THE EQUIVALENCE OF PLANARITY AND CIRCULAR PLANARITY

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1. Transformations

Definition 1.1. For graph with boundary, circular planar graph, see [1] p11.

Definition 1.2. A fixed-ordered circular planar graph is a graph $G$ with boundary nodes $V_1, V_2, \ldots, V_n$, such that there exists a circular planar embedding with these boundary nodes lying on a circle in the circular order $V_1, V_2, \ldots, V_n$.

Remark 1.3. For convenience, let $E(A, B)$ denotes an edge connecting two vertices $A$ and $B$.

Lemma 1.4. Suppose $G(V, V_B, E)$ is a graph with boundary, and $V_B = \{V_1, V_2, \ldots, V_n\}$, let $H(V', E')$ be a graph (not a graph with boundary) such that $V' = V \cap \{P\}$, $E' = E \cap \{E(P, V_1), E(P, V_2), \ldots, E(P, V_n)\}$. $G$ is circular planar if and only if $H$ is planar.

Proof. 1. If $G$ is circular planar, by definition we can embed $G$ in a disc $D$ so that the boundary nodes lie on the bound of $D$. Embed $P$ outside of $D$, then we have a planar embedding of $H$.

2. If $H$ is planar, we can embed it on a plane. Adjust the positions of the adjacent vertices of $P$ (i.e. $V_1, V_2, \ldots, V_n$) topologically so that they lie on a circle $C$, remove $P$ and its adjacent edges, and invert the graph about the circle $C$, then we have a circular planar embedding of $G$. □

Lemma 1.5. Suppose $G(V, V_B, E)$ is a graph with boundary, and $V_B = \{V_1, V_2, \ldots, V_n\}$, let $H(V', E')$ be a graph (not a graph with boundary) such that $V' = V \cap \{P\}$, $E' = E \cap \{E(P, V_1), E(P, V_2), \ldots, E(P, V_n)\} \cap \{E(V_1, V_2), E(V_2, V_3), \ldots, E(V_n, V_1)\}$. $G$ is fixed-ordered circular planar if and only if $H$ is planar.

Proof. The proof is similar to lemma 1.1. □

Remark 1.6. The two lemmas are nameless so far. However, compared to those terminologies we made this year, we might have called them Star-None Transformation and Wheel-None Transformation respectively.

REFERENCES


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