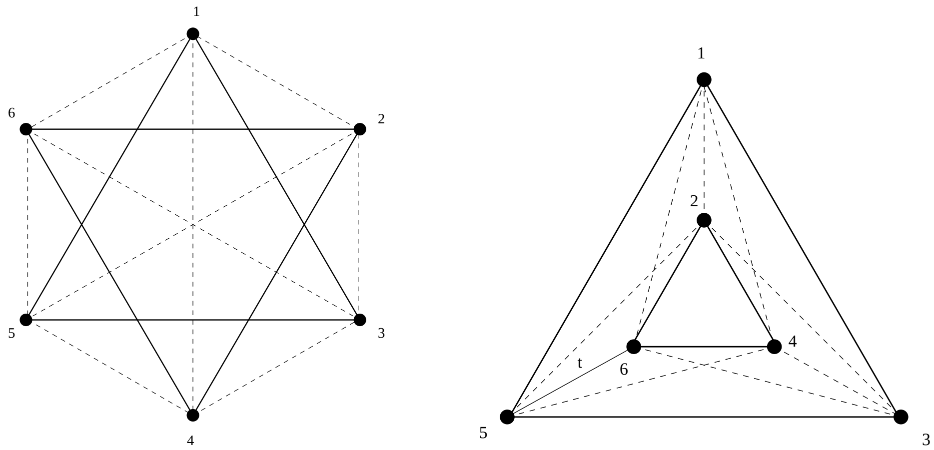

FIGURE 4. Star of David rearranged

the constraints on parameters which are induced by Condition 6 on quadrilaterals with parametrized edges. In this case, the problem of spurious parameters can be avoided by using an extended method, but this is an issue which will arise again in other places.

5. Recovering a Star For Real

Let's take a step back and look at quadrilaterals again in a different light.

A quadrilateral in a network on $\mathcal{K}_n$ corresponds to four positions in the network's Kirchhoff matrix (i.e. the response matrix of the corresponding star-network)—the coordinates (i, j), (i, l), (k, j), and (k, l) (Figure 5). Closing a quadrilateral consists in calculating the value of λ at one of these pairs of coordinates when λ is known for the other three pairs. The ordered pairs correspond to edges in $\mathcal{K}_n$.

The diagonal of the matrix is a peculiar exception: λ_{ii} does not correspond to an edge in $\mathcal{K}_n$, and entries on the diagonal can not be calculated by quadrilateral closure. Equation 6 requires that $i \neq j \neq k \neq l$—but in the matrix, we can just as easily pick out coordinates when some of these values are equal. Of course, because the Kirchhoff matrix is defined differently on the diagonal than on off-diagonal entries, if we tried to apply Equation 6 to the diagonal, we would produce false equations.

So let's shift our attention from the Kirchhoff matrix to another, closely related matrix. We can write the response matrix for a network on an n-star with conductivities $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ as

$$\Lambda_\gamma = -D_\gamma + \frac{\gamma\gamma^\top}{\sigma} \tag{14}$$

where D_γ is the diagonal matrix corresponding to the entries of γ and $\sigma = \sum_k \gamma_k$. Then consider the matrix

$$M_\gamma = \frac{\gamma\gamma^\top}{\sigma} \tag{15}$$

To make my life easier and your life miserable, in the subsequent sections we will use the letter λ to refer to the entries in M_γ—and *not* Λ_γ.

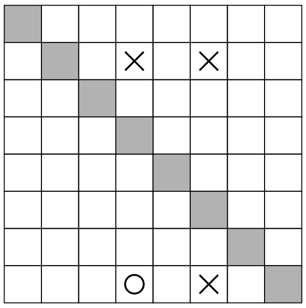

FIGURE 5. A quadrilateral in the response matrix

This matrix is equal to the response matrix of $(\bigstar_n, \gamma)$ (i.e., the Kirchhoff matrix of the corresponding $(\mathcal{K}_n, \lambda)$) everywhere except on the diagonal, and it satisfies Equation 6 on *all* of its entries—in particular, on the diagonal:

$$\lambda_{ii}\lambda_{jk} = \lambda_{ij}\lambda_{ik}. \tag{16}$$

Thus, if we are given a partial conductivity function Λ on $\mathcal{K}_n$, we can use this extension of quadrilateral closure in order to calculate diagonal entries of the corresponding matrix M. Moreover, we can use symmetry to calculate an off-diagonal entry using only *two* diagonal entries, instead of the usual three off-diagonal entries:

$$\lambda_{ii}\lambda_{jj} = \lambda_{ij}\lambda_{ji} = \lambda_{ij}^2 \tag{17}$$

and so

$$\lambda_{ij} = \sqrt{\lambda_{ii}\lambda_{jj}} \tag{18}$$

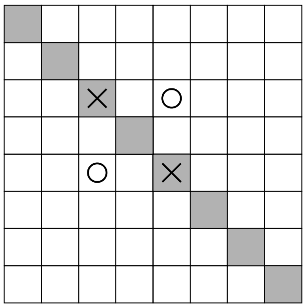

FIGURE 6. Using an extended quadrilateral

We will use the term "extended quadrilateral" to refer either to an ordinary quadrilateral or to a quadrilateral that involves the diagonal. Given a set of ordered pairs (i.e., edges in $\mathcal{K}_n$) $E = \{(i,j)\}$, the *q-closure* of E, written $\overline{E}$, is the set of all pairs (k,l) which can be reached by closing extended quadrilaterals from E. A set Q is called *q-closed* iff $Q = \overline{Q}$.

We will prove below that False Theorem 4.1 is true when we use extended quadrilaterals.

Lemma 5.1. *Every q-closed set Q is a set of ordered pairs of the form*

$$Q = D \times D \cup \bigcup_i (A_i \times B_i \cup B_i \times A_i) \tag{19}$$

where D, A_i, and B_j are pairwise disjoint.

Proof. Let Q be a q-closed set of ordered pairs of nodes in $\mathcal{K}_n$. Let $D = \{d : (d,d) \in Q\}$. By (18), if $(d_1, d_1) \in Q$ and $(d_2, d_2) \in Q$ then it follows that (d_1, d_2) and (d_2, d_1) are also in Q. Thus $D \times D \subset Q$.

Now for each $(a, b) \in Q$, where a and b are not in D, we define

$$A_b = \{x : (x, b) \in Q\}$$
$$B_a = \{y : (a, y) \in Q\}$$

Then the following are facts:

(i) $A_b \times B_a \subset Q$ and $B_a \times A_b \subset Q$
If $(x, y) \in A_b \times B_a$ then (x, b), (a, y), and (a, b) are in Q. By closing the quadrilateral, $(x, y) \in Q$. The second statement is true by symmetry.

(ii) $A_b \cap D = B_a \cap D = \emptyset$
If x is in B_a then $(a, x) \in Q$, and by symmetry $(x, a) \in Q$. If x is in D then $(x, x) \in Q$. Then we close the quadrilateral, so $(a, a) \in Q$—so $a \in D$. But a was chosen not to belong to D.

(iii) $A_b \cap B_a = \emptyset$
If $(x, x) \in A_b \times B_a$ then $x \in D$, and so x is contained in $A_b \cap D$.

(iv) If $a_1, a_2 \in A_b$ then $B_{a_1} = B_{a_2}$. Likewise if $b_1, b_2 \in B_a$ then $A_{b_1} = A_{b_2}$.
We know (a_1, b) and (a_2, b) are in Q. If $\widehat{b} \in B_{a_1}$, then $(a_1, \widehat{b}) \in Q$, so closing the quadrilateral yields $(a_2, \widehat{b}) \in Q$, and so $\widehat{b} \in B_{a_2}$. So $B_{a_1} \subset B_{a_2}$, and conversely by the same argument.

(v) Suppose (a_1, b_1) and (a_2, b_2) are in Q. If $A_{b_1} \cap A_{b_2}$ or $B_{a_1} \cap B_{a_2}$ is non-empty, then $A_{b_1} = A_{b_2}$ and $B_{a_1} = B_{a_2}$.
Say $\widehat{a}$ is in A_{b_1} and A_{b_2}; i.e., $(\widehat{a}, b_1)$ and $(\widehat{a}, b_2)$ are in Q. Then b_1 and b_2 are both contained in $B_{\widehat{a}}$. Thus, by (4), $A_{b_1} = A_{b_2}$. By the same argument, $B_{a_1} = B_{a_2}$.

Thus we have produced a finite collection of sets A_i and B_i which are pairwise disjoint and each disjoint with D. We have shown that $D \times D$ and each product $A_i \times B_i$ and $B_i \times A_i$ is contained in Q. Moreover, every element $(a, b) \in Q$ is either contained in D or else in the product $A_b \times B_a$, so we have equality. □

We can visualize this decomposition as splitting a q-closed set into blocks as in Figure 7.

Thinking of the set in terms of the graph, we can also say it like this: every q-closed set is a union of a complete graph and some bipartite graphs.

One simple consequence of the lemma is that if a q-closed set Q contains a bipartite graph on sets of nodes A and B, and Q also contains an edge connecting two points in A, then Q must contain the complete graph on $A \cup B$.

Let N denote the set of nodes of $\mathcal{K}_n$, and write N^2 for $N \times N$. The response matrix can be thought of as a function on N^2. Let Δ_S signify the diagonal of S^2. A partial function on N^2 is said to be *q-consistent* if it satisfies the extended quadrilateral condition wherever it is defined.

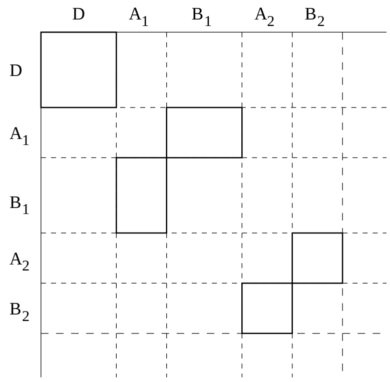

FIGURE 7. Block decomposition of a q-closed set

It follows from Theorem 3.1 that if a function λ is defined on a set containing $N^2 - \Delta_N$, then $(\mathcal{K}_n, \lambda)$ is response equivalent to a star-network $(\bigstar_n, \gamma)$ if and only if λ is q-consistent.

Lemma 5.2. *Say $M \subset N$. Suppose that $\Delta_M \subset X \subset M^2$, and let $\lambda : X \to \mathbb{R}$ be a q-consistent partial function. Then λ has a unique q-consistent extension to M^2; i.e., there exists a unique function $\overline{\lambda} : M^2 \to \mathbb{R}$ such that $\overline{\lambda}|_X = \lambda$ and $\overline{\lambda}$ is q-consistent.*

In particular, if $X \supset \Delta_N$, then λ has a unique q-consistent extension over all of N^2.

Proof. Since X contains the diagonal, the entries λ_{11}, λ_{22}, ..., λ_{nn} are known. Then we can calculate $\overline{\lambda}$ uniquely everywhere in M^2 by

$$\overline{\lambda}_{ij} = \sqrt{\lambda_{ii}\lambda_{jj}} \tag{20}$$

(Note that in particular $\overline{\lambda}_{ii} = \sqrt{\lambda_{ii}\lambda_{ii}} = \lambda_{ii}$.) The extension $\overline{\lambda}$ is clearly q-consistent, since

$$\overline{\lambda}_{ij}\overline{\lambda}_{kl} = \sqrt{\lambda_{ii}\lambda_{jj}\lambda_{kk}\lambda_{ll}} = \sqrt{\lambda_{ii}\lambda_{kk}}\sqrt{\lambda_{jj}\lambda_{ll}} = \overline{\lambda}_{ik}\overline{\lambda}_{jl} \tag{21}$$

Finally, $\overline{\lambda}|_X = \lambda$, since the q-consistency of λ requires that if $(i, j) \in X$ then (since (i, i) and (j, j) are also in X)

$$\lambda_{ij} = \sqrt{\lambda_{ii}\lambda_{jj}} = \overline{\lambda}_{ij} \tag{22}$$

□

The following fact follows from the proof of the lemma:

Corollary 5.3. A partial function λ defined on a set X such that $\Delta_M \subset X \subset M^2$ is q-consistent if and only if $\lambda_{ij} = \sqrt{\lambda_{ii}\lambda_{jj}}$ for each (i, j) in X.

This will also be a handy fact:

Lemma 5.4. *Suppose $X \subset M^2$ and $Y \subset (N - M)^2$. Then if λ_1 is a q-consistent function on X and λ_2 is a q-consistent function on Y, then $\lambda = \lambda_1 \cup \lambda_2$ is a q-consistent function on $X \cup Y$.*

Proof. Every quadrilateral in $X \cup Y$ either has all subscripts contained in X or all subscripts contained in Y. □

Combining the two preceding lemmas, we also obtain

Lemma 5.5. *If $\Delta_M \subset X \subset M^2$, $Y \subset (N-M)^2$, and λ is a q-consistent function on $X \cup Y$, then λ has a unique q-consistent extension to $M^2 \cup Y$.*

Proof. Let $\lambda_1 = \lambda|_X$ and $\lambda_2 = \lambda|_Y$. Use Lemma 5.2 to find $\overline{\lambda_1} : M^2 \to \mathbb{R}$. Then the function $\overline{\lambda} = \overline{\lambda_1} \cup \lambda_2$ is the only possible extension of λ over $M^2 \cup Y$, and by Lemma 5.4 it is q-consistent. □

Let E_0 be a set of edges—i.e., ordered pairs of nodes—in $\mathcal{K}_n$. ($E_0 \subset N \times N$.)

Theorem 5.6. *Suppose λ is a partial conductivity function on $\mathcal{K}_n$ defined on E_0. Suppose also that λ has a q-consistent extension $\overline{\lambda}$ (in particular, λ is itself q-consistent), so $(\mathcal{K}_n, \overline{\lambda})$ comes from a star network $(\bigstar_n, \gamma)$. Then*

1. *The network $(\bigstar_n, \gamma)$ is unique if and only if $\overline{E_0} = N \times N$.*
2. *If γ is not unique, we can determine from λ and E_0 a parametrization of the set of all possible functions γ.*
3. *Suppose that*
$$\overline{E_0} = D^2 \cup (A_1 \times B_1 \cup B_1 \times A_1) \cup \cdots \cup (A_m \times B_m \cup B_m \times A_m)$$
where D, A_i, and B_i are disjoint, and $C = N - D - \bigcup A_i \cup B_i$. Then the number of parameters is equal to $m + |C|$.

Proof. By hypothesis, λ has a q-consistent extension $\overline{\lambda}$ defined on N^2. If $\overline{E}_0 = N^2$ then closing quadrilaterals pins down the possible values of $\overline{\lambda}$ uniquely on all of $N \times N$. Since conductivities on the n-star are in 1-1 correspondence with q-consistent functions on N^2, $(\bigstar_n, \gamma)$ is determined uniquely. The other direction will come out of our proof of parts (2) and (3).

Suppose that $\overline{E}_0$ is a proper subset of $N \times N$. Then using Lemma 5.1, write

$$\overline{E_0} = D^2 \cup (A_1 \times B_1 \cup B_1 \times A_1) \cup \cdots \cup (A_m \times B_m \cup B_m \times A_m) \tag{23}$$

where D, A_i, and B_i are disjoint. Also, let $C = \{c_1, c_2, ..., c_{|C|}\} = N - D - \bigcup A_i \cup B_i$.

We will also write $D_0 = D \times D$ and $D_i = (A_i \cup B_i)^2$. In a final flurry of notation, write

$$S_i = \bigcup_{j=0}^{i} D_j \cup \bigcup_{j=i+1}^{m} (A_j \times B_j) \cup (B_j \times A_j) \tag{24}$$

and let $\mathcal{L}_i$ denote the set of all possible extensions $\overline{\lambda}^{(i)}$ to S_i. That is to say, $\overline{\lambda}^{(i)} : S_i \to \mathbb{R}$ is in $\mathcal{L}_i$ if and only if $\overline{\lambda}^{(i)} = \lambda$ on E_0 and $\overline{\lambda}^{(i)}$ satisfies the extended quadrilateral condition.

The initial set $S_0 = \overline{E_0}$, so $\mathcal{L}_0$ contains one element, the unique extension $\overline{\lambda}$ to $\overline{E_0}$. Now we want to calculate $\mathcal{L}_i$ from $\mathcal{L}_{i-1}$.

Suppose we know $\mathcal{L}_{i-1}$, and we can parametrize the set of functions $\overline{\lambda}^{(i-1)}$ in terms of parameters $t_1, t_2, ..., t_{i-1}$. Let's pick a particular function $\overline{\lambda}^{(i-1)}(t_1, t_2, ..., t_{i-1})$ from $\mathcal{L}_{i-1}$. For the sake of simplicity, in the next few paragraphs we'll refer to this function as λ.

Let's focus our attention on the square D_i. (See Figure 8) We will show that parametrizing one unknown entry in this block will consistently fill out the whole block, and affect nothing outside of D_i.

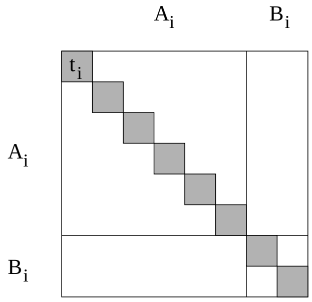

FIGURE 8. Parametrizing a block

For simplicity, we'll use local coordinates in D_i, starting the numbering of nodes with $(1,1)$ in the upper left corner. Let's assign a parameter t_i to $\overline{\lambda}^{(i)}_{11}$. Thus far $\overline{\lambda}^{(i)}$ is q-consistent for any choice of t_i, since $(1,1)$ does not close any quadrilateral from S_{i-1}.

Next we'll assign[2] $\overline{\lambda}^{(i)}$ on the diagonal of $A_i \times A_i$ to be

$$\overline{\lambda}^{(i)}_{aa} = t_i \left(\frac{\lambda_{ba}}{\lambda_{b1}}\right)^2 \tag{25}$$

and the entries on the diagonal of $B_i \times B_i$ by

$$\overline{\lambda}^{(i)}_{bb} = \frac{\lambda_{b1}^2}{t_i} \tag{26}$$

We now have $\overline{\lambda}^{(i)}$ defined on a set that contains the diagonal of D_i. Let's check that this definition is consistent:

$$\sqrt{\overline{\lambda}^{(i)}_{aa}\overline{\lambda}^{(i)}_{bb}} = \sqrt{t_i \left(\frac{\lambda_{ab}}{\lambda_{b1}}\right)^2 \left(\frac{\lambda_{b1}{}^2}{t_i}\right)} = \lambda_{ab} = \overline{\lambda}^{(i)}_{ab} \tag{27}$$

As we pointed out before, this is sufficient for consistency. Thus, by Lemma 5.5, we have a unique extension $\overline{\lambda}^{(i)}$ defined over all of S_i. Expanding our abbreviated notation, this is function is actually unique in terms of the function $\overline{\lambda}^{(i-1)}(t_1, t_2, ..., t_{i-1})$ and the parameter t_i. So the set $\mathcal{L}_i$ contains all of the functions $\overline{\lambda}^{(i)}(t_1, t_2, ..., t_i) : S_i \to \mathbb{R}$.

Continuing this way, we can calculate the parametrized family $\mathcal{L}_m$ of functions $\overline{\lambda}^{(m)}(t_1, t_2, ..., t_m) : S_n \to \mathbb{R}$.

Then, cracking our knuckles before one last bout, we extend the function to the set $S_n \cup \Delta_C$ by adding a parameter t_j to each of the entries (c_j, c_j). By Lemma 5.4, this function is also q-consistent no matter how we choose the parameters. Moreover, $(S_n \cup \Delta_C)$ contains the full diagonal of N^2. So, at long last, we use Lemma 5.2 to obtain an extension $\overline{\lambda}(t_1, t_2, ..., t_{m+|C|})$ defined q-consistently over all of $N \times N$ for each choice of parameters.

[2]These formulae are motivated by quadrilaterals in D_i, but their ultimate justification is that they produce a q-consistent extension, as we will show.

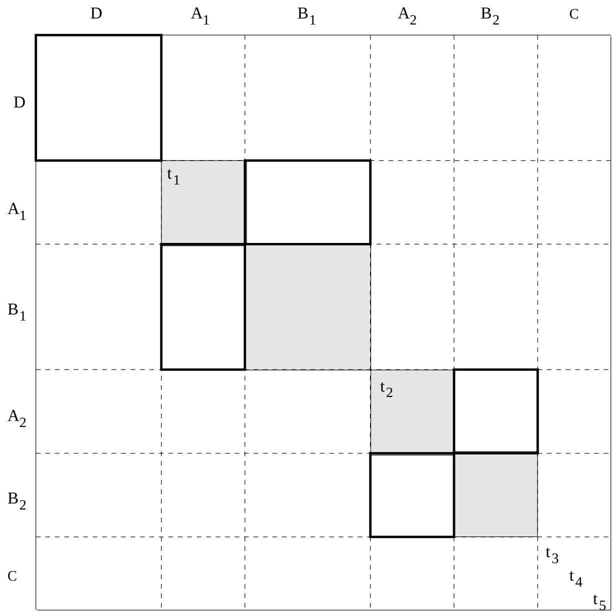

FIGURE 9. Parametrizing the matrix M.

In particular, note that unless $D_0 = N^2$ (in which case we had $E_0 = N^2$) we get at least one free parameter floating around, and so the extension $\overline{\lambda}$ is not unique. ☐

Corollary 5.7. Since the map $H : \mathbf{t} \mapsto \overline{\lambda}(\mathbf{t})$ from parameter-space to conductivity functions is smooth and injective, the set of q-consistent extensions of a partial conductivity function λ, considered as vectors in $\mathbb{R}^{n^2}$, comprise an $(m+|C|)$-dimensional manifold.

The set of q-consistent extensions of λ is diffeomorphic to the set of possible conductivity functions γ on the original n-star that correspond to λ; thus the set of possible conductivity functions is an $(m + |C|)$-dimensional manifold in $\mathbb{R}^n$.

Corollary 5.8. If the ordinary quadrilateral-closure of E_0 is equal to E (all edges of $\mathcal{K}_n$), then it easily follows that the extended q-closure $\overline{E_0} = N \times N$, and so we can calculate γ uniquely.

Example 5.9. Extended quadrilateral closure easily recovers the Star of David graph. The known edges are shown as X's in Figure 10. Every entry on the diagonal can be calculated by closing an extended quadrilateral. Then, by Formula 18, we can calculate λ for the whole matrix.

If we're in an imaginative mood, we can visualize an extended quadrilateral as one that has two or three sides in the graph, and also involves a "side" (or two) connecting a node to itself (Figure 11). So the triangles in the Star of David, which were useless for finding ordinary quadrilaterals, give perfectly good extended quadrilaterals.

Remark 5.10. If a conductivity function is recoverable from a partial function λ by using extended quadrilaterals, but not by using ordinary quadrilaterals, then

	1	2	3	4	5	6
1	O		X		X	
2		O		X		X
3	X		O		X	
4		X		O		X
5	X		X		O	
6		X		X		O

FIGURE 10. Recovering the Star of David.

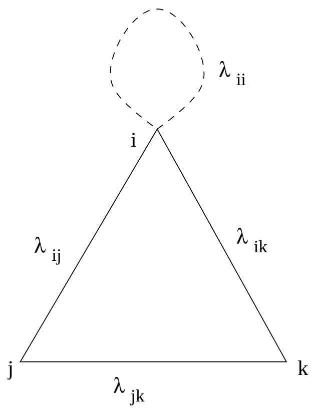

FIGURE 11. An extended quadrilateral

the resulting γ may not be a rational function of λ: in particular, it can involve square roots.

6. The Forward Problem Again

Observation 6.1. Every graph is locally a star.

Consider an arbitrary network $\Gamma = (G, \gamma)$. Every interior node i is the central node of a star, where the rays are simply all the edges that meet at i. We will call this an *embedded star*. If G is a graph with multiplicity, some of the rays of the star may actually comprise multiple edges, but we shall simply ignore this and treat edge-cables in the same way as singleton edges. (Recall that the conductivity function γ is defined not on individual edges in this case, but on cables. Therefore transforming and recovering stars involving cables is identical to when all the rays are singletons.)

We can perform a $\bigstar - \mathcal{K}$ transformation on an embedded star as above. This results in both a new graph and a new conductivity function. Denote the network obtained by performing a $\bigstar - \mathcal{K}$ transformation around node i by $\Sigma_i(\Gamma)$. The new graph will have multiplicity greater than one on any edge between two vertices that had an edge between them in G and were on rays of the star.

Wherever the transformation creates a cable of multiplicity greater than one, the conductivity on the new cable will be equal to the sum of the conductivity from the $\bigstar - \mathcal{K}$ with the original conductivity on that edge. Conductivities of edges between nodes that are not in the transformed star will of course be unaffected by the transformation.

The Kirchhoff matrix of $\Sigma_i(\Gamma)$ is determined by the new conductivity function, which we described by Equation 5. In fact, it is related in a simple way to the original Kirchhoff matrix: we simply take the Schur complement with respect to i, $K/K(i;i)$.

Observation 6.2. Performing a $\bigstar - \mathcal{K}$ transformation around node i is algebraically equivalent to interiorizing i.

Suppose we successively interiorize all of the interior nodes in G. This gives rise to the network $\overline{\Gamma} = \Sigma_{i_n} \circ \cdots \circ \Sigma_{i_1}(\Gamma)$, which has no interior nodes. The response matrix of a network with no interior nodes is equal to its Kirchhoff matrix. Moreover, the Kirchhoff matrix of $\overline{\Gamma}$ is equal to $K/K(i_1;i_1)/K(i_2;i_2)/\cdots/K(i_n;i_n)$, which, as we pointed out, is the response matrix of Γ. Thus the response matrix of $\overline{\Gamma}$ is equal to the response matrix of Γ.

This gives a new geometric interpretation to the solution of the forward problem: the response matrix Λ of Γ is determined by successive $\bigstar - \mathcal{K}$ transformations (which are equivalent to interiorization). This presents a natural approach to the inverse problem: *if we calculate Λ from K by a sequence of $\bigstar - \mathcal{K}$ transformations, we may be able to calculate K from Λ by a sequence of $\mathcal{K} - \bigstar$ transformations.*

7. Solving the Inverse Problem

Algorithm 7.1. Let $\Gamma = (G, \gamma)$ be a connected simple network—i.e., Γ has no parallel connections. For each $k = 1, 2, ..., n$ let

$$\Gamma_k = \Sigma_{i_k} \circ \Sigma i_{k-1} \circ \cdots \circ \Sigma_{i_1}(\Gamma) = \Sigma_{i_k}(\Gamma_{k-1}) \tag{28}$$

The Kirchhoff matrix of Γ_n (same as $\overline{\Gamma}$ in the preceding section) is the response matrix of Γ. Given the Kirchhoff matrix of Γ_k, we will calculate the Kirchhoff matrix of Γ_{k-1}.

Let's say the same thing in slightly different terms. We can read the conductivity function on the cables of Γ_n directly from the response matrix of Γ. Now, given the conductivities of the cables of Γ_k, we intend to calculate the conductivities on the cables of Γ_{k-1}. By induction, we can then determine the conductivities of the cables of $\Gamma_0 = \Gamma$. Since all of Γ's cables are singletons, we will have calculated the conductivity of every edge in Γ.

Consider the networks $\Gamma_k = (G_k, \gamma_k)$, $\Gamma_{k-1} = (G_{k-1}, \gamma_{k-1})$, and the node i_k. The graph G_k contains an embedded $\mathcal{K}_r$ which is response equivalent to the embedded $\bigstar_r$ around i_k in G_{k-1}. Because Σ_{i_k} preserves conductivities outside of the star around i_k, we know γ_{k-1} everywhere except on the star right off the bat. The only problem, then is to recover the star (and the edges connecting the nodes on the star's rays).

Some of the cables in the embedded $\mathcal{K}_r$ are singletons and others contain multiple edges. If a cable c is a singleton, its conductivity came completely from the star. If, on the other hand, c is has higher multiplicity, then one of its edges belongs to the star, and so only part of its conductivity can be attributed to the star. Let $\lambda = \gamma_k|_{\mathcal{K}_r}$, restricting the conductivity function to the embedded complete graph.

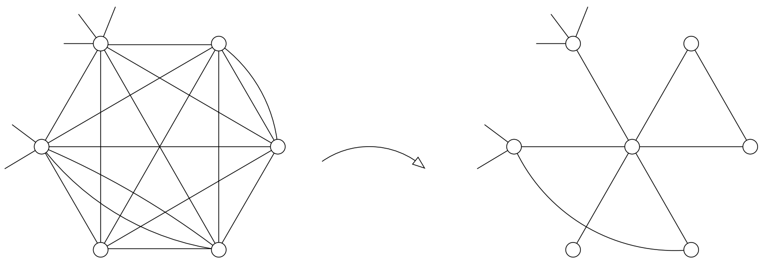

Figure 12. Embedded $\mathcal{K} - \bigstar$.

If all cables were singletons, then because $(\mathcal{K}_r, \lambda)$ comes from a star, satisfies Equation 6, and we can recover the conductivities on the star. If not all cables are singletons, treat the conductivities λ_{ij} on singletons as known, and consider the conductivities on cables with higher multiplicity as unknown. That is, define the function $\lambda^* = \lambda$ on the set of singletons, then use Theorem 5.6 to find either a unique extension $\overline{\lambda^*}$ (if the quadrilateral-closure of the set of singletons is all of $\mathcal{K}_r$), or else a family of possible extensions $\overline{\lambda^*}(t_1, t_2, ..., t_m)$, where $\{t_1, t_2, ..., t_m\}$ is a set of independent parameters.

In either case, we can then calculate the set of all possible conductivity functions γ_{k-1}. Namely, $\gamma_{k-1}(t_1, t_2, ..., t_m)$ is equal to

- γ_k, on edges that do not join nodes in the star around i_k.
- the $\mathcal{K} - \bigstar$ transformation (Formula 7) of $\overline{\lambda^*}$, on rays of the star.
- $\gamma_k - \overline{\lambda^*}(t_1, t_2, ..., t_m)$, on edges connecting rays of the star.

If $\overline{\lambda^*}$ could be determined uniquely, then γ_{k-1} is known uniquely in terms of γ_k. Otherwise, γ_{k-1} is known in terms of γ_k and some set of parameters. If γ_k was already parametrized, then γ_{k-1} inherits those parameters as well.

Following this procedure from Γ_n to Γ_0 will yield a set containing all possible conductivity functions γ on the cables (i.e. the edges) of Γ. If this set contains only one element, we are done: Γ is a recoverable network, and we have successfully recovered its conductivity function.

However, if the set is parametrized, the set we obtain by this process may actually be too large: in order to find the correct set of conductivities, we need to take into account restrictions on the parameters' domains. These restrictions are introduced in two ways:

First, the determinantal condition may introduce polynomial restrictions on the parameters. If, having inherited a set of parameters from Γ_k, we find that the embedded $\mathcal{K}_r$ in Γ_{k-1} contains a quadrilateral of singletons, then these four values of λ must satisfy Condition 6. Since λ_{ij}, λ_{jk}, λ_{kl}, and λ_{il} are rational functions of parameters, Condition 6 introduces a constraint which can be expanded as a polynomial in the parameters. The correct set of possible conductivities includes only those where the parameters satisfy all of these polynomial constraints.

Second, at the end of the day all conductivities should be positive. So all conductivities we obtain that do not satisfy this condition should be discarded from the set of possibilities. We can think of this constraint as introducing further restrictions on the parameters' domains.

The set of conductivities $\gamma_0(t_1, t_2, ..., t_M)$, where the parameters' domains are restricted by the above constraints—certain polynomial equations and inequalities—is the full set of possible conductivities for Γ. Thus Γ is a recoverable network exactly when this set contains a single element. In general, this algorithm in principle presents a means of calculating the complete set of possible conductivites for an arbitrary network.

□

8. An Example

Example 8.1. Consider the graph in Figure 13. The sequence of $\bigstar - \mathcal{K}$ transformations on the graph is shown in Figure 14.

At each stage the node we interiorize has two "spikes", which have no edges connecting them to each other or any other nodes. This means that in the complete graph after interiorization all of the edges to these two spike-nodes are known. But then it's easy to see that the whole matrix M_γ can be recovered.

Suppose we wish to calculate the conductivity function that corresponds to a response matrix

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & \cdots & \lambda_{1,8} \\ \vdots & & \vdots \\ \lambda_{8,1} & \cdots & \lambda_{8,8} \end{bmatrix} \tag{29}$$

This is the Kirchhoff matrix for the last graph in the $\bigstar - \mathcal{K}$ sequence.

Let's consider the extraction of node 9. The partial function λ_9^* on singleton edges is the following:

$$\begin{bmatrix} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} & \lambda_{1,7} & \lambda_{1,8} \\ \lambda_{2,1} & & & & & & & \lambda_{2,8} \\ \lambda_{3,1} & & & & & & & \lambda_{3,8} \\ \lambda_{4,1} & & & & & & & \lambda_{4,8} \\ \lambda_{5,1} & & & & & & & \lambda_{5,8} \\ \lambda_{6,1} & & & & & & & \lambda_{6,8} \\ \lambda_{7,1} & & & & & & & \lambda_{7,8} \\ \lambda_{8,2} & \lambda_{8,2} & \lambda_{8,3} & \lambda_{8,4} & \lambda_{8,5} & \lambda_{8,6} & \lambda_{8,7} & \end{bmatrix} \tag{30}$$

By q-closure, we extend the function over the whole matrix M shown below.

$$M_9 = \begin{bmatrix} \frac{\lambda_{1,8}\lambda_{1,2}}{\lambda_{2,8}} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} & \lambda_{1,7} & \lambda_{1,8} \\ \lambda_{2,1} & \frac{\lambda_{2,8}\lambda_{1,2}}{\lambda_{1,8}} & \frac{\lambda_{2,8}\lambda_{1,3}}{\lambda_{1,8}} & \frac{\lambda_{2,8}\lambda_{1,4}}{\lambda_{1,8}} & \frac{\lambda_{2,8}\lambda_{1,5}}{\lambda_{1,8}} & \frac{\lambda_{2,8}\lambda_{1,6}}{\lambda_{1,8}} & \frac{\lambda_{2,8}\lambda_{1,7}}{\lambda_{1,8}} & \lambda_{2,8} \\ \lambda_{3,1} & \frac{\lambda_{3,8}\lambda_{1,2}}{\lambda_{1,8}} & \frac{\lambda_{3,8}\lambda_{1,3}}{\lambda_{1,8}} & \frac{\lambda_{3,8}\lambda_{1,4}}{\lambda_{1,8}} & \frac{\lambda_{3,8}\lambda_{1,5}}{\lambda_{1,8}} & \frac{\lambda_{3,8}\lambda_{1,6}}{\lambda_{1,8}} & \frac{\lambda_{3,8}\lambda_{1,7}}{\lambda_{1,8}} & \lambda_{3,8} \\ \lambda_{4,1} & \frac{\lambda_{4,8}\lambda_{1,2}}{\lambda_{1,8}} & \frac{\lambda_{4,8}\lambda_{1,3}}{\lambda_{1,8}} & \frac{\lambda_{4,8}\lambda_{1,4}}{\lambda_{1,8}} & \frac{\lambda_{4,8}\lambda_{1,5}}{\lambda_{1,8}} & \frac{\lambda_{4,8}\lambda_{1,6}}{\lambda_{1,8}} & \frac{\lambda_{4,8}\lambda_{1,7}}{\lambda_{1,8}} & \lambda_{4,8} \\ \lambda_{5,1} & \frac{\lambda_{5,8}\lambda_{1,2}}{\lambda_{1,8}} & \frac{\lambda_{5,8}\lambda_{1,3}}{\lambda_{1,8}} & \frac{\lambda_{5,8}\lambda_{1,4}}{\lambda_{1,8}} & \frac{\lambda_{5,8}\lambda_{1,5}}{\lambda_{1,8}} & \frac{\lambda_{5,8}\lambda_{1,6}}{\lambda_{1,8}} & \frac{\lambda_{5,8}\lambda_{1,7}}{\lambda_{1,8}} & \lambda_{5,8} \\ \lambda_{6,1} & \frac{\lambda_{6,8}\lambda_{1,2}}{\lambda_{1,8}} & \frac{\lambda_{6,8}\lambda_{1,3}}{\lambda_{1,8}} & \frac{\lambda_{6,8}\lambda_{1,4}}{\lambda_{1,8}} & \frac{\lambda_{6,8}\lambda_{1,5}}{\lambda_{1,8}} & \frac{\lambda_{6,8}\lambda_{1,6}}{\lambda_{1,8}} & \frac{\lambda_{6,8}\lambda_{1,7}}{\lambda_{1,8}} & \lambda_{6,8} \\ \lambda_{7,1} & \frac{\lambda_{7,8}\lambda_{1,2}}{\lambda_{1,8}} & \frac{\lambda_{7,8}\lambda_{1,3}}{\lambda_{1,8}} & \frac{\lambda_{7,8}\lambda_{1,4}}{\lambda_{1,8}} & \frac{\lambda_{7,8}\lambda_{1,5}}{\lambda_{1,8}} & \frac{\lambda_{7,8}\lambda_{1,6}}{\lambda_{1,8}} & \frac{\lambda_{7,8}\lambda_{1,7}}{\lambda_{1,8}} & \lambda_{7,8} \\ \lambda_{8,1} & \lambda_{8,2} & \lambda_{8,3} & \lambda_{8,4} & \lambda_{8,5} & \lambda_{8,6} & \lambda_{8,7} & \frac{\lambda_{1,8}\lambda_{2,8}}{\lambda_{1,2}} \end{bmatrix} \tag{31}$$

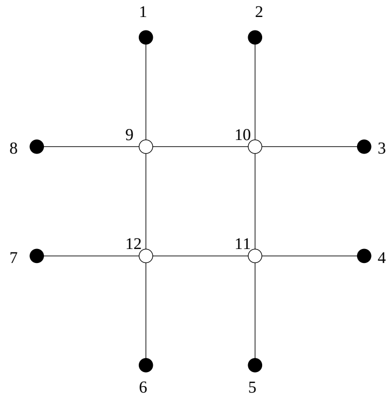

FIGURE 13. Our first victim

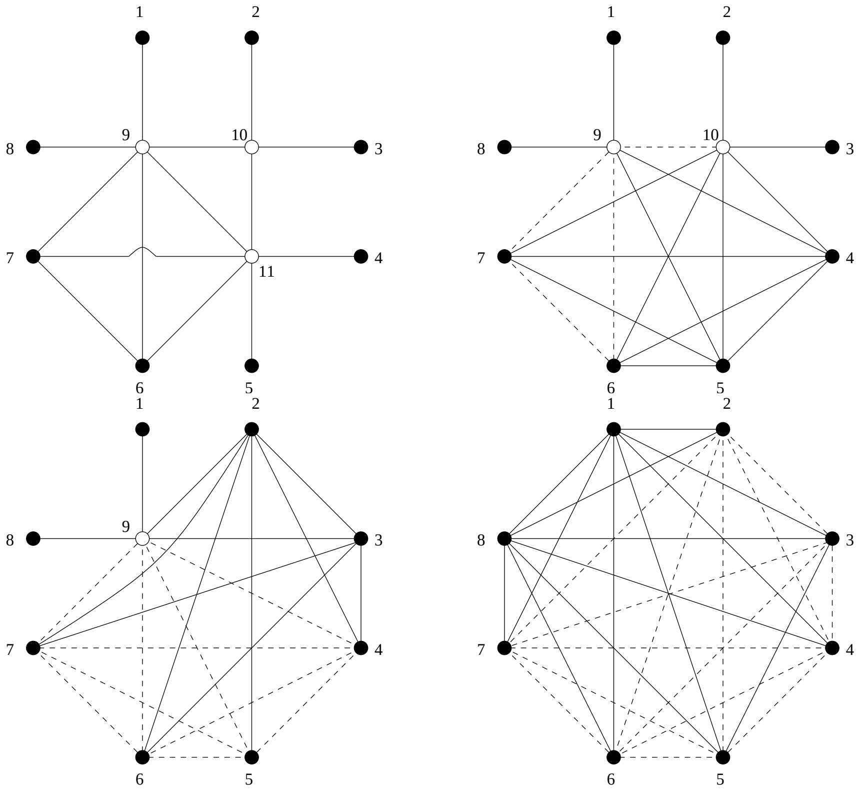

FIGURE 14. Mutilating the poor graph. (Dashed lines represent multiple edges.)

Then we can calculate the Kirchhoff matrix of the graph second-to-last graph, which is the following, except on the diagonal.

(32)

$$\Lambda_9 = \begin{bmatrix} \lambda_{1,1} - \frac{\lambda_{1,8}\lambda_{1,2}}{\lambda_{2,8}} & 0 & \cdots & 0 & 0 & \gamma_1(M_9) \\ 0 & \lambda_{2,2} - \frac{\lambda_{2,8}\lambda_{1,2}}{\lambda_{1,8}} & \cdots & \lambda_{2,7} - \frac{\lambda_{2,8}\lambda_{1,7}}{\lambda_{1,8}} & 0 & \gamma_2(M_9) \\ 0 & \lambda_{3,2} - \frac{\lambda_{3,8}\lambda_{1,2}}{\lambda_{1,8}} & & \lambda_{3,7} - \frac{\lambda_{3,8}\lambda_{1,7}}{\lambda_{1,8}} & 0 & \gamma_3(M_9) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & \lambda_{7,2} - \frac{\lambda_{7,8}\lambda_{1,2}}{\lambda_{1,8}} & \cdots & \lambda_{7,7} - \frac{\lambda_{7,8}\lambda_{1,7}}{\lambda_{1,8}} & 0 & \gamma_7(M_9) \\ 0 & 0 & \cdots & 0 & \lambda_{8,8} - \frac{\lambda_{1,8}\lambda_{2,8}}{\lambda_{1,2}} & \gamma_8(M_9) \\ \gamma_1(M_9) & \gamma_2(M_9) & \cdots & \gamma_7(M_9) & \gamma_8(M_9) & \gamma_9(M_9) \end{bmatrix}$$

where $\gamma(M_9)$ denotes the conductivity function on the star that corresponds to the matrix M_9, namely

$$\gamma_i(M_9) = \sum_{j\neq i} \mu_{ij} + \frac{\mu_{ij}\mu_{ik}}{\mu_{jk}} = \sum_j \mu_{ij} \tag{33}$$

where μ_{ij} is the (i,j)th entry in M_9.

This completes one step. We leave the thrilling algebra of the next three steps to the reader.

9. The Moral of the Story

The algorithm as described in this paper produces good results on a large variety of graphs. Star-complexes (graphs where each internally-connected components ([5]) is a star), graphs where nodes can be ordered so each node has two "spikes" when interiorized, and graphs which in some sense have a low degree of connectivity are straightforward to analyze. Some basic results: any tree with no series connections is recoverable; any star-complex with no loops such that the intersection of each pair of ICC's contains less than three nodes is recoverable.

This approach has also proved fruitful in examining 2-1 graphs that were not well understood in the past. Lovejoy's [3] presents an in-depth examination of some of these graphs. I expect that these methods will soon present the necessary tools to resolve open questions on the existence and nature of n-1 graphs.

The full extent of the terrain that this algorithm deals with efficiently and straightforwardly has not yet been charted.

On the other hand, many recoverable graphs do not produce pleasant results by this algorithm as it currently stands. Graphs with a high degree of connectivity, such as grids and the infamous "circles and rays" graphs ([6]) are often less tractable. While the present recovery algorithm does correctly produce a description of the set of conductivity functions corresponding to a given response matrix, it often does so in parametric form, with a large number of polynomial constraints on the parameters—even when the graph is recoverable. In this form, it is not always easy to determine from the constraints whether or not the graph is recoverable: that is, whether the constraints are resolved uniquely. In these cases the algorithm outlined in this paper effectively reduces one vast mess of polynomial equations to another, less vast but still messy system of polynomial equations.

Parameters that appear in recoverable networks and later drop out because of constraints—called "spurious parameters"—are the largest mystery of this algorithm, and gaining a deep, topologically motivated understanding of when they appear and how they are eliminated is the major outstanding challenge related to this program for incremental recovery. It is known that the constraints that appear are related to determinantal conditions imposed by connections in the graph, but much is left to be explored.

The $\bigstar - \mathcal{K}$ transformation promises to have applications to other problems relating to electrical inverse problems, besides recovery. Because stars can be analyzed by elementary methods where more general graphs cannot, and because general graphs can be constructed from stars, one can examine properties of general graphs in terms of the properties of the corresponding stars.

For example, one may be able to directly prove that the map L is a diffeomorphism for recoverable graphs by composition of the diffeomorphism for stars. $\bigstar$ and $\mathcal{K}$ may as well provide insight into characterizing the set of response matrices of an arbitrary graph. If all of the nodes of a graph are considered to be boundary nodes, then the set of response matrices is simply an e-dimensional vector space, e being the number of edges in the graph. When we interiorize a node, the new set of response matrices is equal to the algebraic $\bigstar - \mathcal{K}$ transformation of the previous set. So the geometry of the set of response matrices can be calculated in terms of the geometry of the response matrices for successive interiorizations.

Remark 9.1. This could go places.

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