APPLICATIONS OF THE STAR-K TOOL

TRACY LOVEJOY

ABSTRACT. The purpose of this paper is to apply the useful tool of star-K transformations to recovery questions in several examples to further the understanding of arbitrary networks. The tools used in this paper are also derived and exposed in this paper. Attempts to generalize the star-K tool into an algorithm for arbitrary graph recovery can be found in [1].

Special emphasis is given to explaining the nature of known $2-1$ networks which have gone without clear explanations for too long.

Contents

1. Introduction

The work in [3] outlines a functionally complete solution to the inverse problem it proposes in the circular planar case. There is very little understood with regard to this problem in the non-circular planar case. In this paper we will derive and use star-K transformations, a tool for examining non-circular planar networks. Whether these techniques can be generalized to form a general recovery algorithm is being currently examined in [1], and it is there I refer the reader for a more general exposition.

A motivating example for this technique is the remarkable light that this tool can shed on known $2-1$ networks. The infamous triangle-in-triangle network has never been so clearly understood. In general, many of the cases of annular graphs exposed in [2] have been examined using this tool to various degrees of success.

Date: August 11, 2003.

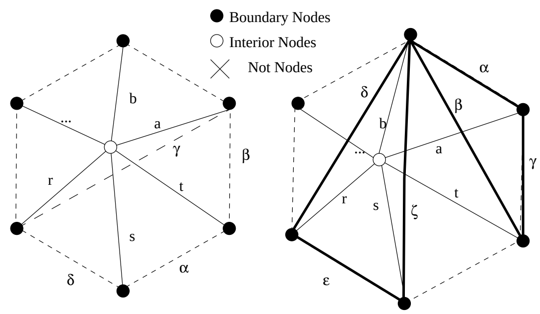

FIGURE 1. quadrilateral and Triangle Conditions

2. The Star-K Transformation with 4 Boundary Nodes

In this section we will examine the Star-K transformation for the case of four boundary nodes. The star with four nodes is also called the plus. K_n is shorthand for the complete graph, a graph with n boundary nodes, no interior nodes, where every node is joined to every other by an edge.

Terms in the response matrix of a star are easily written down by computing the voltage at the only interior node by the weighted average property. Knowing this, the ijth entry in the response matrix is simply the negative of the product of the two conductors joining nodes i and j divided by the sum of the conductors around the interior node. This sum will often be abbreviated σ. Now we want to understand the correspondence between the response matrix of the star and the conductances on K_4. First, the ijth entry in the response matrix is simply the negative of the conductivity on the complete graph joining nodes i and j. Second, when we transform to a K we necessarily pick up some algebraic relations on the conductances. These relations can be thought of as determinants which are zero due to the total lack of two connections in the star, but [1] has a geometric interpretation that is quite clear and useful. That is, the products of opposites sides of a quadrilateral in a K that came from a star are equal. We can quickly prove this with what we have stated thus far. To say

$$\alpha\gamma = \beta\delta \tag{1}$$

is equivalent to saying

$$\frac{st}{\sigma}\frac{ar}{\sigma} = \frac{ta}{\sigma}\frac{rs}{\sigma}.$$

If we call (1) the quadrilateral condition we can rightly call (2) the triangle condition. For any K coming from a star, the product of the legs divided by the base is a constant for any triangle sharing an apex. The proof of this is quite straightforward as well. We simply remark that

$$\frac{\delta\zeta}{\epsilon} = \frac{\alpha\beta}{\gamma} \tag{2}$$

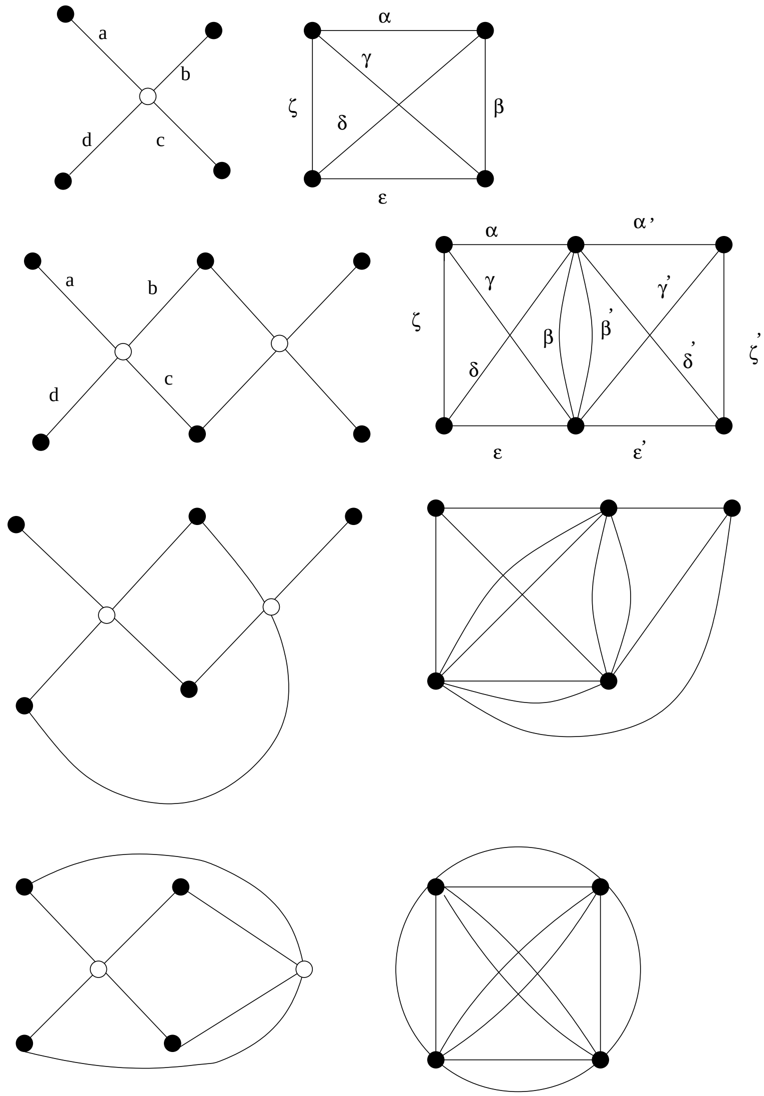

FIGURE 2. The Plus-K and a Few Applications

is equivalent to

$$\frac{br}{\sigma}\frac{bs}{\sigma}\frac{\sigma}{rs} = \frac{ba}{\sigma}\frac{bt}{\sigma}\frac{\sigma}{at}.$$

We can also write down the formula for transforming a star back into a K. If γ_i is the conductor with boundary node i in the star, Σ_i is the sum of the conductors around node i in the K, and K_{ij} is the conductivity on the corresponding edge in the complete graph it is easy to show γ_i is given by the formula

$$\gamma_i = \Sigma_i + \frac{K_{ij}K_{ik}}{K_{jk}}. \tag{3}$$

For the case of a four node K-star the transformation is shown below. Note: roman letters are on the star, Greek on the K, as in the diagram.

$$a = \alpha + \gamma + \zeta + \frac{\alpha\zeta}{\delta} \qquad b = \alpha + \delta + \beta + \frac{\beta\delta}{\epsilon} \tag{4}$$

$$c = \beta + \gamma + \epsilon + \frac{\beta\gamma}{\alpha} \qquad d = \zeta + \delta + \epsilon + \frac{\zeta\delta}{\epsilon} \tag{5}$$

We can also write down the quadrilateral relations,

$$\alpha\epsilon = \gamma\delta \quad \beta\zeta = \alpha\epsilon \quad \alpha\epsilon = \gamma\delta. \tag{6}$$

These equations let us quickly recover a plus from a K_4 as in Figure 2. Figure 2 also shows two plus-graphs joined at two boundary nodes. We see that when we make the transformation to the K_4 on both pluses we get one parallel edge. However, this edge can be eliminated using the quadrilateral condition which shows that the graph can still be recovered.

Next in Figure 2, is two pluses joined at three boundary nodes. This is a more interesting case because the graph is not circular planar so we can not use our circular planar tools to examine it. When we transform both pluses into K's we get three parallel edges. Clearly then, we can not use the quadrilateral condition to recover all the parallel edges and this graph is not recoverable. Furthermore, if we fix one parameter the others can be determined; from this we conclude that the solution space is one dimensional. Lastly, shown is two pluses joined at all boundary nodes. In this case we get all six parallel edges. We have to specify four parameters before we can determine all the conductances so the solution space is four dimensional.

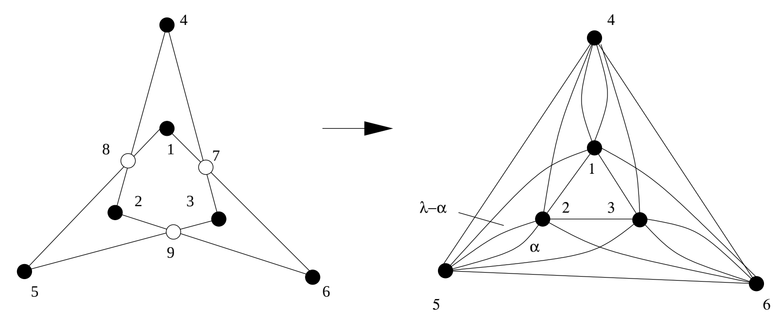

Figure 3. The Triangle in Triangle with Three plus-K Transformations

3. The Triangle in Triangle and other Two to One Graphs

Certain graphs have been found in previous years to have the special property that they could generate the same response matrix for exactly two sets of conductances. The "Triangle in Triangle" was the first of these and was put forth in [2] where an explicit quadratic formula for the conductances was found in terms of entries in the response matrix. Using several pages of manipulations worked by hand the terms in this quadratic formula were approximately 20 terms in length. In this section we use the plus-K tool to derive a simpler quadratic equation and effectively explain the nature of this $2-1$ graph. First we draw the triangle in triangle graph

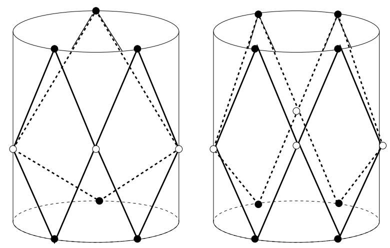

FIGURE 4. The Triangle-In-Triangle and Square-In-Square Graphs Embedded on the Cylinder

to clearly show how it is a sum of three plus-graphs. Then we transform each plus into a K.

Using the quadrilateral condition we can see that the parallel edges cannot be found within a single K_4 so we assign a parameter to one of the edges, say α. Then the other edge must be $\lambda_{25} - \alpha$ so that the sum of the parallel edges is λ_{25}. The quadrilateral relation gives us one of the edges joining node 1 to node 4 as the product of edges 1,2 and 4,5 divided by $\lambda_{25} - \alpha$, or $\frac{\lambda_{12}\lambda_{45}}{\lambda_{25}-\alpha}$. As before, the other edge must be $\lambda_{14} - \frac{\lambda_{12}\lambda_{45}}{\lambda_{25}-\alpha}$. In turn, we can solve for the edges joining node 3 to node 6 with the product of edges 1,3 and 4,6 divided by $\lambda_{14} - \frac{\lambda_{12}\lambda_{45}}{\lambda_{25}-\alpha}$. So one edge is

$$\frac{\lambda_{13}\lambda_{46}}{\lambda_{14} - \frac{\lambda_{12}\lambda_{45}}{\lambda_{25}-\alpha}} \text{ and the other edge is } \lambda_{36} - \frac{\lambda_{13}\lambda_{46}}{\lambda_{14} - \frac{\lambda_{12}\lambda_{45}}{\lambda_{25}-\alpha}}.$$

Applying the quadrilateral condition one final time gives us

$$\lambda_{23}\lambda_{56} = \alpha(\lambda_{36} - \frac{\lambda_{13}\lambda_{46}}{\lambda_{14} - \frac{\lambda_{12}\lambda_{45}}{\lambda_{25}-\alpha}}). \tag{7}$$

This quadratic equation can also be written in the more familiar form,

$$\alpha^2[\lambda_{13}\lambda_{46}-\lambda_{36}\lambda_{14}]+\alpha[\lambda_{14}\lambda_{23}\lambda_{56}+\lambda_{36}\lambda_{14}\lambda_{25}-\lambda_{36}\lambda_{12}\lambda_{46}-\lambda_{13}\lambda_{46}\lambda_{25}]+\lambda_{23}\lambda_{56}\lambda_{12}\lambda_{45} = 0.$$

In this form we can see that we do indeed have a quadratic because the coefficient of α^2 is a non-zero determinant. This determinant corresponds to a connection which is present in the network.

3.1. **The Locus of Degenerate Points.** and other interesting characteristics of this graph have led to previous attempts to understand $2-1$ behavior. Many such attempts were made before the star-K tool was discovered so we will give some mention of these here. First, when the discriminant of this quadratic equals zero we get a locus of points where the conductances can be determined exactly from the response matrix. This discriminant is given by $D = b^2 - 4ac$, or

$$D = [\lambda_{14}\lambda_{23}\lambda_{56}+\lambda_{36}\lambda_{14}\lambda_{25}-\lambda_{36}\lambda_{12}\lambda_{46}-\lambda_{13}\lambda_{46}\lambda_{25}]^2-4[\lambda_{13}\lambda_{46}-\lambda_{36}\lambda_{14}][\lambda_{23}\lambda_{56}\lambda_{12}\lambda_{45}].$$

Another way to examine this behavior is from a topological point of view. If we take a parameterization of α as $\alpha = -\frac{b}{2a} + t\frac{D}{2a}$ with $t \in [-1, 1]$ and then compute the response matrix $\Lambda(t)$, then $\Lambda(t)$ forms a closed curve as t varies.

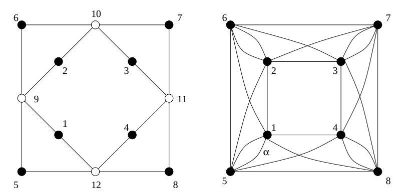

FIGURE 5. The Square in Square Graph and the Star-K Equivalent Graph

3.2. **Other Two to One and 2^n to One Graphs.** were also known to exist in previous years, but without any precise tools to examine them the algebra involved was very complex and in some cases misleading. For example, some networks that seem to have every property of a $2-1$ network in an algebraic sense can actually be shown using the star-K tool to be recoverable. The conjecture we may now be able to prove is that an "n-gon in an n-gon graph" is $2-1$. An "n-gon in an n-gon graph" is a graph consisting of n plus-graphs joined at two boundary nodes such that they form a chain that loops back to the original plus-graph. This can be visualized as in Figure [4] as n diamonds embedded on the cylinder. Annular graphs like those in [2] can in general can be embedded on the cylinder (no caps on the top or bottom) with their boundary nodes on the two boundary circles. Refer to the appendix for a note on the embedding of these graphs on surfaces with boundary.

3.3. **The Square in Square Graph.** is also $2-1$. We will show this by explicitly finding the coefficient of the α^2 term and showing that it is also a non-zero determinant. By a very close analogy to the triangle-in-triangle calculation we can write down the terminating continued fraction version of the quadratic for the square-in-square easily. If we fix the parameter shown in Figure 5 then the quadratic in continued fraction form is

$$\alpha\left(\lambda_{48}-\cfrac{\lambda_{34}\lambda_{78}}{\lambda_{37}-\cfrac{\lambda_{23}\lambda_{67}}{\lambda_{62}-\cfrac{\lambda_{12}\lambda_{65}}{\lambda_{15}-\alpha}}}\right)=\lambda_{14}\lambda_{58}. \tag{8}$$

When we clear denominators we can see the coefficient of the α^2 term is

$$[\lambda_{48}\lambda_{23}\lambda_{67}+\lambda_{34}\lambda_{78}\lambda_{62}-\lambda_{48}\lambda_{37}\lambda_{62}]. \tag{9}$$

As in the triangle-in-triangle case, this is also a non-zero subdeterminant of entries in the response matrix. There must be some reason for this, though it is not yet known. To see that it is so, note that D(2,4,7;3,6,8) $=\lambda_{23}(\lambda_{46}\lambda_{78}-\lambda_{76}\lambda_{48})+\lambda_{34}(\lambda_{26}\lambda_{78}-\lambda_{76}\lambda_{28})+\lambda_{73}(\lambda_{26}\lambda_{78}-\lambda_{67}\lambda_{28})$ is equal to equation 9 if λ_{46} and λ_{28} are zero. They are zero in the response matrix of the square-in-square because those pairs of nodes are not connected through the interior. There is only one way to make the connection (2,4,7;3,6,8) so this determinant is non-zero. At this point we could speculate that the coefficient of the α^2 term in the pentagon-in-pentagon graph shown in Figure 6 will be D(2,8,4,10;7,3,9,5).

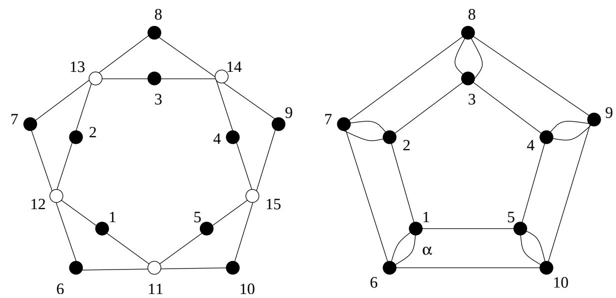

FIGURE 6. The Pentagon in Pentagon Graph and the Star-K Equivalent Graph

3.4. **The Pentagon-in-Pentagon Graph.** shown in Figure 6 has an associated quadratic that can be easily written down in its terminating continued fraction form as

$$\alpha\left(\lambda_{5,10}-\cfrac{\lambda_{45}\lambda_{9,10}}{\lambda_{49}-\cfrac{\lambda_{34}\lambda_{89}}{\lambda_{38}-\cfrac{\lambda_{23}\lambda_{78}}{\lambda_{27}-\cfrac{\lambda_{12}\lambda_{67}}{\lambda_{16}-\alpha}}}}\right)=\lambda_{15}\lambda_{6,10}. \tag{10}$$

Again, we can clear denominators and this will show the coefficient of the α^2 term to be

$$[\lambda_{5,10}\lambda_{49}\lambda_{38}\lambda_{27}+\lambda_{5,10}\lambda_{34}\lambda_{89}\lambda_{27}-\lambda_{5,10}\lambda_{49}\lambda_{23}\lambda_{78}-\lambda_{38}\lambda_{45}\lambda_{9,10}\lambda_{27}+\lambda_{45}\lambda_{9,10}\lambda_{23}\lambda_{78}].$$

That is, in fact, D(2,8,4,10;7,3,9,5) because of the zeros in the response matrix. There is only one way to make this connection as before which guarantees we do in fact have a quadratic. This leads us to make the following conjecture.

Theorem 3.1. *If you number an n-gon-in-n-gon graph clockwise around the inside then clockwise around the outside from the same starting side, then assign a parameter to one of the edges joining nodes 1 and n+1, then*

$$D(2, n+3, 4, n+5, ..., n-1, 2n; n+2, 3, n+4, 5, ..., 2n-1, n)$$

is the coefficient of the quadratic term.

3.5. **This section outlines a proof by recursive definition of coefficients of a Linear Fractional Transformation.** which is equivalent to our terminating continued fraction. First, we need to re-write equations (7), (8) and (10) in the explicit form of a terminating continued fraction. Figure 7 will be our guide. We want to write the terminating continued fraction in a form closely analogous to that in Chrystal's book so, we use λ_n equals the product of the actual λ's in the response matrix on the top and bottom of the the quadrilateral above λ_n, and μ_n is the λ in

FIGURE 7. A Chain of Plus Graphs that Result in a Continued Fraction

the response matrix corresponding to the parallel connection which it is labeled in Figure 7. Using the quadrilateral condition we get a terminating continued fraction that reads

$$\alpha = \mu_4 - \cfrac{\lambda_4}{\mu_3 - \cfrac{\lambda_3}{\mu_2 - \cfrac{\lambda_2}{\mu_1 - \cfrac{\lambda_1}{\alpha}}}}. \tag{11}$$

We will be relying on the recursive formulas in Chrystal's book so we will also write down that his notation for terminating continued fractions is

$$\frac{p_n}{q_n} = a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \cfrac{b_5}{a_5} \cdots}}}. \tag{12}$$

We may also want to stray from this convention and examine what we get if relabel Chrystal's a's and b's so that the numbering is as follows

$$a_5 + \cfrac{b_5}{a_4 + \cfrac{b_4}{a_3 + \cfrac{b_3}{a_2 + \cfrac{b_2}{a_1}\cdots}}}. \tag{13}$$

Chrystal shows that p_n and q_n can be defined recursively by identical recursion formulas; p_n and q_n differ only because of their initial conditions: $p_0 = 1, p_1 = a_1; q_1 = 1, q_2 = a_2$. The recursion formulas are

$$p_n = a_n p_{n-1} + b_n p_{n-2} \quad \text{and,} \tag{14}$$

$$q_n = a_n q_{n-1} + b_n q_{n-2}. \tag{15}$$

When we compare (11) and (12) we find that our α takes the place of Chrystal's a_n. Since a_n only appears in the $p_n th$ and $q_n th$ terms, and in equation (11) we have $\frac{p_n}{q_n} = \alpha$ we can write our terminating continued fraction in the form of a linear fractional transformation (LFT). This LFT is

$$\alpha = \frac{\alpha p_{n-1} + \lambda_1 p_{n-2}}{\alpha q_{n-1} + \lambda_1 q_{n-2}}. \tag{16}$$

In this form we can easily write down the quadratic that corresponds to this LFT. Remarkably, the coefficient of α^2 is simply q_{n-1}. The discriminant, which we will also want to examine is

$$[\lambda_1 q_{n-1} + p_{n-1}]^2 - 4\lambda_1 p_{n-2} q_{n-1}. \tag{17}$$

We can use the work outlined in Chrystal to equate p_n and q_n to certain determinants. Chrystal defines a *continuant* denoted by $\mathbf{K}(i, n)$ where $\mathbf{K}(1, n) = p_n$, $\mathbf{K}(2, n) = q_n$. This is useful because Crystal derives a determinantal expression for $\mathbf{K}(1, n)$;

$$K(1,n) = \begin{vmatrix} a_1 & b_2 & 0 & 0 & 0 & \dots & 0 & 0 \\ -1 & a_2 & b_3 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & a_3 & b_4 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & a_4 & b_5 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & -1 & a_{n-1} & b_n \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & a_n \end{vmatrix}.$$

In this form we can see that there is no effect in the value of $\mathbf{K}$(1,n) if we reverse the numbering of the a's and b's. This is like the renumbering we did in equation (13). Now we have an exact analogy, up to a sign, to the continued fraction of equation (11) that came from our chain of plus-graphs. Now, also note that our λ_n's in equation (11) were the product of two actual λ's in the response matrix so we are going to instead write them with one term above and the other term below the diagonal and state without proof that this does not change the value of the determinant. After doing this we can write out **K**(2,n-1) which we remember is the coefficient of the α^2 term. Thus, for the case of equation (11) representing the square-in-square graph, with $a_4 = \mu_3, a_3 = \mu_2, a_2 = \mu_1, \lambda_3 = b_4, \lambda_2 = b_3$,

$$K(2,4) = \begin{vmatrix} a_4 & b_4 & 0 \\ -1 & a_3 & b_3 \\ 0 & -1 & a_2 \end{vmatrix} = \begin{vmatrix} \lambda_{48} & \lambda_{34} & 0 \\ \lambda_{78} & \lambda_{37} & \lambda_{67} \\ 0 & \lambda_{23} & \lambda_{26} \end{vmatrix}.$$

This determinant, because λ_{28} and λ_{64} are zero, corresponds to the connection $(8,3,6;4,7,2)$ which can also be written as the connection (2,7,4;6,3,8). This connection exists in the original graph, so this determinant is non-zero. This is the same conclusion as we reached by hand and showed in equation (9), but at that point it was just coincidence. Now we can write out the coefficient of α^2 for an arbitrarily long chain of n plus-graphs:

$$\begin{vmatrix} \lambda_{2,n+2} & \lambda_{2,3} & & 0 & 0 & \dots & 0 & 0 \\ \lambda_{n+2,n+3} & \lambda_{3,n+3} & \dots & 0 & 0 & & 0 & 0 \\ & \vdots & \ddots & & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \lambda_{i,n+i} & \lambda_{i,i+1} & & 0 & 0 \\ 0 & 0 & & \lambda_{n+i,n+i+1} & \lambda_{i+1,n+i+1} & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots & \ddots & \vdots & \\ 0 & 0 & & 0 & 0 & \dots & \lambda_{n-1,2n-1} & \lambda_{n-1,n} \\ 0 & 0 & \dots & 0 & 0 & 0 & \lambda_{2n-1,2n} & \lambda_{n,2n} \end{vmatrix}.$$

This determinant corresponds to the connection

$$(2, n+3, 4, n+5, ..., n-1, 2n; n+2, 3, n+4, 5, ..., 2n-1, n),$$

which we know to be non-zero because the connection exists. This guarantees that we have a genuine quadratic term in every case. It remains only to be shown that the discriminant is positive and not always zero to show that we sometimes have two real, positive solutions.

A topic for future study, besides examining the discriminant, would be to take the limit as n goes to infinity in the matrix above and see if you can come up with a meaning associated with the result. This ”circle-in-circle” graph may have interesting properties.

3.6. **The Race Track Graph.** also has an associated equation that looks quadratic, but more analysis is needed to show which terms do or do not vanish. Many other seemingly $2-1$ graphs can be constructed by joining plus-graphs together into various chains that loop back on themselves. The method of examining the terminating continued fraction form of the resultant quadratic equation may prove useful in these case as well. The difference is that in each case we get a product of continued fractions, or some more complicated behavior.

4. A Recoverable Flower

can be found with the star-K tool. By flower, I mean to say a graph with no boundary spikes, and no boundary to boundary connections. Figure 8 can be shown, using star-K transofrmations to be recoverable.

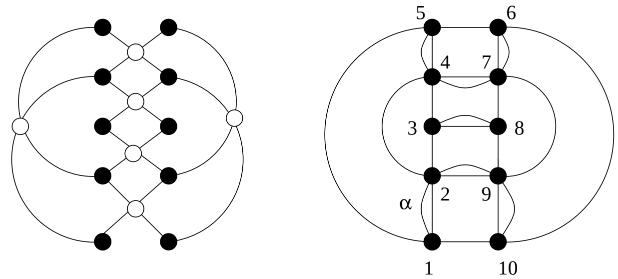

Figure 8. The Race Track Graph and Its Star-K Equivalent

Figure 9. A Recoverable Flower and Its Star-K Equivalent

5. Circles and Rays (Incomplete)

The algebra of this section is quickly turning out to be very complex. The goal is to produce a straightforward exposition of two circles three rays and two circles four rays. Perhaps it is a task for another day, but we have drawn a sequence of Star-K transformations on both graphs that could serve as a guide.

References

[1] Jeffrey Russell. Solving the Inverse Problem. 2003.
[2] Ernie (John) Esser. On Annular Graphs. 2000.
[3] Curtis, B., and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.

University of Washington, Seattle WA
E-mail address: oedipus@u.washington.edu

Figure 10. The 2 Circle 3 Ray Graph and Its Star-K Equivalent

FIGURE 11. The 2 Circle 4 Ray Graph and Its Star-K Equivalent