THE INVERSE CONDUCTIVITY PROBLEM FOR ANNULAR NETWORKS

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1. The Forward and Inverse Problem

Suppose $G = (V, V_B, E)$ is a simple graph in which V is the set of nodes, the set of boundary nodes V_B is a nonempty subset of V, $I = V - V_B$ is the set of interior nodes, and E is the set of edges. Let γ be a positive edge conductivity function defined on E. Then $\Gamma = (G, \gamma)$ is called a resistor network. Choose an indexing of the nodes where boundary nodes precede interior nodes. Then the Kirchhoff matrix K is defined as follows.

$$K_{ij} = \begin{cases} -\gamma_{ij} & \text{if the edge } ij \text{ exists,} \\ \sum_{k \neq i} \gamma_{ik} & \text{if } i = j. \end{cases}$$

The Kirchhoff matrix K maps voltages to currents. The forward problem is given K compute the response matrix $\Lambda = ((\lambda_{ij}))$, the map from boundary voltages to boundary currents such that Kirchhoff's Law is satisfied at the interior nodes. If K is expressed in the block form

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where A corresponds to the boundary nodes, then

$$\Lambda = A - BC^{-1}B^T.$$

The inverse problem is given Λ and G recover K.

2. Connections

If p and q are boundary nodes, a path from p to q is a sequence of edges $\{pr_1, r_1r_2, \ldots, r_hq\}$ where $r_1, \ldots, r_h$ are interior nodes. If $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are disjoint sets of boundary nodes, then a k-connection $P \leftrightarrow Q$ is a set of disjoint paths, p_i to $q_{\tau(i)}$ for $1 \leq i \leq k$ where τ is a permutation.

The main result from connections used in this paper is as follows. If there is only one valid permutation for a connection $P \leftrightarrow Q$ then $\det \Lambda(P; Q) \neq 0$.

3. Relations in Annular Networks

Consider graphs such as mentioned in the next section which have n circles and $2n - 3$ rays. For example, let us look at the following numbering of nodes. Then there is the following relation.

$$\lambda_{17}\lambda_{28}\lambda_{39}\lambda_{4,10}\lambda_{56} - \lambda_{1,10}\lambda_{59}\lambda_{48}\lambda_{37}\lambda_{26} = 0 \tag{1}$$

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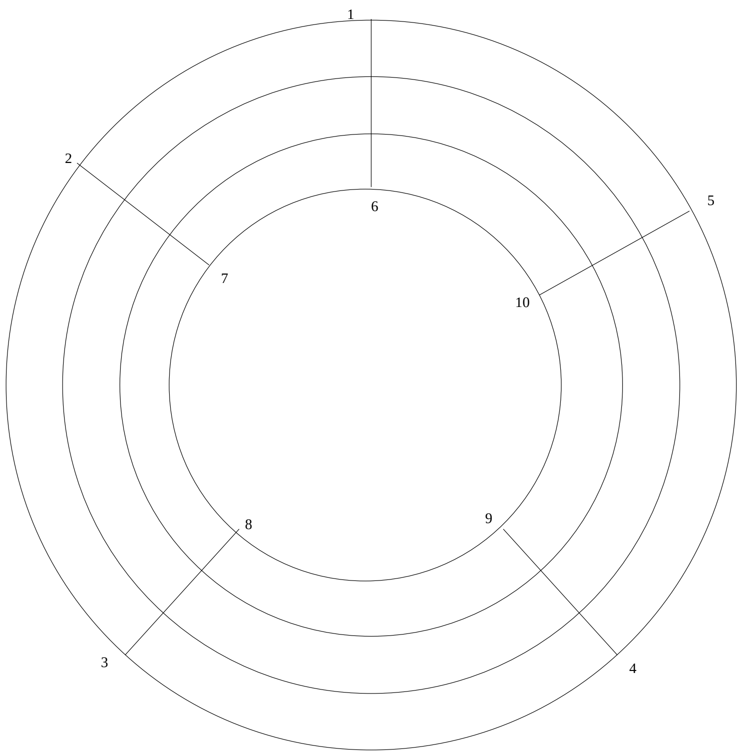


Analygously, in the case of 3 circles and 3 rays there is the following relation

$$\lambda_{15}\lambda_{26}\lambda_{34} - \lambda_{16}\lambda_{35}\lambda_{24}. \tag{2}$$

The conjecture for n circles and $2n-3$ rays where n is odd should now be obvious.

In the 3 circles and 3 rays case we can see why this might by true by first conjecturing that the resistive network is unrecoverable. Then consider the following boundary conditions. We number the boundary nodes as follows. There is clearly a connection from boundary node 5 to boundary node 3. Hence, the boundary conditions are realizable. From the boundary conditions we get

$$\lambda_{51} + \alpha\lambda_{53} = 0 \tag{3}$$

$$\lambda_{21} + \alpha\lambda_{23} = -\gamma_{12} - \alpha\gamma_{23}. \tag{4}$$

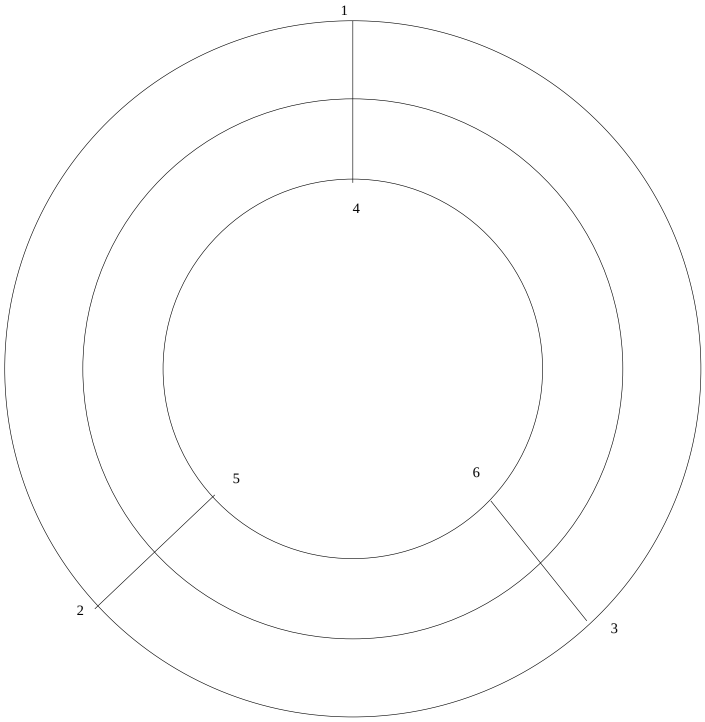
Plugging in for α in the second equation gives

$$\lambda_{53}\gamma_{12} - \lambda_{51}\gamma_{23} = \lambda_{51}\lambda_{23} - \lambda_{53}\lambda_{21}. \tag{5}$$

If we rotate the boundary conditions and repeat the same computations we'll get 2 other equations, one with γ_{23} and γ_{13}, the other with γ_{12} and γ_{13}. If we now view this as 3 equations in the 3 unknowns: $\gamma_{12}, \gamma_{13}, \gamma_{23}$ and compute the determinant of the coefficient matrix and set it equal to 0 we get exactly

$$\lambda_{15}\lambda_{26}\lambda_{34} - \lambda_{16}\lambda_{35}\lambda_{24}. \tag{6}$$

That the determinant should be 0 follows from our guess that the resistive network is unrecoverable. For if the determinant were nonzero then the network would necessarily be recoverable.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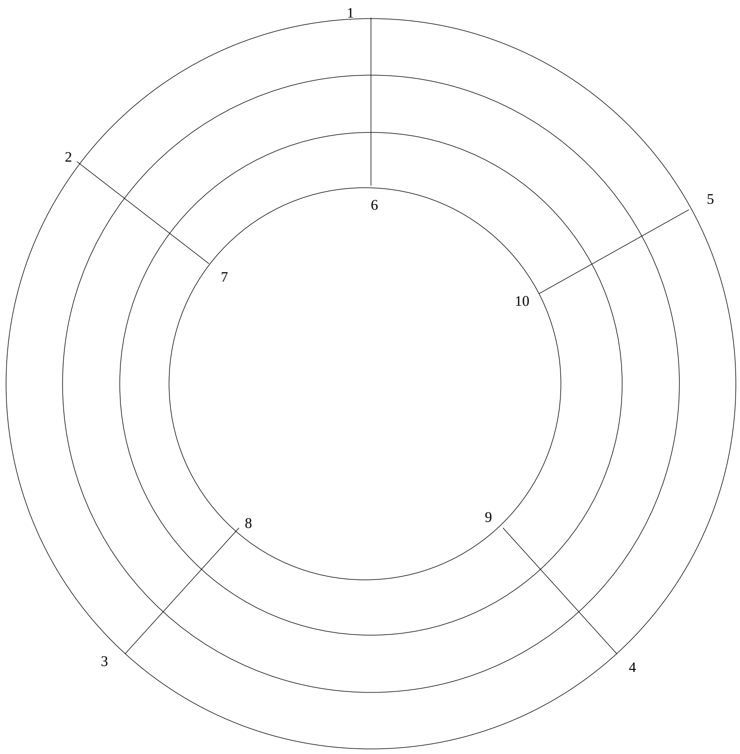
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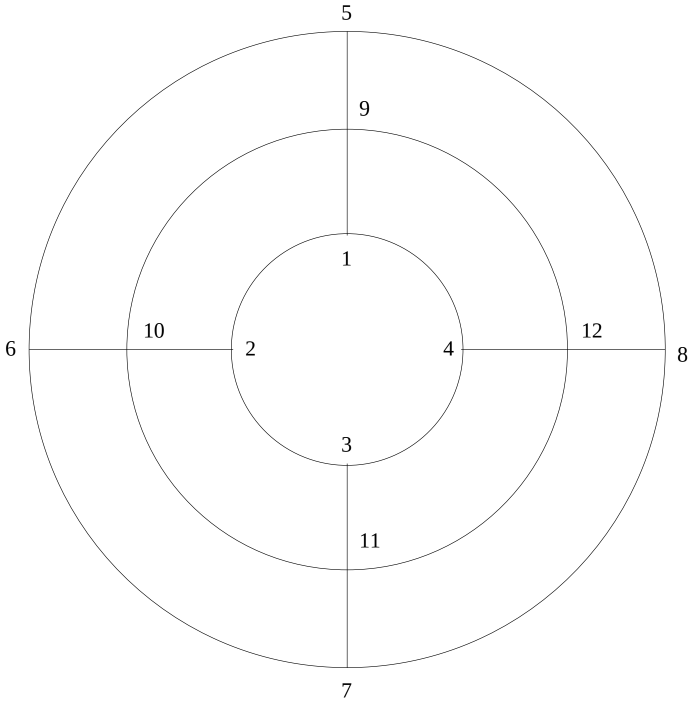

FIGURE 1. $G(3,4)$: An indexing of the nodes

4. Recoverable Annular Networks

Figures 1 and 3 are examples of $G(n, 2(n-1))$ graphs which have n circles and $2(n-1)$ rays. Networks with such graphs where the nodes on the outer and inner circles are boundary nodes are recoverable.

Boundary to boundary, boundary to interior, and interior to interior edges will be abbreviated $\partial - \partial$, $\partial - int$, and $int - int$, respectively.

To recover the network shown in Figure 1 begin by applying the boundary conditions shown in Figure 2 which will recover the $\partial - \partial$ conductivity $\gamma_{7,8}$. The currents flowing into node 9 and the currents flowing out of node 12 are determined by the boundary conditions, therefore α and β are uniquely determined and the boundary conditions are realizable. This is also shown by the existence of only one possible correspondence in the connection $(1,8) \leftrightarrow (3,6)$ which shows that

$$\begin{vmatrix} \lambda_{38} & \lambda_{31} \\ \lambda_{68} & \lambda_{61} \end{vmatrix} \neq 0.$$

Thus

$$\lambda_{38}\alpha + \lambda_{31}\beta = -\lambda_{35}$$
$$\lambda_{68}\alpha + \lambda_{61}\beta = -\lambda_{65}$$

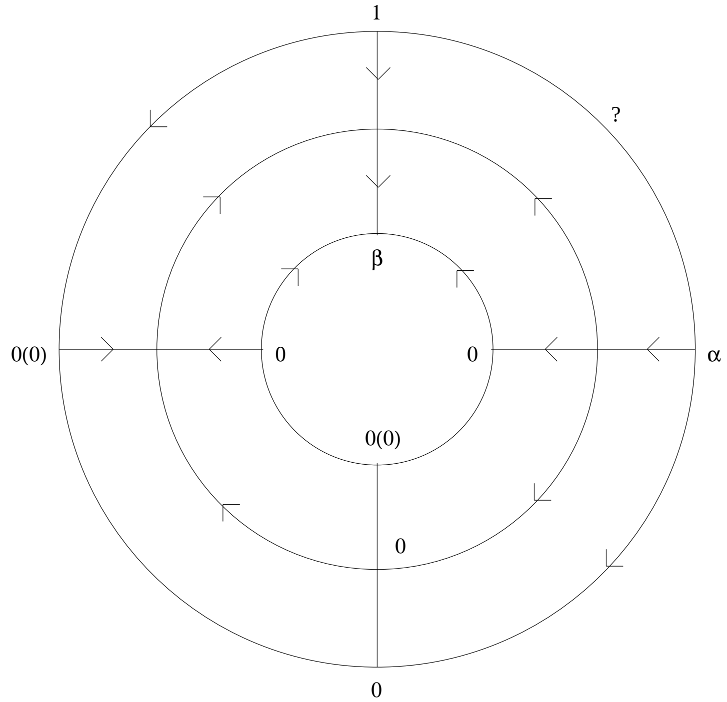

FIGURE 2. $G(3,4)$: Boundary conditions and resulting current flows

from which

$$\alpha = \frac{\lambda_{31}\lambda_{65} - \lambda_{35}\lambda_{61}}{\lambda_{38}\lambda_{61} - \lambda_{31}\lambda_{68}}$$
$$\beta = \frac{\lambda_{35}\lambda_{68} - \lambda_{38}\lambda_{65}}{\lambda_{38}\lambda_{61} - \lambda_{31}\lambda_{68}}.$$

Since there is only one possible correspondence in the connection $(3,6) \leftrightarrow (1,5)$, $\alpha \neq 0$ and

$$-\alpha\gamma_{7,8} = \lambda_{75} + \lambda_{78}\alpha + \lambda_{71}\beta.$$

Hence

$$\gamma_{7,8} = \lambda_{78} + \lambda_{75}\frac{\lambda_{31}\lambda_{68} - \lambda_{38}\lambda_{61}}{\lambda_{31}\lambda_{65} - \lambda_{35}\lambda_{61}} + \lambda_{71}\frac{\lambda_{38}\lambda_{65} - \lambda_{35}\lambda_{68}}{\lambda_{31}\lambda_{65} - \lambda_{35}\lambda_{61}}.$$

Now the $\partial - int$ conductivity $\gamma_{7,11}$ can be recovered by changing the voltage at node 7 to 1. Then denoting the new resulting α and β by α' and β', respectively,

$$\gamma_{7,11} = \alpha'\lambda_{78} + \beta'\lambda_{71} + \lambda_{75} + \lambda_{77} + \gamma_{7,8}(\alpha' - 1)$$

where

$$\lambda_{38}\alpha' + \lambda_{31}\beta' = -\lambda_{35} - \lambda_{37}$$
$$\lambda_{68}\alpha' + \lambda_{61}\beta' = -\lambda_{65} - \lambda_{67}$$

from which $\gamma_{7,11}$ can be computed.

Now to recover the $int - int$ edges apply the original boundary conditions shown in Figure 2. The middle circle has three unknown currents and three independent

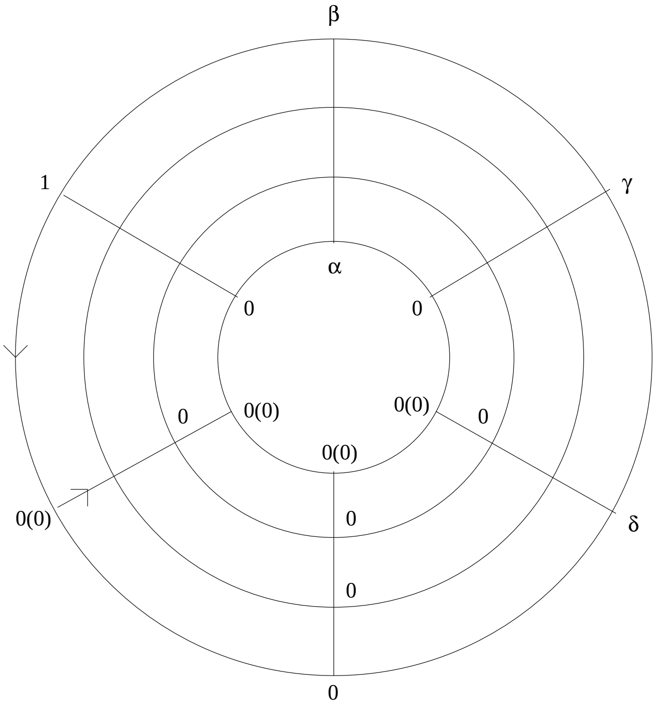

FIGURE 3. $G(4, 6)$: First set of boundary conditions

equations from Kirchhoff's current law applied at nodes 9, 10, and 12. From those currents the $int - int$ edge conductivities can be computed.

Recovering arbitrarily large n circles and $2(n-1)$ rays networks requires a new set of boundary conditions in addition to a generalization of the boundary conditions in Figure 2.

For example, the 4 circles and 6 rays network can be recovered using the boundary conditions shown in Figures 3 and 4. The $\partial - \partial$ conductivities can be recovered using the boundary conditions shown in Figure 3 whereas the rest of the conductivities can be recovered using the boundary conditions in Figure 4.

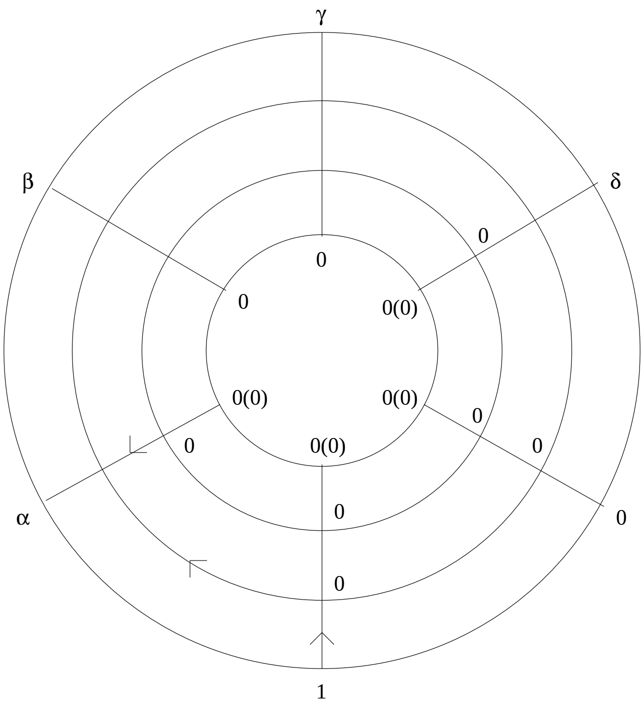

FIGURE 4. $G(4,6)$: Second set of boundary conditions