EXTENDING THE MEDIAL GRAPH TO THE NON-CIRCULAR-PLANAR CASE

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Abstract. The following paper examines a method for extending the definition of a medial graph to fit non-circular-planar graphs-with-boundary. Specifically, we examine the construction of a medial graph for graphs-with-boundary embeddable on a surface-with-boundary of genus 1. After establishing a convention for drawing these medial graphs, we show that this convention 'makes sense' in that it results in a two-colorable graph. In the remainder of the paper we study the effects of certain geodesic transformations on these new medial graphs, enumerate some of the complications which arise in studying these structures, and make a conjecture regarding the recoverability of resistor networks embeddable on surfaces-with-boundary of genus 1.

1. Extending the Medial Graph

In attempting to define a medial graph for non-circular-planar graphs-with-boundary, we must first define a convention for drawing such graphs. As in the circular planar case, we choose to draw our graph with the boundary nodes in some circular ordering on a boundary circle. The edges and interior nodes, however, will be embedded on a surface-with-boundary of certain genus having as its boundary this boundary circle (See Figure 1a below).

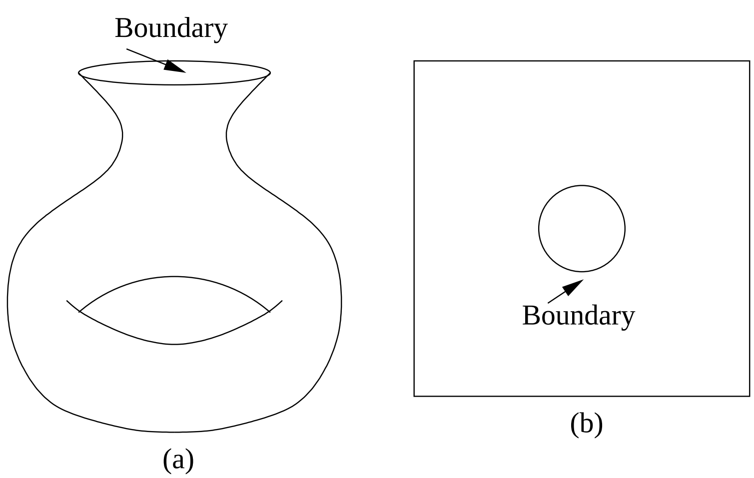

Figure 1. (a) A surface-with-boundary of genus 1. (b) An alternate way of viewing the same surface.

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Using this convention, we make the following definition:

Definition 1.1. *The* genus *of a graph-with-boundary will be defined as the minimal genus surface-with-boundary on which that graph can be embedded without edge crossings, placing the boundary nodes in circular order on the surface's boundary.*

As shorthand, we will refer to a circular planar graph as a $G-0$ graph, a graph-with-boundary of genus 1 as a $G-1$ graph, etc.

In the rest of this paper we will be concerned primarily with $G-1$ graphs, and so we now describe a convention for drawing such graphs on the plane. In Figure 1b, we see a square with a circle drawn in the center. If we identify the right edge of the square with the left edge of the square and the top edge of the square with the bottom edge of the square, then this figure becomes a torus. If we now identify the circle as a circular boundary to this surface, we have the surface depicted in Figure 1a. Thus a simple method of embedding a $G-1$ graph on the surface of genus 1 with circular boundary is to:

(1) Draw the structure depicted in Figure 1b.
(2) On the circle, draw the boundary nodes in circular order.
(3) Outside the circle, draw the interior nodes.
(4) Draw the edges of the graph. If the edges cross, change the positioning of the interior nodes. By definition, a $G-1$ graph should be embeddable on this structure without edge crossings.

We will use this drawing convention often as a means for visualizing the medial graph and layout of $G-1$ graphs.

With this understanding of how $G-n$ graphs are embedded on surfaces of genus n with circular boundary, we are in a position to define the medial graph for these graphs-with-boundary.

Definition 1.2. *Given an embedding of a $G-n$ graph on a surface of genus n (as described above) the* medial graph *of the given graph-with-boundary is constructed in the same way as for the circular planar case, with one additional rule: all geodesic segments must be drawn so they can be contracted on the embedding surface to the node they subtend.*

In drawing the medial graph of a $G-n$ graph, $n > 0$, the genus of the surface on which the graph is embedded adds some ambiguity to the construction of the medial graph's geodesics. The above rule removes some of this ambiguity and (as shown in the next section) guarantees the fact that the medial graph will 'make sense' in that it will be properly two-colored in a way analagous to that of the circular planar medial graph.

2. Two-Colorability of the Medial Graph

The extension of the medial graph makes little sense unless the two-colorability of the circular planar case extends analagously to graphs of other genuses, and so we now demonstrate that two-colorability does in fact apply to our extended convention for drawing the medial graph.

Theorem 2.1. *The medial graph, as defined above, is two-colorable for graphs of any genus. Moreover, coloring the node-bearing cells black and the empty cells white is a proper two-coloring of these medial graphs.*

Proof. Given a graph-with-boundary embedded on some surface as described above, we draw the medial graph using our accepted convention. Next, we color the node-bearing cells black and the empty cells white. Assume this is not a proper two-coloring of the medial graph. Then either we have two adjacent white cells or two adjacent black cells.

Assume we have two adjacent white cells. Then there exists a geodesic segment (a peice of a geodesic between edges of the graph) with empty cells to either side. But this implies that this geodesic segment can not be contracted to any nodes, which is a contradiction by our convention for drawing the medial graph.

Next, assume that we have two adjacent black cells. Then there exists a geodesic segment with node-bearing cells to either side. The geodesic segment may be either closed or non-closed, so we first assume that the geodesic segment is closed. Then this implies that the node on the interior of the closed geodesic segment corresponds to a spike, while the node on the exterior of the closed geodesic segment must have a loop which intersects this spike (see Figure 2a). But this is a contradiction, because we assume that our graph embedding has no edge crossings.

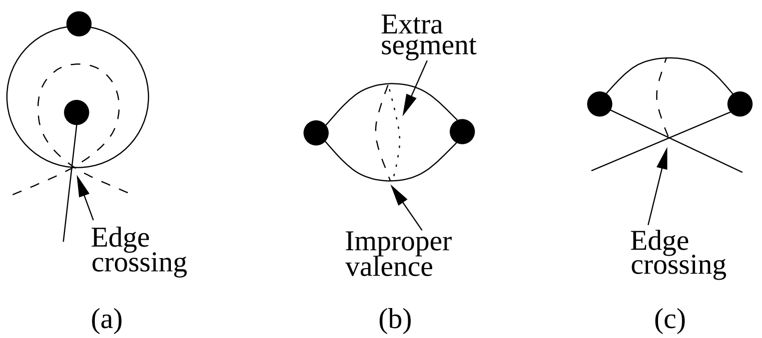

FIGURE 2. The many failures that result when one assumes that two-colorability does not hold. In each figure, the geodesic segment is marked by a dashed line.

Now we assume that the geodesic segment is non-closed. Then we are left with two potential cases. Either the geodesic segment we are examining is part of a parallel connection between the two nodes under examination, or it is not. If it is part of such a parallel connection, then the medial graph will have improper valence unless there is another geodesic segment connected between the same endpoints (see Figure 2b). This extra segment solves the coloring problem and is a contradiction of our assumption that the two black cells were adjacent. So we assume that our geodesic segment is not part of a parallel connection. A non-closed geodesic segment must connect two edges of the graph sharing as a vertex the node to which the segment is contractable. As this geodesic segment forms a portion of two cells surrounding different nodes and is thus contractable to each node, it must connect two of the graph's edges for each node. But the two edges connected by the geodesic segment corresponding to one node can not be the same edges as those corresponding to the other node, as this would imply that the geodesic segment were part of a parallel connection between the two nodes. Thus, the geodesic segment connects at least 3 edges of the graph, which implies an edge crossing and thus another contradiction (see Figure 2c).

Therefore, we must conclude that our original coloring of the graph was a proper two-coloring. □

3. Geodesic Transformations

On $G - n$ graphs, $n > 0$, there is an interesting geodesic transformation we can make on the medial graph: moving a geodesic across a hole in the surface. After studying this, however, it becomes clear that many such transformations lead either to medial graphs that are not two-colorable or medial graphs that are but do not correspond to well-defined graphs. An example of one such loss of two-colorability is shown in figure 3 below.

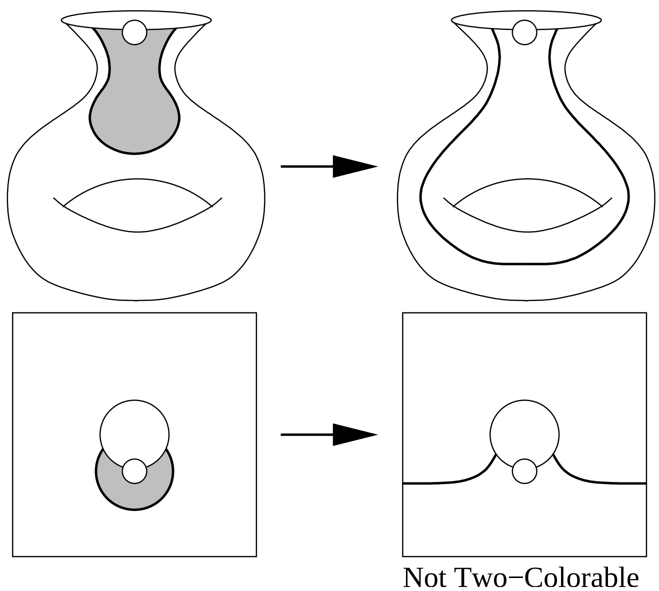

Figure 3. On the left is a depiction of the medial graph for the graph consisting of a single boundary node. On the right is a depection of the same graph with the geodesic drawn around the center hole of the embedding surface. Note that the medial graph on the right is not two-colorable.

4. Complications and a Conjecture

There are many problems which arise in studying this extended definition of the medial graph, but before enumerating several examples it is necessary to introduce some terminology.

Definition 4.1. *A* lens *is a structure formed by two geodesics intersecting at two separate points. If the double intersection of these geodesics forms the boundary of a simple region on the surface on which the medial graph is embedded, then this lens will be called a* region bounding lens *or RB lens for short. If the double intersection* *of these geodesics does not bound a simple region on this surface, then this lens will be called a* non-region bounding lens *or NRB lens for short.*

Now, we explore some of the difficulties in evaluating these extended medial graphs as a tool for checking the recoverability of non-circular-planar resistor networks:

(1) Lenses are not necessarily RB lenses. They can be embedded through a hole so that they do not bound a simple region.
(2) Lenses are not all bad. NRB lenses exist in many recoverable networks, including the n-circle $2n$-ray annular network.
(3) Bubbles (geodesics that never connect to the boundary circle) are everywhere.
(4) Parallel (and probably series) connections can exist without the existence of RB lenses, and even without the existence of NRB lenses (see Figure 4a for an example of a parallel connection showing up in a bubble).

That being said, we can make the following conjecture:

Conjecture 4.1. $G-1$ *graphs representing resistor networks whose medial graphs contain bubbles (Figure 4a), RB lenses (Figure 4b), or linked NRB lenses (3 geodesics forming 2 NRB lenses which intersect to bound a simple region; Figure 4c) are non-recoverable.*

It must be noted that this conjecture is based simply on a current absence of counter-examples. Clearly the existence of an RB lens is a sign of the same problems it signals in the circular planar case, but whether or not bubbles and linked NRB lenses signify non-recoverability has yet to be established. An interesting case to study would be the 3-circle 5-ray annular network. The medial graph for this network (under one convention for ordering the boundary nodes) contains many linked NRB lenses and thus a proof of the recoverability of this network would bring immediate alteration to the above conjecture.

References

[1] Curtis, Edward B. & Morrow, James A. “Inverse Problems for Electrical Networks.” Series on Applied Mathematics - Vol. 13. World Scientific, ©2000.

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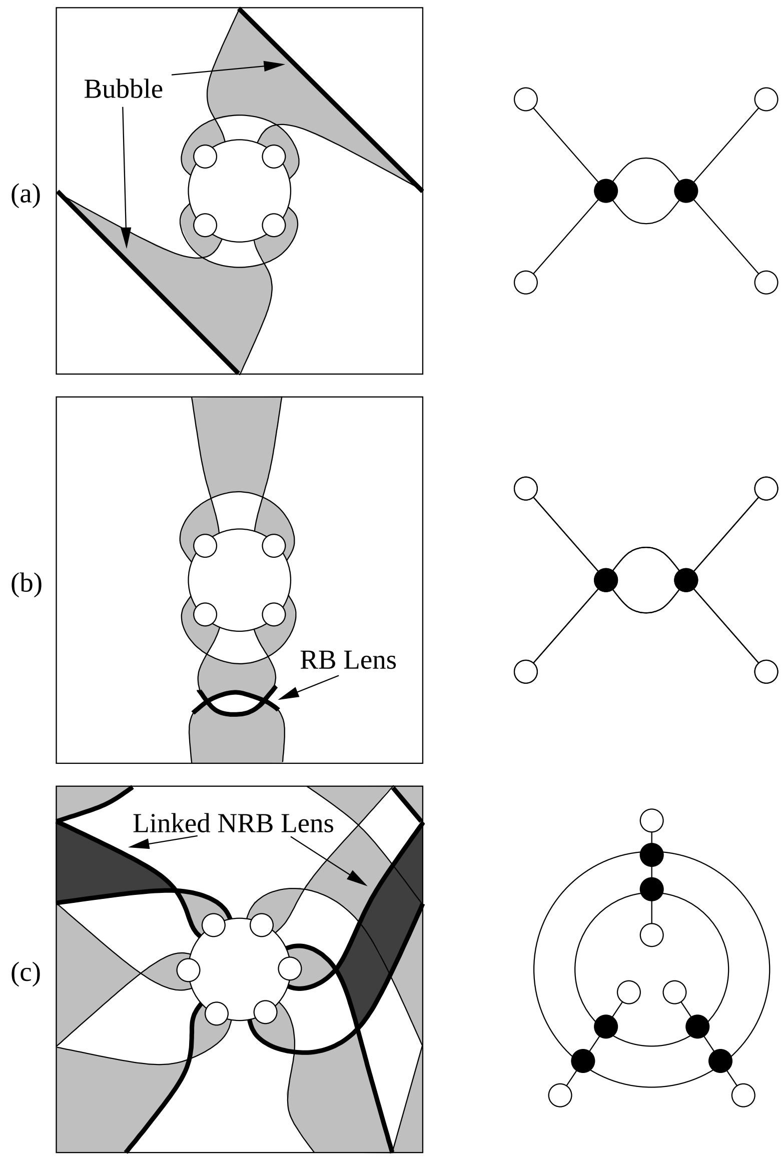

FIGURE 4. A set of graphs (right) whose corresponding medial graphs (left) demonstrate interesting structures. (a) A geodesic bubble, marked in bold. (b) An RB lens, where the sections of the geodesics coresponding to the lens are in bold. (c) Linked NRB lenses, where the geodesics of the linked NRB lens are in bold and the bounded region is darkly shaded.