THE EQUIVALENCE OF PLANARITY AND CIRCULAR PLANARITY

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1. Transformations

Definition 1.1. For *graph with boundary, circular planar graph*, see [1] p11.

Definition 1.2. A *fix-ordered circular planar graph* is a graph G with boundary nodes $V_1, V_2, \ldots, V_n$, such that there exists a circular planar embedding with these boundary nodes lying on a circle in the circular order $V_1, V_2, \ldots, V_n$.

Remark 1.3. For convienence, let $E(A, B)$ denotes an edge connecting two verteice A and B.

Lemma 1.4. *Suppose $G(V, V_B, E)$ is a graph with boundary, and $V_B = \{V_1, V_2, \ldots, V_n\}$, let $H(V', E')$ be a graph (not a graph with boudary) such that $V' = V \cap \{P\}, E' = E \cap \{E(P, V_1), E(P, V_2), \ldots, E(P, V_n)\}$. G is circular planar if and only if H is planar.*

Proof. 1. If G is circular planar, by definition we can embed G in a disc D so that the boundary nodes lie on the bound of D. Embed P outside of D, then we have a planar embedding of H.
2. If H is planar, we can embed it on a plane. Adjust the positions of the adjacent vertices of P (i.e. $V_1, V_2, \ldots, V_n$) topologically so that they lie on a circle C, remove P and its adjacent edges, and invert the graph about the circle C, then we have a circular planar embedding of G. □

Lemma 1.5. *Suppose $G(V, V_B, E)$ is a graph with boundary, and $V_B = \{V_1, V_2, \ldots, V_n\}$, let $H(V', E')$ be a graph (not a graph with boudary) such that $V' = V \cap \{P\}, E' = E \cap \{E(P, V_1), E(P, V_2), \ldots, E(P, V_n)\} \cap \{E(V_1, V_2), E(V_2, V_3), \ldots, E(V_n, V_1)\}$. G is fixed-ordered circular planar if and only if H is planar.*

Proof. The proof is similar to lemma 1.1. □

Remark 1.6. The two lemmas are nameless so far. However, compared to those terminologies we made this year, we might have called them *Star-None Transformation* and *Wheel-None Transformation* respectively.

References

[1] Curtis, Edward B. and James A. Morrow. "Inverse Problems for Electrical Networks." Series on Applied Mathematics, Vol 13; 2000.

Date: August 2003.