MODIFICATION OF THE BOUNDARY SET FOR SOLVING THE INVERSE PROBLEM

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Abstract. This paper examines how modifications to the boundary set of a graph can affect the recoverability of that graph. First, we show that boundary promotion in a recoverable graph yields a recoverable graph. Second we will show that promotion in the case of a series connection or an interior pendant can make these structures recoverable. Next we examine the effect of boundary promotion on the medial graph. Lastly we present a new interpretation of the medial graph which can be applied to all planar networks. Taken together, these results allow us to produce new classes of recoverable, non-circular planar graphs, as well as methods for their recovery.

Contents

1. Introduction

We will use the definitions and notion presented in [1]. Some additional preliminaries are necessary:

Definition 1.1. The process of adding an interior node to the boundary set of a graph will be referred to as *promoting* that node to the boundary. The process of removing a node from the boundary set will be referred to as *demoting* or *interiorizing* that node.

If G is a graph, then the notation G^+ will refer to that graph with one or more interior nodes promoted to the boundary. Similarly, G^- will denote G with one or more boundary nodes interiorized. Exactly which nodes are affected will be made clear by context.

This paper asks questions about the recoverability of electrical networks, which consist of a graph G taken together with a set of conductances γ. However, the recoverability of a network is only dependant on the underlying graph, not on the

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particular set of conductances chosen. Therefore throughout this paper we will simply refer to a network as a graph G and assume that there is a particular γ on that graph. When we perform promotions, demotions or other constructions on a graph, we assume that the new graph has the same underlying conductances except as noted.

We will follow the convention of drawing the boundary nodes of a graph as white circles and the interior nodes as black circles. When drawing a medial graph, those faces which correspond to vertices in the original graph will be colored black, other faces will be colored white. Two faces in the medial graph are *adjacent* if they share a common edge. A face is *on the boundary* if one of its edges is part of the boundary circle.

If A is a matrix, then $A[p;p]$ will represent A with the pth row and pth column removed. $A(p;:)$ and $A(:;p)$ will denote the pth row or pth column of A taken as a vector, respectively.

2. Effect of Boundary Promotion on Recoverable Graphs

The first obvious question is how promoting a node to the boundary affects a graph which is already recoverable.

Theorem 2.1. *Let $G = \{V, V_b, E\}$ be a recoverable graph with boundary V_b. If $G^+ = \{V, V_b^+, E\}$ where $V_b \subset V_b^+$, then G^+ is recoverable.*

Proof. Let Λ and Λ^+ be the response matrices for G and G^+, respectively. Let D be the set $V_b^+ - V_b$. Assume we are given G^+ and Λ^+. Note that we can find Λ by the Schur Complement $\Lambda = \Lambda^+/\Lambda^+(D;D)$. Hence we can recover the conductances in G since we know G is recoverable from Λ. But the conductances in G are exactly the conductances in G^+, hence G^+ is recoverable. $\square$

This result fits with our intuition, since adding a node to the boundary set effectively gives us more information about a graph whose conductances we could already recover. This theorem gives us a method for constructing examples of recoverable graphs. Of particular interest is the ability to take a circular planar graph, which we can easily determine to be recoverable, and through boundary node promotion produce a possibly non-circular planar graph which we know to be recoverable.

This theorem can also be interpreted to give the following:

Corollary 2.2. *Let $G = \{V, V_b, E\}$ be a graph with boundary V_b. If there exists a graph $G^- = \{V, V_b^-, E\}$ where $V_b^- \subset V_b$ and G^- is recoverable, then G is recoverable.*

Proof. This statement follows directly from Theorem 1.1. $\square$

When stated in this manner we have produced a useful tool for checking the recoverability of certain graphs. This is of particular use in reducing a planar but non-circular graph to a circular case by demoting particular boundary nodes. However, this tool is limited in use since it requires that we be able to ignore some information about the original graph and still have enough data to determine that it is recoverable.

3. Effect of Boundary Promotion on Interior Pendants

We want to be able to say what nodes in a non-recoverable graph need to be promoted in order to make the graph recoverable. We continue by taking a relatively simple structure which we know causes a graph to be non-recoverable, an interior pendant.

A *pendant* is defined to be a one-valent interior node, along with its incident edge. It is clear that the conductance of the edge in the pendant cannot be calculated from the graph and its response matrix. Our question is, under what circumstances does the promotion of the node in a pendant produce a recoverable graph? We now present a simple case.

Theorem 3.1. *Let G be a graph which contains an interior pendant consisting of a node p incident with an edge e. If G is recoverable when node p and edge e are removed from the graph, then G is recoverable when p is promoted to the boundary.*

Proof. Let G be our graph with node p incident with edge e and adjacent to node q. Let G^+ be the graph G with p promoted to the boundary, with response matrix Λ^+. We must show that G^+ is recoverable. Since we assumed G was recoverable with p and e removed, we can interiorize p and compute the conductance of every edge in G^+ except for e. Let α be the conductance of edge e. We have two cases, depending on whether or not node q is a boundary node.

If q is on the boundary, we can simply read off α from the response matrix as λ^+_{pp}.

If q is in the interior, then we can set up an equation for α using the Kirchhoff matrix K^+. We know K^+ has the form:

$$K^+ = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \begin{array}{c} \\ \\ \\ p \\ \\ \\ q \end{array} \begin{array}{c} \begin{array}{cccccc} & & p & & & q \end{array} \\ \left(\begin{array}{ccc|ccc} & & 0 & & & \\ & & 0 & & & \\ 0 & 0 & \alpha & 0 & 0 & -\alpha \\ \hline & & 0 & & & \\ & & 0 & & & \\ & & -\alpha & & & \alpha+\sigma \end{array}\right) \end{array}$$

where α is the only unknown and σ is the sum of the conductances of edges incident to q, excluding e. Recall that $\Lambda^+ = A - BC^{-1}B^T$. Thus we have

$$\lambda^+_{pp} = A_{pp} - B(p;:)C^{-1}B^T(:;p) = \alpha - (-\alpha)(C^{-1})_{qq}(-\alpha) = \alpha - \alpha^2(C^{-1})_{qq}$$

We know by Kramer's Rule that

$$(C^{-1})_{qq} = (-1)^{q+q}\frac{\det C[q;q]}{\det C} = \frac{\det C[q;q]}{\det C} = \frac{\det C[q;q]}{\alpha \det C[q;q] + \det C'}$$

where C' is the matrix C after subtracting α from C_{qq}. Substituting into the previous equation, we have

$$\lambda^+_{pp} = \alpha - \frac{\alpha^2 \det C[q;q]}{\alpha \det C([q;q]) + \det C'}$$

Solving for α yields

$$\alpha = \frac{\lambda^+_{pp} \det C'}{\det C' - \lambda^+_{pp} \det C[q;q]}$$

We argue that since we assumed that Λ^+ was a response matrix which was produced from a graph with strictly positive conductances, α as given by the above equation will always exist and is strictly positive. □

It should be noted that our assumption in this case is too strong; the graph need not be recoverable without the pendant in order for this promotion to yield a recoverable graph. For example, the graph of Figure 1 is not recoverable when its pendant is removed, but is recoverable when the node is promoted to the boundary.

FIGURE 1.

There are several ways to interpret this result, all of which deserve some discussion. First is as the theorem is presented: it is a tool for deciding which nodes in a non-recoverable graph need to be promoted to have the resulting graph be recoverable. Second is a construction to produce recoverable graphs: it tells us that if we are given a recoverable graph, we may add a pendant with a boundary node and we will still have a recoverable graph. Lastly we have a tool for determining if a given graph is recoverable: if the graph is recoverable with a boundary pendant removed, then the graph is recoverable.

4. Effect of Boundary Promotion on Series Connections

This section asks the same questions as those presented in Section 3, but the object of our attention is now another simple non-recoverable structure, a series connection. A *series connection* is defined to be an interior node of degree two, including its incident edges.

Theorem 4.1. *Let G be a graph which contains a series connection consisting of an interior node p with adjacent nodes q and r. Assume G is recoverable when p and its incident edges are removed and replaced by a single edge joining q and r. If G^+ is the graph G with p promoted to the boundary, then G^+ is recoverable.*

Proof. We must show that G^+ is recoverable given its response matrix Λ^+. Let the conductance of edge pq be α and the conductance of edge pr be β. We can interiorize p by taking the Schur Complement of λ_{pp} in Λ^+. Since edges pq and pr are now connected in series by an interior node, we can remove p, pq and pr from our graph and add edge qr with conductance $\frac{\alpha\beta}{\alpha+\beta}$. Call the resulting graph G'. By assumption G' is recoverable, so we can compute the conductance of every edge in G^+ other than α and β. We also obtain the following relation between α and β:

$$c = \frac{\alpha\beta}{\alpha + \beta}$$

where c is the conductance of edge qr in G' as computed by the recovery algorithm.

Since G' is recoverable, we know we can find voltages $\vec{v}$ on the boundary of G' such that current flows across edge qr. These voltages produce unique voltages at the interior nodes, specifically at q and r. Call these voltages v_q and v_r respectively. We know these values are distinct since current is flowing in edge qr. We now impose the same voltages on the boundary of G^+, with the additional voltage v_p at p such that the current at p is zero. v_p can be computed from Λ^+ as $\vec{v} \cdot \Lambda^+(p;:)$. Hence the voltages at q and r are v_q and v_r. This yields:

$$0 = \alpha(v_p - v_q) + \beta(v_p - v_r)$$

Using these two equations we can derive α and β.

$$\alpha = \frac{-c(v_q - v_r)}{v_p - v_q} \qquad \beta = \frac{c(v_q - v_r)}{v_p - v_r}$$

Since we can argue that v_p, v_q and v_r are all distinct with either $v_q < v_p < v_r$ or $v_q > v_p > v_r$, it is clear that α and β always exist and are positive. $\square$

Again we note that our assumption in this case is too strong. The graph need not be recoverable with the series connection replaced by an edge in order for the promotion to yield a recoverable graph. Figure 2 is an example of such a graph.

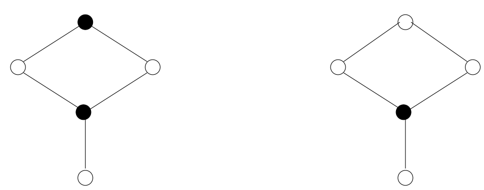

FIGURE 2.

As with the pendant case, Theorem 4.1 can be interpreted in multiple ways. First, given a non-recoverable graph containing series connections we can determine some of the nodes that must be promoted to make the graph recoverable. Second, we know that given a recoverable graph, one can remove an edge and replace it with a promoted series connection and obtain a recoverable graph. Lastly, if we wish to know if a given graph is recoverable, we can remove any two-valent boundary node and its incident edges and replace them with a single edge. If the resulting graph is recoverable, then the original graph is recoverable.

5. Effect of Boundary Promotion and Demotion on Medial Graphs

It is important to note that the results of the preceding sections hold for any graph. However, the effect of promoting or interiorizing boundary nodes in circular planar graphs can be understood visually by examining medial graphs.

Clearly, demoting a boundary node in a circular planar graph will yield a circular planar graph. However, the corresponding statement for node promotion is not necessarily true.

Definition 5.1. Given a fixed drawing of a planar graph G, the *border* of G is the set of nodes which lie on the outer (unbounded) face of the graph.

Theorem 5.2. *For a fixed embedding of a circular planar graph, the only nodes which can be promoted to the boundary and have the resulting graph be circular (with the same drawing) are the interior nodes on the border. These nodes correspond to black faces in the medial graph which are not on the boundary, but are adjacent to white faces which are on the boundary.*

Proof. This fact is clear from inspection of the two graphs and the construction of the medial graph. □

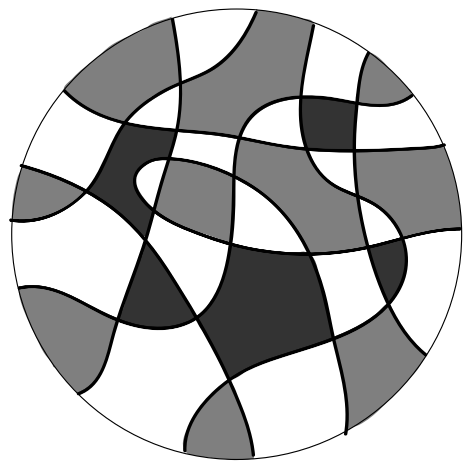

FIGURE 3. Nodes in the medial graph which can be promoted.

The effect of promoting such a node is as follows: The geodesic separating the two faces is split into two geodesics, each with one new endpoint in the former interior of the white face. This extends the black face to the boundary and divides the white face into two white faces on either side of the black face. Note that all other geodesics remain fixed.

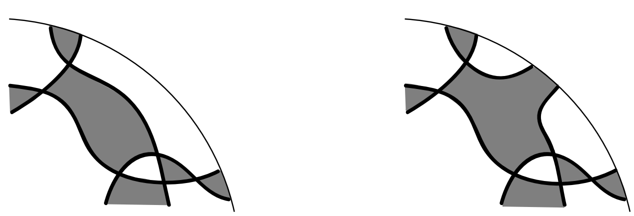

FIGURE 4. Effect of promotion on the medial graph.

This process of dividing a geodesic helps us determine which interior nodes in a non-recoverable circular planar graph must be promoted for the result to be recoverable. We maintain the definition of a *lens* in the medial graph as a pair of geodesics intersecting more than once. A *loop* is a lens formed by a geodesic intersecting itself. A *braid* is a set of two or more lenses formed by two geodesics (or a single geodesic) intersecting more than twice.

Definition 5.3. A lens *bounds* a vertex if that vertex corresponds to a face in the medial graph which lies in the region bordered by the two geodesics forming the lens between their intersections. If a geodesic forming the lens contains a loop (or braided loop) on the exterior of the lens, then the lens also bounds the vertices bounded by that loop.

This definition may not suffice for a full theory on this subject, but it is clear enough for the results we will present here.

Theorem 5.4. *Assume G is a non-recoverable circular planar graph with fixed embedding, with I being the set of interior vertices on the border. Moreover, assume the medial graph of G contains a lens which bounds at least one of the faces corresponding to a vertex in I. Promoting any node in I bounded by the lens will produce a new medial graph which does not contain that lens, and also contains no new lenses. Moreover, promoting any node in I not bounded by the lens will produce a new medial graph which still contains the lens.*

Proof. Note that every node in I which is bounded by a lens corresponds to a face in the medial graph which is bordered by one of the geodesics forming the lens. Moreover, the geodesic forming the lens is the one which separates the node's face from the white face on the boundary. Thus if that node is promoted, one of the geodesics forming the lens will be split into two between the intersections forming the lens. Thus the new medial graph does not contain that lens. If a node not in I is promoted, it will not affect the geodesics forming the lens between their intersections, thus the new medial graph will still contain that lens. $\square$

Similarly, we can use demotion and the medial graph to determine if any boundary information is unnecessary for recovery. It is possible that some boundary nodes provide no useful information, or at least no information that is not attainable from the other boundary data.

Theorem 5.5. *Let G be a recoverable circular planar graph. Let p be a vertex in the boundary of G. If p is interiorized, G^- will be recoverable if and only if the two geodesics on either side of p's corresponding face in the medial graph:*

(1) *Are not the same geodesic.*
(2) *Do not intersect.*
(3) *Do not intersect a common geodesic.*

Proof. The process of interiorizing p causes the two geodesics bordering the corresponding face to join into a single geodesic. It is clear that if any of the listed conditions were to fail, then the resulting medial graph with contain a lens, so G^- will not be recoverable. Conversely, if the above conditions hold, since the process of demoting p leaves all other geodesics fixed, the new medial graph will not contain any new lenses. Thus G^- is recoverable. $\square$

6. A New Interpretation of the Medial Graph

When the earlier results of this paper are combined with our knowledge of the medial graph, these results can give us insight into the recoverability of many planar (but not necessarily circular) graphs.

Our definition of the border of a graph enables us to distinguish between a graph's circular and non-circular components. We can treat those boundary nodes which are on the border the same as we would in a circular graph. It is those boundary nodes which are not on the border (which make that graph non-circular) that we wish to find a way to address. We now present a new definition of the medial graph which attempts to reconcile this issue.

Definition 6.1. The *circular medial graph* of a circular planar graph is the medial graph drawn as laid out in [1]. The *closed medial graph* of a planar graph is the circular medial graph drawn considering every node to be an interior node. The *open medial graph* of a planar graph is the circular medial graph drawn considering those boundary nodes not on the border to be interior nodes.

Our definitions of lens, loop, braid and bounding a vertex are the same as those in Section 5. While the open and closed medial graphs effectively contain the same information, drawing the open graph as opposed to the closed will reduce the number of lenses and make the graph easier to interpret. From here on the term medial graph with no descriptor will refer to the open medial graph.

It is clear that when using the open or closed medial graph we do not have the condition that the graph is recoverable if and only if its medial graph is lensless. However, using the results obtained so far in this paper we can examine relationships between the lenses in the open medial graph and the boundary nodes not on the border to obtain information about the recoverability of a graph.

Recall that $Y-\Delta$ transformations on a graph have a correspondence with moving geodesics on the medial graph. If a face on the medial graph is bordered by exactly three geodesics, moving one geodesic across the intersection of the other two corresponds to a $Y-\Delta$ or $\Delta-Y$ transformation in the original graph, as illustrated in Figure 5. Since $Y-\Delta$ transformations do not affect the recoverability of the graph, no lenses are created or destroyed in the medial graph by this process.

FIGURE 5. A $Y-\Delta$ transformation on the medial graph.

A $Y-\Delta$ transformation cannot be performed if the node at the center of a Y is a boundary node. In the medial graph, this corresponds to not being able to push a geodesic across a face which represents a boundary node.

Also recall that given a lens in the medial graph which does not contain any other lenses, we can perform a series of geodesic $Y-\Delta$ transformations to produce an equivalent medial graph where the lens is not crossed by any other geodesics (bounds only one face of the medial graph). This process is referred to as *emptying* the lens. An empty lens corresponds to a series connection, parallel connection, interior pendant or loop in the original graph.

Theorem 6.2. *If the open medial graph of a planar graph G contains a lens which does not contain any other lenses and does not bound a boundary node, then G is not recoverable.*

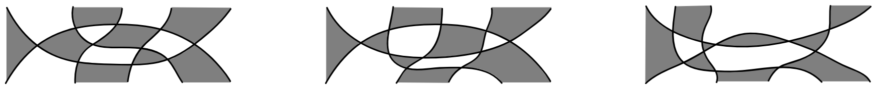

FIGURE 6. Emptying a lens.

Proof. G is non-recoverable for exactly the same reasons that a circular graph whose circular medial graph has lenses is not recoverable. Since the lens contains no boundary nodes, we can empty the lens as we would in the circular case. This shows that G is $Y - \Delta$ equivalent to a graph which contains a series connection, a parallel connection, interior pendant or loop. Since each of these structures causes that graph to be non-recoverable, G is not recoverable. □

This theorem provides us with a necessary condition for a planar graph to be recoverable. Next we provide a sufficient condition for planar graphs with some assumptions.

Theorem 6.3. *Let G be a planar graph such that the open medial graph of G contains no loops or braids. Additionally assume that none of the lenses in the medial graph of G intersect. If every lens in the medial graph bounds at least one boundary node, then G is recoverable.*

Proof. If there are any lenses in the medial graph of G which bound more than one boundary node, we interiorize arbitrary boundary nodes until each lens bounds exactly one. We will show the resulting graph, G^-, is recoverable, which by Theorem 2.1 implies that G is recoverable.

Choose an arbitrary lens in G^-. We can empty that lens such that the only face it bounds is the one corresponding to the boundary node it contained. Since none of the lenses of G intersected or formed braids, we can perform this process on all the lenses of G^- simultaneously. Doing so produces no new lenses in the medial graph. We have now produced a graph whose lenses all correspond to series connections. By applying Theorem 4.1 recursively, we conclude that G^- is recoverable. □

This theorem can actually be stated for a slightly more general case. It is only necessary that the graph be $Y - \Delta$ equivalent to one which has no intersecting lenses, and every lens in the original graph must bound a boundary node which is not bounded by any other lens.

Example 6.4. The graph of Figure 7 is an example of a non-circular graph which we can now show is recoverable. This graph's medial graph is also shown in Figure 7. The darkened faces in the medial graph coorespond to boundary nodes not on the border.

We can perform $Y - \Delta$ transformations so that none of the lenses intersect, and the resulting graph satisfies the hypothesis of Theorem 6.3. After emptying each of the lenses, we see that our original graph is equivalent to the graph of Figure 8.

Replacing each of the boundary nodes not on the border and their incident edges with single edges produces the critical circular planar graph of Figure 9. Thus by applying Theorems 6.3 and 4.1 we can conclude that our original graph was recoverable.

These results are far from a general theory discussing how this definition of the medial graph relates to a graph's recoverability. However, they do provide

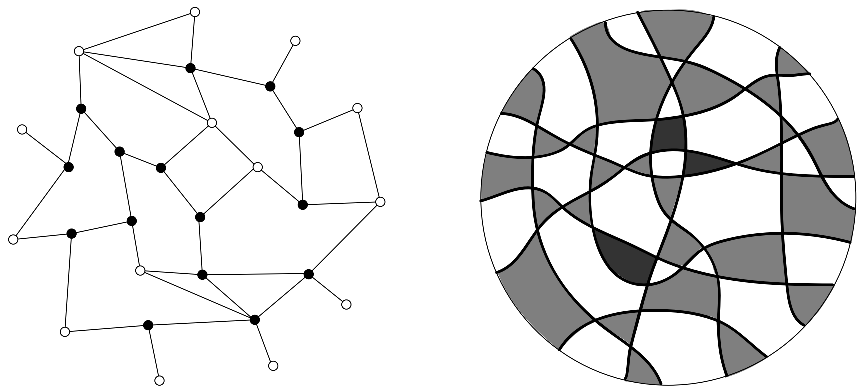

FIGURE 7. A non-circular planar graph.

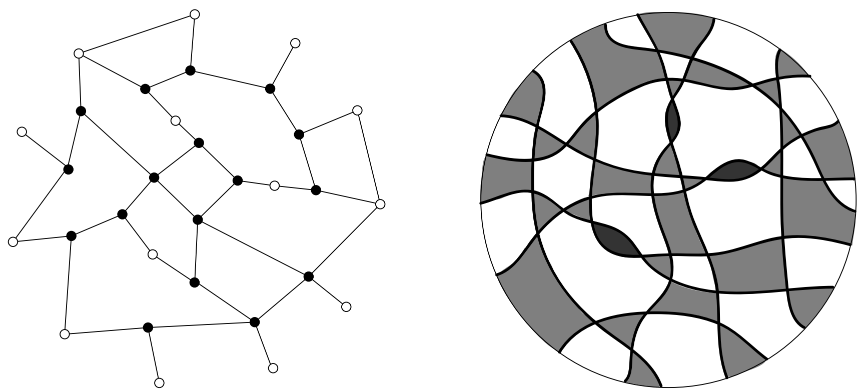

FIGURE 8. A non-circular planar graph.

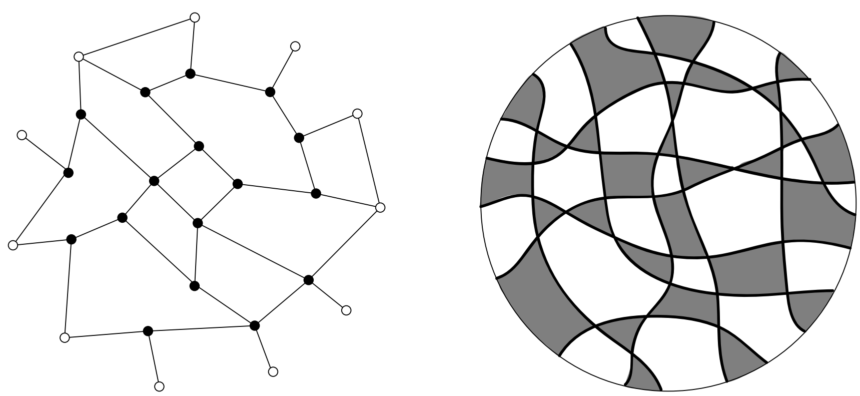

FIGURE 9. A non-circular planar graph.

useful information in certain cases, which lends confidence that this approach is worthwhile.

Conjecture 6.5. A planar graph G is recoverable if and only if the boundary nodes of G satisfy some relation with the lenses of the closed medial graph of G.

What the prescribed relation might be is left open. A naive hypothesis is that every lens in the medial graph must bound at least one boundary node.

7. Future Research

The following questions remain open, and answers to them may provide insight into solving the inverse problem:

(1) Given a non-recoverable graph, what is a minimum set of boundary nodes that must be promoted to generate a recoverable graph?
(2) Given a recoverable graph, what is a maximum set of boundary nodes that can be interiorized and still generate a recoverable graph?
(3) Given a planar, but non-circular graph, what is a minimum set of boundary nodes that must be demoted to produce a circular graph?
(4) Section 5 discusses how boundary promotion on the border can make some structures recoverable. Do the same promotions make these structures recoverable when not on the border?
(5) Does Conjecture 6.3 hold? If so, what is the relationship between lenses and boundary nodes?

References

[1] Edward B. Curtis and James A. Morrow. "Inverse Problems for Electrical Networks." Series on applied mathematics – Vol. 13. World Scientific, ©2000.

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