RECOVERING NETWORKS WITH NEGATIVE CONDUCTIVITES

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Abstract. It is known that a critical circular planar graph can be recovered if conducitivies are restricted to positive real numbers. If the range of conductivities is extended to all nonzero real numbers, we are still able to recover the network if certain conditions are satisfied.

Contents

1. Negative Conductivites

Let $G = (V, V_b, E)$ be a graph for which V is the set of vertices, V_b is a subset of V denoting boundary vertices, and E is the set of edges. Let γ be the conductivity function on E, which assigns each edge e to a real number, not necessarily positive. The Kirchoff matrix K is the matrix defined as follows. $K_{i,j} = -\gamma(i,j)$, where $\gamma(i,j)$ is the conductivity of the edge joining nodes i and j, and $i \neq j$. $K_{i,j} = 0$ is no edge joins nodes i and j. If $i = j$, $K_{i,j} = \sum_{j \neq i} \gamma(i,j)$ As before, Λ_γ denotes the response matrix, which is the linear map which sends the voltage to a current

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on the boundary nodes. We compute $\Lambda = K/(I;I)$, that is, we take the Schur complement of K with respect to the entries corresponding to the interior nodes.

K has the block structure:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{1}$$

Λ is defined if and only if $det(C) \neq 0$. This identity holds when G is a connected graph and $\gamma > 0$ for all edges. However, when γ is allowed to range over all real numbers, we are not always able to calculate Λ. The simplest example is a graph with two boundary nodes, labeled 1 and 2, and one interior node, labelled 3. Let $\gamma(1,3) = 1$, $\gamma(2,3) = -1$, and $\gamma(1,2) = 0$. Then the network response cannot be calculated.

2. Recovering the Y Graph

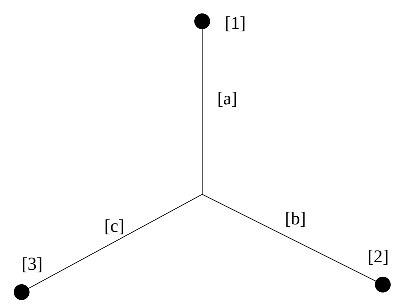

Figure 1. The Y Graph

Consider the simple Y graph, with boundary nodes 1, 2, and 3 and interior node 4. The conductivities of the edges are labelled a, b, and c. (See Figure 1 on page 2.) We get the Kirchoff matrix and response matrix,

$$K = \begin{bmatrix} a & 0 & 0 & -a \\ 0 & b & 0 & -b \\ 0 & 0 & c & -c \\ -a & -b & -c & a+b+c \end{bmatrix} \tag{2}$$

$$\Lambda = \frac{1}{a+b+c} \begin{bmatrix} ab+ac & -ab & -ac \\ -ab & ab+bc & -bc \\ -ac & -bc & ac+bc \end{bmatrix} \tag{3}$$

We can calculate Λ if and only if $a+b+c \neq 0$. Alternately, suppose we are given a response matrix $\Lambda = [\lambda_{i,j}]$. Then we can use the response matrix shown above to get three equations,

$$\frac{-ab}{a+b+c} = \lambda_{1,2}$$

$$\frac{-ac}{a+b+c} = \lambda_{1,3}$$

$$\frac{-bc}{a+b+c} = \lambda_{2,3}$$

Solving these equations yields,

$$a = \frac{\lambda_{1,2}\lambda_{1,3} + \lambda_{1,2}\lambda_{2,3} + \lambda_{1,3}\lambda_{2,3}}{\lambda_{2,3}}$$

$$b = \frac{\lambda_{1,2}\lambda_{1,3} + \lambda_{1,2}\lambda_{2,3} + \lambda_{1,3}\lambda_{2,3}}{\lambda_{1,3}}$$

$$c = \frac{\lambda_{1,2}\lambda_{1,3} + \lambda_{1,2}\lambda_{2,3} + \lambda_{1,3}\lambda_{2,3}}{\lambda_{1,2}}$$

These equations may be solved under the conditions $\lambda_{1,2} \neq 0$, $\lambda_{1,3} \neq 0$, and $\lambda_{2,3} \neq 0$, which imply $a \neq 0$, $b \neq 0$, and $c \neq 0$. The last three equations, then, are the conditions under which the Y graph may be uniquely recovered from Λ. A operation similar to the $Y - \Delta$ transformation exists when we allow $\gamma \leq 0$, but when we use this operation to recover a network, we must verify $a + b + c \neq 0$, $a \neq 0$, $b \neq 0$, and $c \neq 0$.

3. Recovering the Kite Graph

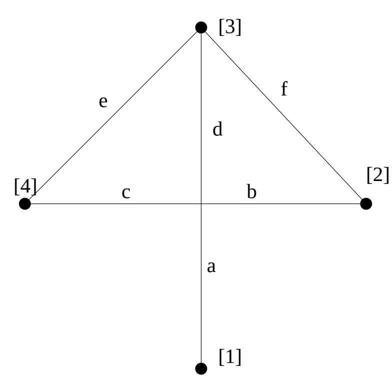

Figure 2. The Kite Graph

Before we can recover the Towers of Hanoi graph with four boundary vertices, Σ_4, we should look at the "kite" graph, which can be obtained from Σ_4 by doing a $Y - \Delta$ transformation. (See Figure 2 on page 3.) The Kirchoff matrix and the upper triangular portion of the response matrix are given as follows:

$$K = \begin{bmatrix} a & 0 & 0 & 0 & -a \\ 0 & b+f & -f & 0 & -b \\ 0 & -f & d+e+f & -e & -d \\ 0 & 0 & -e & e+c & -c \\ -a & -b & -d & -c & a+b+c+d \end{bmatrix} \tag{4}$$

$$\Lambda = \begin{bmatrix} \Sigma & 0 & 0 & 0 \\ & \Sigma & -e & 0 \\ & & \Sigma & -f \\ & & & \Sigma \end{bmatrix} - \frac{1}{a+b+c+d} \begin{bmatrix} \Sigma & ab & ad & ac \\ & \Sigma & bd & bc \\ & & \Sigma & cd \\ & & & \Sigma \end{bmatrix} \tag{5}$$

Suppose we are given a response matrix $\Lambda = \lambda_{i,j}$. Then we can use the following four equation to solve for a, b, c, and d. Let $\sigma = a + b + c + d$.

$$\frac{-ab}{\sigma} = \lambda_{1,2}$$

$$\frac{-ad}{\sigma} = \lambda_{1,3}$$

$$\frac{-ac}{\sigma} = \lambda_{1,4}$$

$$\frac{-bc}{\sigma} = \lambda_{2,4}$$

If $\lambda_{1,2} \neq 0$, $\lambda_{1,4} \neq 0$, and $\lambda_{2,4} \neq 0$, we can solve for a, b, c, and d uniquely. These conditions imply $a \neq 0$, $b \neq 0$, and $c \neq 0$. The values of e and f immediately follow. Note that $\lambda_{1,3} = 0$ is acceptable, as is $d = 0$. It is also possible to recover the network when $e = 0$ or $f = 0$. So, $a \neq 0$, $b \neq 0$, $c \neq 0$, and $a+b+c+d \neq 0$ are necessary and sufficient conditions for recovering the network.

4. Recovering the Top-hat Graph

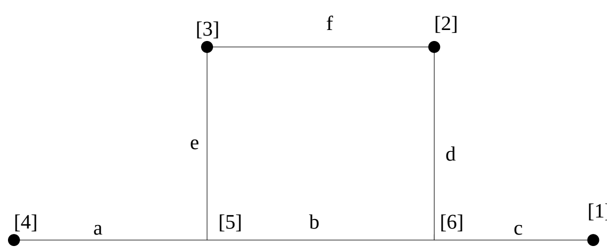

Figure 3. The Top-hat Graph

The top hat graph, pictured in Figure 3 on page 4, is the Towers of Hanoi graph with four boundary nodes, also denoted Σ_4. Trying to recover the Top-hat with a system of equations, as we did for the kite, would be cumbersome because there are two interior nodes. However, we can eliminate one of the interior nodes by performing a $Y - \Delta$ transformation around node 5. Then we have the kite graph, which we know how to recover and the conditions under which we may do so.

Therefore, sufficient conditions for recovering the top-hat include $a \neq 0$, $b \neq 0$, $e \neq 0$, and $a + b + e \neq 0$ for making the $Y - \Delta$ transformation. Each edge of the new Δ has non-zero conductivity. To recover kite, we furthermore need $c \neq 0$ and $e \neq 0$. We also need to know that the sum of the conductivities of the edges adjacent to node 6 is not 0 after the transformation is performed. After making the

$Y-\Delta$ transformation, the sum becomes $\frac{ab+ac+ad+bc+bd+be+ce+de}{a+b+e} \neq 0$ if and only if $\det K(5,6;5,6) \neq 0$, where K is the Kirchoff matrix of the Top-hat graph.

The algorithm for recovering the top-hat is as follows. Treat the response matrix as the response matrix of the kite, and recover the kit. Then perform a $\Delta - Y$ transformation around nodes 3, 4, and 5. If the hyptheses of the algorithm are not satisfied, the algorithm will fail to provide a unique γ for the top-hat graph.

Alternately, it is possible $a+b+e=0$ but $b+c+d \neq 0$. In that case we can relabel rows and columns of Λ so node 4 corresponds with the tail of the kite instead of node 1. Then we can recover the kite and perform the $\Delta - Y$ transformation around nodes 1, 2, and 5.

If $a+b+e \neq 0$ and $b+c+d \neq 0$, we write $a=-b-e$ and $c=-b-d$. Then

$$\Lambda = \frac{1}{b}\begin{bmatrix} \Sigma & 0 & -be-de & (b+d)(b+e) \\ & \Sigma & -bf+de & -bd-de \\ & & \Sigma & 0 \\ & & & \Sigma \end{bmatrix} \tag{6}$$

We can solve this equation uniquely for b, d, e, and f if and only if $b \neq 0$.

5. The Well Connected Graph with 5 Nodes

In recovering the Towers of Hanoi graph with five boundary nodes, Σ_5, we use a similar procedure to reduce the number of interior nodes to one. First perform a $Y-\Delta$ transformation of the left and right interior nodes. Then eliminate the top interior node by transforming the kite into a complete graph with four vertices. Each of these operations is invertible, provided that $\gamma_{i,j} \neq 0, i \neq j$ and the appropriate subdeterminants of K are nonzero. Relabel the remaining interior node 6, and use the given response matrix to recover the modified network. The response matrix of the modified graph is as follows, with $\sigma = \gamma_{1,6}+\gamma_{2,6}+\gamma_{3,6}+\gamma_{4,6}+\gamma_{5,6}$

$$\Lambda = \begin{bmatrix} \Sigma & -\gamma_{1,2} & 0 & 0 & 0 \\ & \Sigma & -\gamma_{2,3} & -\gamma_{2,4} & 0 \\ & & \Sigma & -\gamma_{3,4} & 0 \\ & & & \Sigma & -\gamma_{4,5} \\ & & & & \Sigma \end{bmatrix} - \frac{1}{\sigma}\begin{bmatrix} \Sigma & \gamma_{1,6}\gamma_{2,6} & \gamma_{1,6}\gamma_{3,6} & \gamma_{1,6}\gamma_{4,6} & \gamma_{1,6}\gamma_{5,6} \\ & \Sigma & \gamma_{2,6}\gamma_{3,6} & \gamma_{2,6}\gamma_{4,6} & \gamma_{2,6}\gamma_{5,6} \\ & & \Sigma & \gamma_{3,6}\gamma_{4,6} & \gamma_{3,6}\gamma_{5,6} \\ & & & \Sigma & \gamma_{4,6}\gamma_{5,6} \\ & & & & \Sigma \end{bmatrix} \tag{7}$$

Therefore, we can use $\lambda_{1,3}$, $\lambda_{1,4}$, $\lambda_{1,5}$, $\lambda_{2,5}$, and $\lambda_{3,5}$ to solve for $\gamma_{1,6}$, $\gamma_{2,6}$, $\gamma_{3,6}$, $\gamma_{4,6}$, and $\gamma_{5,6}$. Then we use the remaining entries of Λ to solve for the rest of the conductances. The equations will have a unique solution if $\sigma \neq 0$ and $\gamma_{1,6} \neq 0$, $\gamma_{3,6} \neq 0$, and $\gamma_{5,6} \neq 0$. The former is guaranteed by $\det K(I;I) \neq 0$, where I denotes the set of interior nodes. The latter condition is true because $\gamma_{i,j} \neq 0, i \neq j$ on the original graph, and this property is preserved through the transformations except possibly on $\gamma_{2,6}$ and $\gamma_{4,6}$. Therefore, we have the following sufficient conditions for recovery of Σ_5:

- $\gamma_{i,j} \neq 0, i \neq j$
- $\det K(I;I) \neq 0$
- $\sum_{j\ \ i} \gamma_{i,j} \neq 0$, for each interior node except possibly the center node.

6. Important Formulas

The approach of making transformations and using a system of equations on a graph with one interior node may work for networks with more than four boundary nodes, but it becomes very cumbersome. It is time for a new approach. We first need the following formula from [1]. P and Q are disjoint sets of boundary nodes, each of size k. The first summation is over the permutation group S_k, and the second is over all paths α in a connection from P_i to $Q_{\tau(i)}$. E_α is the set of edges used in α. J_α is the set of interior nodes not used on α, while D_α is $\det K(J_\alpha, J_\alpha)$.

$$\det \Lambda(P;Q) \cdot \det K(I;I) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau)\{ \sum_{\substack{\alpha \\ \tau_\alpha = \alpha}} \prod_{e \in E_\alpha} \gamma(e) \cdot D_\alpha \} \tag{8}$$

We also need the boundary edge and boundary spike formulas. $P = (p_1, \ldots, p_k)$, $Q = (q_1, \ldots, q_k)$ $P' = (p, p_1, \ldots, p_k)$, $Q' = (Q, Q_1, \ldots, Q_k)$, and pq is a boundary edge. A $(k+1)$ connection from (P', Q') is broken by deleting pq. As given in [1], (P', Q') is a circular pair, but for the proof of the formula it is sufficient that $\Lambda(P;Q) \neq 0$. Then,

$$\gamma(pq) = -\Lambda(p;q) + \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;q) \tag{9}$$

Similarly, let pr be a boundary spike between boundary node p and interior node r. Suppose the contraction of pr breaks a connection between two disjoint sets of boundary nodes P, Q, and $\Lambda(P;Q) \neq 0$. (P, Q) is not necessarily a circular pair. Then,

$$\gamma(pr) = \Lambda(p;p) - \Lambda(p;Q) \cdot \Lambda(P;Q)^{-1} \cdot \Lambda(P;q) \tag{10}$$

We will see that, for a critical graph, we can recover a boundary edge or a boundary spike is we merely know that the conductivities are nonzero. However, we cannot use any broken connection (P, Q) for this; we need to insure $\Lambda(P;Q) \neq 0$. Using the determinental identity above, we have three sufficient conditions for $\Lambda(P;Q) \neq 0$.

- There is only one τ so that there is a set of paths α from P_i to $Q_\tau(i)$.
- For the fixed τ, there is only one α joining (P, Q).
- $J_\alpha = \emptyset$ guarantees $D_\alpha \neq 0$.

If we have a method of constructing the connection (P, Q), we can use a process similar to the process for recovering a circular planar network with positive conductivities. Recover the boundary edges and boundary spikes using the given formulas, the remove the edges and contract the spikes, performing the appropriate operations on the response matrix. Because the graph is critical, the process will terminate with every edge recovered.

7. Minimal Paths and Connections

Suppose a graph G has boundary nodes p and q that can be joined by possibly many paths through the interior. Consider p to be to counter-clockwise of q, though there may be other boundary nodes between them. Which node is considered clockwise and which is considered counter-clockwise is arbitrary for the purpose of definition. Consider pq to divide G into two components.

Definition 7.1. The minimal path pq is the path which minimizes the component of the graph counter-clockwise to p. It will be denoted $\min(pq)$.

Note that $\min(pq)$ and $\min(qp)$ are generally different paths. $\min(pq)$ can be constructed by taking the first edge counter-clockwise from p that can lead to q, and by taking the first edge that leads to q counter-clockwise from the previous edge until the path reaches q.

Suppose $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ are ordered sets of boundary nodes. Unlike the conventional idea of a connection, a minimal connection must connect p_i to q_i for $1 \leq i \leq k$. The minimal connection is constructed as follows. First construct the minimal path from p_1 to q_1. When the first $i-1$ paths are drawn, draw the minimal path from p_i to q_i, but the paths from which the minimal path can be chosen are restricted to those that were not used by any previous path.

8. Building a Maximal Connection

We are now ready to build a connection with nonzero subdeterminant that is broken by deleting a boundary edge or contracting a boundary spike. We will start with the boundary edge case, though both are similar. Suppose we have boundary edge pq, with p immediate clockwise to q. Because the graph is circular planar, the various edges can be thought to divide the graph into cells. For every interior node on the border the cell bounded by pq, draw the minimal path to the boundary with respect to the pq to get a set of minimal paths. If a given path has one endpoint p_k clockwise to p, and one endpoint q_k counter-clockwise to q, then it is $\min(p_k q_k)$. For all such paths obtain in this manner, consider $p_k \in P$ and $q_k \in Q$ as the corresponding endpoint. For most critical circular planar graphs, the number of such paths obtained will be significantly less than the number of interior nodes used to obtain them, because several interior nodes will lead to the same path.

It is possible that two distinct paths share one endpoint, which will lead to problems in constructing the connection. In that case, adjoin a boundary spike of known conductivity to the problematic boundary node, and interiorize that node. Then combine the two paths into one longer path.

When a set of minimal paths is laid, collect all the interior nodes that are in the new cells next to the previous set of minimal paths. Then use them to lay another set of minimal paths, subject to the restriction that the new paths do not intersect the old paths. Classify the endpoints of the new paths in P and Q in the same manner as before. Note that, after each step, the paths form the minimal connection $\min(P, Q)$. The process terminates when no more paths can be laid.

9. There is only one Permutation

The first of the three conditions we need to verify, to conclude that the connection we constructed has nonzero determinant, is that it can only be formed through one permutation. Suppose we start with boundary node p in the original spike and look for a member of Q with which it may be connected. It can be connected with the original q by the construction of the connection. Now we will traverse the graph counter-clockwise in search of another mate for p. The next node counter-clockwise is another element of Q. If we connect it to p, the original q will have no open path to another element of P. Because every P we encounter on our counter-clockwise search corresponds to a Q we already found, there will always be one more Q than

P left in the portion of the graph counter-clockwise to p after the path is drawn. So, p must be joined to q.

Suppose we have verified that the first endpoints of the first n sets of paths laid must be connected in the manner constructed. If multiple paths were laid on the $(n+1)$th step, the endpoints from one path cannot be connected to the endpoints of another path except possible through paths already laid, so we can consider each path and all subsequent paths in the regions marked by the paths on the $(n+1)$th step separately. If we do that, then the endpoints of the paths on the $(n+1)$th step must connect in the manner constructed by the same argument by which p and q were connected. Therefore, only one $\tau \in S_k$ will be nontrivial in (8).

10. No interior nodes skipped over . . . Almost

We need to verify that no interior nodes were skipped in the process of laying all the paths down. Suppose there is one, trapped between the nth and $(n+1)$th layers of paths. Consider the region of the graph bounded by the boundary circle and the aforementioned paths. Then consider a portion of that new boundary, only the $(n+1)$th layer of paths and the boundary circle. Suppose the lost interior node has two paths to the restricted interior. If the two paths lead to the same path on the $(n+1)$th layer, then that path would not be minimal. If the interior node leads to two separate paths, then they would have been joined into one path. If one leads to a path and the other to a boundary, then the path would have not have been minimal. If the interior node goes to the boundary circle at two distinct points, then an additional path would have been drawn. Therefore, this interior node can have at most one path that does not lead to the nth layer path. We will soon see that in this case, the graph is not critical.

11. Criticality

There are several things that can prevent the constructed subdeterminant from being nonzero, but each one of them implies a non-critical graph. First, suppose there is an edge between two non-consecutive nodes in a path. To see that a graph with this property is not critical, construct a subgraph as follows. First, include all the nodes in the path that are between the extra edge. Then include all edges from these nodes to nodes in a previously laid path, and all the nodes in the previous path that are between the ones included. Continue adding previous layers until no more layers are added, or until the process reaches the first path. Assume that there are no interior nodes not used in a path (proven next) and that this instance of an extra edge is the first one.

Boundary nodes will be designated as follows: All nodes that were either boundary nodes in the original graph, or connect to a node not included in a subgraph. All edges are included that join two nodes in the subgraph. By RESULT, such a graph is critical if the original graph is critical. Therefore, we will show that this subgraph is not critical and thereby show the original graph was not critical.

If a connection between a circular pair exists, it may be connected using minimal connections between successive pairs of consecutive nodes, provided that a pair of nodes is removed from the connection once the path between them has been drawn. Start with either end of the circular pair, drawing minimal paths between neighboring members of P and Q. Continue the process until the node using the higher original path (O Path) moves to the other side of the subgraph. At that point, start drawing the minimal paths in from the other end of the circular pair and complete the connection.

The first path in the circular pair (CP Path) will not use any paths in the original set of paths that are not between the two endpoints. Otherwise, the CP Path would not be minimal. The node using the higher O Path in the next two nodes of the circular pair will use a higher O Path than the previous, so it has no reason for it to use a higher O Path than that of the higher node. This will remain true for the entire process, so no CP Path needs to be drawn higher than the highest O Path. Hence, the extra edge is unnecessary in the subgraph graph for any circular pair, and the original graph is not critical. We conclude that a critical circular planar graph will have no extra edges along any minimal paths constructed in the manner described.

The argument that there cannot be any unused interior nodes, as described in the previous section, is similar. For our subgraph, include the offending interior node(s), whatever one area above the nth subgraph it/they connect(s) to, and the same subnetwork below the nth subgraph that we included before. For the sake of simplicity, consider whatever network of interior nodes trapped between the paths to be a single node. This single node connects to some other node with a boundary spike. We want to show that the subgraph is not critical and, in particular, that we can contract the said boundary spike.

If contracting the boundary spike breaks a connection, that connection must not include the spike itself. So, any circular pair that requires the spike must not involve the spike. By the same argument outlined above, no such circular pair needs any portion of the subgraph above the nth path, so we can contract the boundary spike without breaking a connection. So, there are no interior nodes not used by the set of paths laid by the above procedure.

12. Boundary Spikes

It is worth noting the difference between boundary edges and boundary spikes. Suppose the network has a boundary spike pr, with p as the boundary node. In constructing the set of paths in the maximal connection, starting with pr, we want to draw the first minimal path over r and skip p in the first path. In constructing the subgraph used in the proofs that there are no extra edges or unused interior nodes, include p if the subgraph includes r and two endpoints of the first path that are not r. Then the argument will work as outlined above.

13. Uniqueness of Paths

We have to show that the given set of paths is the only set of paths that will connect the nodes of P and Q. Let $p_k q_k$ be the first pair of nodes that might be connected with an alternate path. The original path given is a minimal path, so it cannot use any lower nodes. It also does not have any "shortcuts", or extra edges it can use to achieve the same connection with the same interior nodes, albeit in a different order. Then $p_k q_k$ must use a higher interior node. Because there are no interior nodes not used in a path, the new path $p_k q_k$ interferes with the path $p_{k+1} q_{k+1}$. The necessarily modified $p_{k+1} q_{k+1}$ interferes with $p_{k+2} q_{k+2}$, and so on to the final path. Suppose there are n paths, so $p_n q_n$ must be drawn differently. There are no extra edges or interior nodes for $p_n q_n$ to use, and no further paths for

$p_n q_n$ to borrow from, so it cannot be drawn. So, we conclude there is no alternate set of paths to join (P, Q) other than the set prescribed by the construction.

If we build (P, Q) over a boundary edge pq, deleting the edge must break the connection (P, Q) because pq was used as a path. If we build (P, Q) over a boundary spike pr, contracting pr breaks the connection (P, Q) because the connection used what was previously interior node r.

14. Contracting Boundary Spikes: An Annoying Problem

We have no problem obtaining a boundary spike, but contracting the spike is another matter. Suppose we adjoin a boundary spike of conductivity ξ to boundary node 1. If the old response matrix looks like this:

$$\begin{bmatrix} \lambda_{1,1} & a \\ b & C \end{bmatrix} \tag{11}$$

The new one looks like this, with $\delta = \lambda_{1,1} + \xi$:

$$\begin{bmatrix} \xi - \frac{\xi^2}{\delta} & \frac{a\xi}{\delta} \\ \frac{b\xi}{\delta} & C - \frac{ab}{\delta} \end{bmatrix} \tag{12}$$

If we recover a boundary spike ξ, then we adjoin a boundary spike $-\xi$ to contract the spike. Therefore, $\lambda_{1,1} = \xi$ will result in a bad response matrix. This is equivalent to the condition $\det K(I'; I') = 0$, where I' refers to the interior nodes after the contraction is done. Therefore, a sufficient condition to avoid the boundary spike problem is that all principle minors of $K(I; I)$ have nonzero determinant. Unfortunately, we do not always have that condition. A simple example is the top-hat network, for which every conductivity is 1 except that of the center edge, whose conductivity is -2.

Sometimes it is possible to contract one boundary spike, and a previous boundary uncontractable boundary spike becomes contractable. Also, it is always possible to delete boundary edges of known conductivity.

15. Changing and Moving Boundary Spikes

Suppose a network has no boundary edges, and no boundary spike may be contracted. Suppose there are two boundary spikes p and q such that $\lambda_{pq} \neq 0$. We may adjoin a boundary spike at node q, and such an operation has the following effect on λ_{pp}, with δ defined as before.

$$new\lambda_{pp} = old\lambda_{pp} - \frac{\lambda_{pq}^2}{\delta} \tag{13}$$

Because λ_{pp} has changed, it is now possible to contract the boundary spike at node p.

It is possible that $\lambda_{p,q} = 0$ for all pairs of boundary spikes p and q, but we still desire to contract the spike at p. If there is another node p so $\lambda_{p,q} \neq 0$, then we can adjoin a boundary spike at node q and contract the spike at node p. Unfortunately, this operation does not reduce the edge count, but it remains useful. Also, we will soon see that we can always perform this operation, because a full row in the response matrix cannot be 0.

16. Boundary Antennas

A problem with adjoining a single boundary spike, where a spike did not exist, is that the result graph might not be critical. We will use boundary antannes instead.

Definition 16.1. An antenna is a pair of boundary spikes that share a common vertex.

Fact 16.2. Adjoining an antenna to a critical graph results in a critical graph.

This can be observed by studying the effects of adjoining an antenna on the medial graph.

We can adjoin a boundary antenna at any desired boundary node by first adjoining a boundary pendant, and then adjoining a boundary spike at the original boundary node.

Theorem 16.3. *A response matrix does not have a row consisting entirely of* 0*'s, after possible addition of boundary antennas.*

Proof. Suppose row p has no non-zero entries. Build a full set of minimal paths, with the first path's endpoints at p and a boundary node adjacent to p. At any point in the construction of the paths we are required to adjoin a boundary edge, adjoin an antenna instead to preserve criticality. Label the endpoints of the paths P and Q. As shown before, $\Lambda(P, Q) \neq 0$. Therefore, some entry of row p in Λ is nonzero. □

Note that no antennas were adjoined to nodes adjacent to p. Also, if $p \in P$, the nonzero entry guaranteed by the proof is in Q.

17. Getting a new boundary edge

We are "stuck" in recovering the graph is there are no boundary edges, and no boundary spike is contractable. However, we have observed that we can remove the boundary spike(s) around an interior node at the possible expense of creating a new boundary spike elsewhere. Suppose we were able to delete every boundary spike. Then the resulting graph would have a boundary edge. Then, this boundary edge can be created by contracting the boundary spike(s) around two interior nodes. If one of the interior nodes has an antenna, it is sufficient to contract one of the spikes to create the boundary edge.

Suppose p and q are adjacent boundary spikes such that contracting p and q will result in a boundary edge. By the theorem of the previous section, we can contract p while possibly creating new antennas, and possibly modifying the value of the boundary spike at q. If the boundary spike at q cannot then be contracted, create the full set of minimal paths such that the first path has endpoints at q and the neighbor of q that is not p. No new boundary spikes will appear at p. If we classify q in the set Q, $p \in Q$ if p is the endpoint of any path. Therefore, the nonzero entry of Λ guaranteed by this process is not p. Adjoin an antenna at this entry, and contract q. Then we have created a new boundary edge pq, which we can calculate and delete because the modified graph is still critical. Also, calculate and delete any other boundary edges that might have resulted from this process.

18. Inductive Conclusion

Suppose we have a critical circular planar network. We can recover any of the boundary edges and boundary spikes by finding a maximal connection that is broken by deleting the edge or contracting the spike. Suppose, after some steps, the network consists no boundary edges, n unknown edges, only boundary spikes known, and every boundary spike known. After performing the process described above to create and recover a boundary edge, the network will have $n-1$ or fewer unknown edges, only boundary spikes known, all boundary spikes known, no boundary edges, and remain critical. Therefore, the process is guaranteed to result in a fully recovered network.

References

[1] Curtis, Edward B. & James A. Morrow., *Inverse Problems for Electrical Networks*, Series of Applied Mathematics - Vol. 13. World Scientific, ©2000.

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