Unique Solutions for the Dirichlet Inverse Problem on Schrodinger Networks

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Abstract

The set of solutions to an inverse problem is useful,interesting, and significant when the solution is either unique or has a finite number of solutions. This paper explores the question of determining when a network will have a finite number of solutions to Dirichlet inverse problem for the discrete Schrodinger equation.

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1 Introduction

A solution to the continuous Schrodinger equation with real valued q on a domain Ω is a function u such that $S_q u = \Delta u - qu = 0$ in the interior of Ω. To discretize the Schrodinger equation, consider a graph, $\Gamma = \{V, V_\partial, E\}$ and real valued functions u and q, where V is the set of all vertices, $V_\partial \subseteq V$ is the set of boundary vertices, E is the set of edges of the graph such that there are no two edges whose endpoints are the same pair vertices, and u and q are vertex functions. Additionally, define $V_{int} = V \setminus V_\partial$ to be the set of interior vertices and $\mathcal{N}(i)$ to be the set of vertices which are adjacent to the vertex i, then u is a solution to the discrete Schrodinger equation if for all interior nodes,

$$S_{d_q} u(i) = \left(\sum_{j \in \mathcal{N}(i)} u(j) - u(i) \right) - q(i)u(i) \tag{1}$$

is zero. The function u will be referred to as the state function and q will be referred to as the potential function; additionally, let $|V| = n$ and $|V_\partial| = d$. For the problems considered in this paper the Dirichlet data will be the value of u on the boundary and the Neumann data will be the value of (1) for boundary vertices i.

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As there are only a finite number of vertices in the graph, the Schrodinger operator can be expressed as a matrix S, where $Su = \begin{bmatrix} \phi \\ 0 \end{bmatrix}$ and ϕ is the Neumann data. S can most easily be expressed as the sum of two matrices, K_1 and I_q.

$$K_{1_{i,j}} = \begin{cases} 0 & i \neq j \text{ and } e_{i,j} \notin E \\ -1 & i \neq j \text{ and } e_{i,j} \in E \\ |\mathcal{N}(i)| & i = j \end{cases}, \quad I_{q_{i,j}} = \begin{cases} 0 & i \neq j \\ q(i) & i = j \end{cases}$$

Then it can easily be seen that $S = -(K_1 + I_q)$ is the Schrodinger operator. S is a symmetric matrix because if the edge $e_{i,j}$ exists, then the edge $e_{j,i}$ exists, and is the same edge; additionally, per convention, the first d rows and columns S will represent the boundary vertices and the remaining $n-d$ rows and columns will represent the interior vertices. This will allow S to be written in the following block form,

$$S = -(K_1 + I_q) = -\begin{bmatrix} A + I_{q|_\partial} & B \\ B^T & C + I_{q|_{int}} \end{bmatrix}$$

where A is a $d \times d$ dimensional matrix representing boundary-boundary connections, B is a $d \times (n-d)$ dimensional matrix representing boundary-interior connections, and C is an $(n-d) \times (n-d)$ dimensional matrix representing interior-interior connections.

2 Existence and Uniqueness of Some Forward and Inverse Problems

As there are two vertex functions, there are several types of forward and inverse problems which can be considered. The first inverse problem is if the state function is known everywhere, can the potential function be determined everywhere. For this problem, the answer is yes as long as u is non-zero everywhere.

Proposition 2.1. Given a graph, Neumann data, ψ, defined on the boundary, and a state function u, which is nonzero at all vertices, then there is a unique potential function q, which satisfies the Schrodinger equation at all interior nodes and satisfies the given Neumann data on the boundary.

Proof. Let $q = -I_{\frac{1}{u}}\left(\begin{bmatrix} \psi \\ 0 \end{bmatrix} + K_1 u\right)$, q exists as u is nonzero at all nodes. Moreover, as $-(K_1 + I_q)u = -K_1 u - I_q u = -K_1 u - I_u q = -K_1 u - I_u\left(-I_{\frac{1}{u}}\left(\begin{bmatrix} \psi \\ 0 \end{bmatrix} + K_1 u\right)\right) = -K_1 u + I\left(\begin{bmatrix} \psi \\ 0 \end{bmatrix} + K_1 u\right) = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$, q satisfies the Schrodinger equation, with state function u. To show uniqueness, assume that q_1 and q_2 both are potentials which satisfy the Schrodinger equation, for the same state function, then $-(K_1 + I_{q_1})u = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$ and $-(K_1 + I_{q_2})u = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$. Then $(I_{q_1} - I_{q_2})u = 0 \Rightarrow I_u(q_1 - q_2) = 0$; however, as u is nonzero at all nodes, I_u is nonsingular, so $q_1 - q_2 = 0$. Thus q is unique. □

Even though this is a simple answer, it is one of the least likely scenarios, it is more likely that the state function will be unknown. Once again, a state function can be found, but the condition is less general than in Proposition 2.1, as it needs S to be invertible.

Proposition 2.2. Given a graph, Neumann data, ψ, on the boundary, and a potential function q, such that $K_1 + I_q$ is invertible, then there exists a unique state function u, which satisfies the Schrodinger equation at all interior nodes and satisfies the given Neumann data on the boundary.

Proof. As $K_1 + I_q$ is invertible, it follows that $-(K_1 + I_q)u = \begin{bmatrix} \psi \\ 0 \end{bmatrix}$ has a unique solution. $\square$

This forward problem will be referred to as the Neumann problem and will be used in this form for Section 3, but a special case will be considered in Section 4, where the potential function is zero on the boundary. The final forward problem which is considered will be referred to as the Dirichlet problem, and it has a unique solution when $C + I_{q|_{int}}$ is invertible.

Proposition 2.3. Given a graph, a potential function q such that $C + I_{q|_{int}}$ is invertible, and the state on the boundary, ϕ, then there exists a unique state function u which satisfies the Schrodinger equation on the interior.

Proof. Let $u = \begin{bmatrix} \phi \\ -\left(C + I_{q|_{int}}\right)^{-1} B^T \phi \end{bmatrix}$. Then u is a solution to the Schrodinger equation as

$$\begin{aligned} -\left(K_1 + I_q\right) u &= -\begin{bmatrix} A + I_{q|_\partial} & B \\ B^T & C + I_{q|_{int}} \end{bmatrix} \begin{bmatrix} \phi \\ -\left(C + I_{q|_{int}}\right)^{-1} B^T \phi \end{bmatrix} \\ &= -\begin{bmatrix} (A + I_{q|_\partial})\phi - B(C + I_{q|_{int}})^{-1} B^T \phi \\ B^T \phi - (C + I_{q|_{int}})(C + I_{q|_{int}})^{-1} B^T \phi \end{bmatrix} = -\begin{bmatrix} (-A - I_{q|_\partial} + B(C + I_{q|_{int}})^{-1} B^T)\phi \\ 0 \end{bmatrix} \end{aligned}$$

Now, assume that $u = \begin{bmatrix} \phi \\ x \end{bmatrix}$ is a solution to the Schrodinger equation, then $(C + I_{q|_{int}})x = -B^T\phi$, and as $C + I_{q|_{int}}$ is invertible, x is unique. $\square$

Most of the paper will consider the inverse Dirichlet problem, and since the Dirichlet problem has a unique solution when $C + I_{q|_{int}}$ is invertible, assume that throughout the rest of this paper $C + I_{q|_{int}}$ is invertible whenever it is encountered. Since $C + I_{q|_{int}}$ is invertible, the Dirichlet to Neumann map to be the matrix $-((A + I_{q|_\partial}) - B(C + I_{q|_{int}})B^T)$, this will be referred to as the response matrix for a graph; moreover, this is the Schur complement of the matrix S with respect to the interior vertices. Additionally, the matrix $(C + I_{q|_{int}})^{-1}B^T$ will occur often when dealing with the inverse Dirichlet problem and thus let $D = (C + I_{q|_{int}})^{-1}B^T$.

3 The Inverse Dirichlet Problem

3.1 Recovery of the Potential Function

The inverse problem which will be considered for much of the rest of the paper will be as follows: given a graph and the Dirichlet to Neumann map, find the potential function q for the graph.

Definition 3.1. For a set graph, let $\mathcal{R}$ be an equivalence relation on the potential functions where $C + I_{q|_{int}}$ is invertible, where $q_1 \mathcal{R} q_2$ if the response matrix for q_1 equals the response matrix of q_2. A graph is <u>uniquely recoverable</u> if the cardinality of every equivalence class is 1, and <u>nearly recoverable</u> if the cardinality of every equivalence class is finite. Moreover <u>recoverable</u> will refer to either nearly recoverable or uniquely recoverable.

Fact 3.1. For a given graph, the set of potentials q such that $C + I_{q|_{int}}$ is invertible is the same for all other graphs related by adding or removing boundary-boundary edges.

Conjecture 3.1. If two graphs, Γ and Γ' have the same set of q's for which $C + I_{q_{int}}$ is invertible, then the two graphs have the same interior-interior connections, and the valence of each interior node is the same.

Proposition 3.1. A graph, Γ is recoverable iff any other graph found by adding or removing boundary-boundary edges from Γ is recoverable.

Proof. Consider graphs Γ and Γ', where the only difference between Γ and Γ' is that Γ' has a boundary-boundary connection between nodes i and j, while Γ does not. Given a potential function, let the response matrix for Γ be Λ_q and the response matrix for Γ' be Λ'_q. For a given potential function consider the differences between Λ_q and Λ'_q. $\Lambda_q = -((A + I_{q|_\partial}) - B(C + I_{q|_{int}})B^T$ and $\Lambda'_q = -((A' + I_{q|_\partial}) - B'(C' + I_{q|_{int}})(B')^T$. Moreover, as there is no difference in the boundary-interior and boundary-boundary connections between Γ and Γ', $B = B'$ and $C = C'$. Additionally, the only difference between A and A' is that $a_{i,i} + 1 = a'_{i,i}$, $a_{j,j} + 1 = a'_{j,j}$, and $a_{i,j} = a_{j,i} = 0$ while $a'_{i,j} = a'_{j,i} = -1$. Let E be the matrix where $E_{k,l} = \begin{cases} 0 & k \notin \{i,j\} \text{ or } l \notin \{i,j\} \\ -1 & k = l = i \text{ or } k = l = j \\ 1 & (k = i \text{ and } l = j) \text{ or } (k = j \text{ and } l = i) \end{cases}$. Then $A + E = A'$, thus $\Lambda'_q = -E - ((A + I_{q|_\partial}) - B(C + I_{q|_{int}})^{-1}B^T) \Rightarrow \Lambda'_q + E = \Lambda_q$. Note that E is fixed and independent of q. Now assume that we are given a response matrix, Λ_q, for Γ and Γ' is recoverable, then the response matrix Λ'_q can be calculated, and as Γ' is recoverable, the equivalence class of potentials can be recovered. Similarly, if a response matrix, Λ'_q, for Γ' is given, and Γ is recoverable, then the equivalence class of potentials can be recovered. $\square$

The importance of the preceding proposition is that it simplifies all calculations in the remainder of this paper. As adding or removing boundary-boundary edges neither aids nor hinders recoverability, whenever it is convenient the assumption will be made that there are no boundary-boundary connections, making A a diagonal matrix where the diagonal elements are the valence of the boundary nodes. For the following proofs, it will be helpful to use the determinantal identity, a more complete description and a proof which can be generalized to Schrodinger networks can be found in [3].

$$\det(-\Lambda(P;Q)) * \det(C + I_{q|_{int}}) = (-1)^k \sum_{\tau \in S_k} sgn(\tau) \left\{ \sum_{\substack{\alpha \in \mathcal{C}(P;Q) \\ \tau_\alpha = \tau}} \prod_{e_{l,m} \in \alpha} S_{l,m} * \det(-S(J_\alpha)) \right\} \quad (2)$$

where P and Q are distinct subsets of the boundary vertices, $k = |P|$, S_k is the group of permutations on k elements, $\mathcal{C}(P;Q)$ is the set of simultaneous connections between the elements of P and Q, J_α is the set of interior vertices which are not used in path α, and $-S(J_\alpha)$ is the principal sumbatrix of $C + I_{q|_{int}}$ consisting of the vertices of J_α.

Proposition 3.2. Let j be a boundary node and i be an interior node of the graph Γ, then $D_{i,j}(q) = 0$, iff j is not connected to i through the interior.

Proof. To prove this proposition, first a lemma will be used to first show that an element, $\lambda_{i,j}(q)$, of the response matrix is zero iff there is no path between the vertices i and j.

Lemma 3.1. In the response matrix Λ_q, an element $\lambda_{i,j}(q)$ is the zero polynomial iff there is no path through the interior between nodes i and j.

Proof. To prove the reverse direction, note that by the determinantal identity,

$$-\lambda_{i,j}(q) * \det\left(C + I_{q|_{int}}\right) = (-1)^k \sum_{\tau \in S_k} sgn(\tau) \left\{ \sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau}} \prod_{e_{l,m} \in \alpha} S_{l,m} * \det\left(-S(J_\alpha)\right) \right\}$$

If there is no connection, then the right hand side of the equation is zero as there are no connections, and as $\det\left(C + I_{q|_{int}}\right)$ is nonzero, then $\lambda_{i,j}(q) = 0$.

To prove the forward direction, assume that $\lambda_{i,j}(q) = 0$, but there is a path between nodes i and j. Additionally, as there is only one permutation in S_1 and for all paths between i and j, for each edge $e_{l,m}$ on this path, $K_{l,m} = 1$. Then, by the determinantal identity,

$$0 = \sum_{\alpha \in \mathcal{C}(i;j)} \det\left(-S(J_\alpha)\right)$$

Now $\sum_{\alpha \in \mathcal{C}(i;j)} \det -S(J_\alpha)$ is a polynomial in $q_{d+1}, \cdots, q_n$ where d is the number of boundary nodes. Now, let m be the degree of the terms of highest degree, then the coefficient of each term of degree m is positive. Therefore the sum is only zero when $\mathcal{C} = \emptyset$, in other words, when there is no path between nodes i and j. $\square$

Now to the proof of the proposition. Assume wlog, that we renumber the interior nodes so that node i is node $d+1$, where d is the number of boundary nodes, then the matrix $K + I_q$ can be written as:

$$\left[\begin{array}{ccc|cccc} & A & & & B & & \\ \hline & K_{1_{i,j}} & & K_{1i,i} + q_i & x_{d+2} & \cdots & x_n \\ \cdots & y_{d+2} & \cdots & x_{d+2} & & & \\ & \vdots & & \vdots & & G + I_{q'|_{int}} & \\ & y_n & & x_n & & & \end{array}\right] \text{ and } C + I_{q|_{int}} = \left[\begin{array}{c|ccc} K_{1_{i,i}} + q_i & x_{d+2} & \cdots & x_n \\ \hline x_{d+2} & & & \\ \vdots & & G + I_{q'|_{int}} & \\ x_n & & & \end{array}\right]$$

Where $K_{1_{i,j}}, y_{d+2}, \cdots, y_n$ is the j^{th} row of B, and $K_{1_{i,i}} + q_i, x_{d+2}, \cdots, x_n$ is the 1^{st} row of $C + I_{q|_{int}}$. Let Γ' be the graph with the same edges as Γ, but node i has been promoted to be a boundary node. Then note that there exists a path through the interior between i and j in Γ iff there exists a path through the interior between i and j in Γ'. By Lemma 3.1, then $\lambda'_{i,j}(q) = 0$ iff there is no path through the interior between i and j. Moreover, note that

$$D_{i,j}(q) = \frac{\det \left[\begin{array}{c|ccc} K_{1_{i,j}} & x_{d+2} & \cdots & x_n \\ \hline y_{d+2} & & & \\ \vdots & & G + I_{q'|_{int}} & \\ y_n & & & \end{array}\right]}{\det\left[C + I_{q|_{int}}\right]}, \quad \lambda_{i,j}(q)' = \frac{\det - \left[\begin{array}{c|ccc} K_{1_{i,j}} & x_{d+2} & \cdots & x_n \\ \hline y_{d+2} & & & \\ \vdots & & G + I_{q'|_{int}} & \\ y_n & & & \end{array}\right]}{\det - \left[G + I_{q'|_{int}}\right]}$$

$$\lambda_{i,i}(q)' = \frac{\det - \left[C + I_{q|_{int}}\right]}{\det - \left[G + I_{q'|_{int}}\right]}$$

It is easily seen that $\frac{\lambda'_{i,j}(q)}{\lambda'_{i,i}(q)} = -D_{i,j}(q)$. Moreover, $\lambda'_{i,i}(q)$ is never the zero polynomial because $C + I_{q|_{int}}$ is invertible. This implies that $D_{i,j}(q) = 0$ iff $\lambda'_{i,j}(q) = 0$ so, $D_{i,j}(q) = 0$ iff there does not exist a path through the interior between boundary node j and interior node i. $\square$

Definition 3.2. A directed path between two vertices, i and j, is an ordered list of internal vertices, $G = \{g_1, \cdots, g_s\}$ such that for all k, $1 \leq k \leq s-1$, the edge $e_{g_k, g_{k+1}}$ exists, $g_1 = i$, and $g_s = j$. An undirected path between two vertices, i and j, is the ordered list of internal vertices together with the list in reverse order. An undirected path is a subpath of a directed path if one of the ordered lists of the undirected path is a subpath of the directed path.

Definition 3.3. As $\mathcal{C}(i;j)$ is the set of connections between boundary vertices, i and j, let $\mathcal{C}'(i;j)$ be the set with multiplicity of all undirected paths from i to j after removing i and j from the path, and $\{\emptyset\}$ for a direct connection between i and j.

Definition 3.4. For a set of internal vertices, $G = \{g_1, \cdots, g_s\}$, then $\mathcal{C}_G(i;j)$ is the set of paths between vertices i and j which use exactly the vertices of $G \cup \{i,j\}$. $\mathcal{C}'_G(i;j) \subset \mathcal{C}'(i;j)$ is the set of paths in $\mathcal{C}'_G(i;j)$ which use exactly the vertices of G. Note that $|\mathcal{C}_G(i;j)| = |\mathcal{C}'_G(i;j)|$.

Theorem 3.1. $\lambda_{i,j}(q) = \lambda_{k,l}(q)$ iff $\mathcal{C}'(i;j) = \mathcal{C}'(k;l)$

Proof. It is easily seen that if $\mathcal{C}'(i;j) = \mathcal{C}'(k;l)$, then $\lambda_{i,j}(q) = \lambda_{k,l}(q)$ through the determinantal identity.

To prove the forward direction, two lemmas will be used, first that if $\lambda_{i,j}(q) = \lambda_{k,l}(q)$, then for all sets of interior nodes, G, $|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k;l)|$ and secondly, if for all sets of interior nodes, G,$|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k;l)|$ then $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$.

Lemma 3.2. Assume that $\lambda_{i,j}(q) = \lambda_{k,l}(q)$, then for all sets of interior nodes, G, $|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k;l)|$

Proof. Note that by the determinantal identity,

$$- \lambda_{i,j} * \det(C + I_{q|_{int}}) = (-1)^m \sum_{\tau \in S_m} sgn(\tau) \left\{ \sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau}} \prod_{e_{p,q} \in \alpha} S_{p,q} * \det(-S(J_\alpha)) \right\}$$

$$= - \lambda_{k,l} * \det(C + I_{q|_{int}}) = (-1)^m \sum_{\tau \in S_m} sgn(\tau) \left\{ \sum_{\substack{\beta \in \mathcal{C}(k;l) \\ \tau_\beta = \tau}} \prod_{e_{p,q} \in \beta} S_{p,q} * \det(-S(J_\beta)) \right\}$$

Moreover, as $|P| = 1$ for both equations, there is only one element in S_1, and $S_{p,q} = 1$ for every edge in every path, the above equations can be reduced to

$$\lambda_{i,j} * \det(C + I_{q|_{int}}) = \sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau}} \det(-S(J_\alpha))$$

$$= \lambda_{k,l} * \det(C + I_{q|_{int}}) = \sum_{\substack{\beta \in \mathcal{C}(k;l) \\ \tau_\beta = \tau}} \det(-S(J_\beta))$$

$$\Rightarrow \sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau}} \det(-S(J_\alpha)) = \sum_{\substack{\beta \in \mathcal{C}(k;l) \\ \tau_\beta = \tau}} \det(-S(J_\beta)) \tag{3}$$

Note that $J_\alpha = V_{int} \setminus G_\alpha$, where G_α is the set of internal vertices of the path α. Moreover, note that the leading term of $\det(-S(J_\alpha))$ is the monomial consisting of all elements of J_α (and thus has degree $|V_{int}| - |G|$).

Proof by induction on $|G|$. Base step 0: assume that $|G| = 0$, then $G = \emptyset$ and thus corresponds to an edge.

Case 1: assume $|\mathcal{C}_G(i;j)| = 0$, then there is no direct connection between vertices i and j. Moreover, this implies that the degree of both polynomials is strictly less than $|V_{int}|$, by the above note there cannot be a direct connection between k and l. Thus $|\mathcal{C}_G(k;l)| = 0$.

Case 2: assume now that $|\mathcal{C}_G(i;j)| = c_G \neq 0$, then this implies that there is an edge between vertices i and j. By construction, there are no pairs of parallel edges and thus $|\mathcal{C}_G(i;j)| = c_G = 1$. Then the leading term of the LHS (left hand side) of (3) is the monomial $q_{d+1} \cdots q_n$, which implies by the note above, that $|\mathcal{C}_G(k;l)| \neq 0 \Rightarrow |\mathcal{C}_G(k;l)| = 1$. Thus $|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k;l)|$.

Inductive step: now, assume that for all sets of interior vertices G such that $|G| = 0, \cdots, s$, $|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k,j)|$. Let $G = \{g_1, \cdots, g_{s+1}\}$ be a set of internal vertices such that $|G| = s+1$. Then

$$\sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau \\ |G_\alpha| \leq s}} \det(-S(J_\alpha)) + \sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau \\ |G_\alpha| > s}} \det(-S(J_\alpha)) = \sum_{\substack{\beta \in \mathcal{C}(k;l) \\ \tau_\beta = \tau \\ |G_\beta| \leq s}} \det(-S(J_\beta)) + \sum_{\substack{\beta \in \mathcal{C}(k;l) \\ \tau_\beta = \tau \\ |G_\beta| > s}} \det(-S(J_\beta))$$

As $|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k,j)|$ for all G such that $|G| \leq s$, by assumption, it is easily seen that

$$\sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau \\ |G_\alpha| \leq s}} \det(-S(J_\alpha)) = \sum_{\substack{\beta \in \mathcal{C}(k;l) \\ \tau_\beta = \tau \\ |G_\beta| \leq s}} \det(-S(J_\beta))$$

$$\Rightarrow \sum_{\substack{\alpha \in \mathcal{C}(i;j) \\ \tau_\alpha = \tau \\ |G_\alpha| > s}} \det(-S(J_\alpha)) = \sum_{\substack{\beta \in \mathcal{C}(k;l) \\ \tau_\beta = \tau \\ |G_\beta| > s}} \det(-S(J_\beta)) \tag{4}$$

Case 1: assume that $|\mathcal{C}_G(i;j)| = 0$, then on the LHS of (4) there is no monomial $g_1 \cdots g_{s+1}$. Moreover, this implies that there is no monomial $g_1 \cdots g_{s+1}$ on the RHS (right hand side) of (4), and thus $|\mathcal{C}_G(i;j)| = 0$.

Case 2: now assume that $|\mathcal{C}_G(i;j)| = c_G \neq 0$, then the coefficient on the LHS of the monomial $g_1 \cdots g_n$ is c_G. Which implies that coefficient of the monomial $g_1 \cdots g_{s+1}$ on the RHS of (4) is c_G. Thus $|\mathcal{C}_G(k;l)| = c_G$. Therefore $|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k;l)|$. ◻

Lemma 3.3. If for all sets of interior nodes, G,$|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k;l)|$ then $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$

Proof. Proof by induction on $|G|$. Base step 0: assume first that $|G| = 0$, then $G = \emptyset$ and corresponds to an edge.

Case 1: if $|\mathcal{C}_G(i;j)| = 0$, then there is no edge between i and j and thus $\mathcal{C}'_G(i;j) = \emptyset$. As $|\mathcal{C}_G(k;l)| = |\mathcal{C}_G(i;j)| = 0$, this also implies that $\mathcal{C}'_G(k;l) = \emptyset$.

Case 2: assume that $|\mathcal{C}_G(i;j)| = c_G \neq 0$, then as this corresponds to an edge between i and j and there are no parallel connections, $|\mathcal{C}_G(i;j)| = c_G = 1$, as $|\mathcal{C}_G(k,l)| = |\mathcal{C}_G(i;j)| = 1$, this implies that the edge between k and l exists. Thus $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$.

Base step 1: now assume that G is a set of internal vertices such that $|G| = 1$, then $G = \{g_1\}$.

Case 1: if $|\mathcal{C}_G(i;j)| = 0$, then either the edge between i and g_1 does not exist, or the edge between j and g_1 does not exist. Moreover, as $|\mathcal{C}_G(k;l)| = |\mathcal{C}_G(i;j)| = 0$, this implies that either the edge between k and g_1 or the edge between l and g_1 does not exist, and thus $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l) = \emptyset$.

Case 2: now, if $|\mathcal{C}_G(i;j)| = c_G \neq 0$, then as there is only one possible way to connect the nodes i and j through one interior node, $|\mathcal{C}_G(i;j)| = 1$. Additionally, as $|\mathcal{C}_G(k;l)| = |\mathcal{C}_G(i;j)| = 1$ and there is only one possible way to connect the nodes k and j through one interior node, $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$.

Base step 3: now assume that G is a set of internal nodes such that $|G| = 2$, then $G = \{g_1, g_2\}$.

Case 1: if the edge between g_1 and g_2 does not exist, then it is easy to see that $|\mathcal{C}_G(k;l)| = |\mathcal{C}_G(i;j)| = 0$ and thus $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$.

Assume that the edge between g_1 and g_2 exists.

Case 2: now let $|\mathcal{C}_G(k;l)| = 0$, then neither (the edge between i and g_1 and the edge between j and g_2 exist) nor (the edge between i and g_2 and the edge between j and g_1 exist). Moreover, as $|\mathcal{C}_G(k;l)| = |\mathcal{C}_G(i;j)| = 0$, then neither (the edge between k and g_1 and the edge between l and g_2 exist) and (the edge between k and g_2 and the edge between l and g_1 exist). Thus $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l) = \emptyset$.

Case 3: now let $|\mathcal{C}_G(k;l)| = 1$, then either (the edge between i and g_1 and the edge between j and g_2 exist) or (the edge between i and g_2 and the edge between j and g_1 exist), but not both. As $|\mathcal{C}_G(k;l)| = |\mathcal{C}_G(i;j)| = 1$, either (the edge between k and g_1 and the edge between l and g_2 exist) or (the edge between k and g_2 and the edge between l and g_1 exist), but not both. It is easily seen that in either case, $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$.

Case 4: finally, let $|\mathcal{C}_G(k;l)| = 2$, then the edges between i and g_1 and i and g_2 both exist and the edges between j and g_1 and j and g_2 both exist. Additionally, as $|\mathcal{C}_G(k;l)| = |\mathcal{C}_G(i;j)| = 2$, then the edges between k and g_1 and k and g_2 both exist and the edges between l and g_1 and l and g_2 both exist. Then it is easily seen that $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$ and the undirected paths have the same multiplicity in each.

Inductive step: finally, assume that for all sets of internal vertices, G, such that $|G| \le s$, assume that $|\mathcal{C}_G(i;j)| = |\mathcal{C}_G(k;l)| \Rightarrow \mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l)$. Now let G be a set of internal nodes such that $|G| = s+1$.

Case 1: if $|\mathcal{C}_G(i;j)| = 0$, then it is easily seen that $|\mathcal{C}_G(k;l)| = 0$ and that $\mathcal{C}'_G(i;j) = \mathcal{C}'_G(k,l) = \emptyset$.

Case 2: Assume that $|\mathcal{C}_G(i;j)| \neq 0$, then let $\alpha \in \mathcal{C}_G(i;j)$ and $\beta \in \mathcal{C}_G(k;l)$. Let $\alpha' \in \mathcal{C}'_G(i;j)$, such that α' is a subpath of α, and $\beta' \in \mathcal{C}'_G(k;lj)$, such that β' is a subpath of β. Now, let α_1 be the second vertex of α (and thus the first vertex after i), similarly let α_{s+1} be the second to last vertex of α (and thus the last vertex before j). Similarly define β_1 and β_{s+1}. Now, if either ($\alpha_1 = \beta_1$ and $\alpha_{s+1} = \beta_{s+1}$) or ($\alpha_1 = \beta_{s+1}$ and $\alpha_{s+1} = \beta_1$), then it is easily seen that $\alpha' \in \mathcal{C}'_G(k;l)$. Assume that neither $\alpha_1 = \beta_1$ and $\alpha_{s+1} = \beta_{s+1}$) nor ($\alpha_1 = \beta_{s+1}$ and $\alpha_{s+1} = \beta_1$) occurs. Then the undirected subpath with endpoints of α_1 and α_{s+1} form a proper subpath of β', let this subpath be called π. As the edge between i and α_1 and the edge between j and α_{s+1} exist, then as π is a path of shorter length than s, by our inductive hypothesis, it must also be a connection between k and l, then this implies that either (the edge between k and α_1 and the edge between l and α_{s+1} exist) or (the edge between k and α_{s+1} and l and α_1 exist), thus $\alpha \in \mathcal{C}'_G(k,l)$. Therefore, $\mathcal{C}'_G(i,j) \subseteq \mathcal{C}'_G(k,l)$, and by symmetry, $\mathcal{C}'_G(i,j) \supseteq \mathcal{C}'_G(k,l)$, thus $\mathcal{C}'_G(i,j) = \mathcal{C}'_G(k,l)$. □

From the two lemmas above, it is easily seen that if $\lambda_{i,j}(q) = \lambda_{k,l}(q)$, then $\mathcal{C}'(i;j) = \mathcal{C}'(k;l)$ as $\mathcal{C}'(i;j) = \bigcup_{G \subseteq V_{int}} \mathcal{C}'_G(i;j)$ □

Corollary 3.1. Let $R = \{(r_1, r_2) | r_1 \neq r_2, r_1 \in V_\partial, r_2 \in V_\partial\}$ and T be defined similarly. $\sum_{I \in R} \lambda_I(q) = \sum_{J \in T} \lambda_J(q)$ iff $\bigcup_{I \in R} \mathcal{C}'(I) = \bigcup_{J \in T} \mathcal{C}'(J)$ with multiplicity.

Proof. The proof follows the exact method as Theorem 3.1. □

4 Future Work and Conjectures

Future Work 4.1. Prove Conjecture 3.1. It is nearly proved and might be done by the end of the day, using Bezout's theorem.

Future Work 4.2. Find a graph theoretic reason to the case when $\lambda_{i,j}\lambda_{k,l} = \lambda_{m,n}\lambda_{o,p}$. There is one known case where this occurs and it can be related to the determinantal identity, but not much else is known.

Future Work 4.3. Some special cases on the value of the potential function (especially on the boundary) are worth considering, for instance if $q|_\partial = 0$.

Conjecture 4.1. All relations between elements of the response matrix can be found through the determinantal identity and that a graph is nearly recoverable if there are n independent solutions equations in the response matrix

References

[1] Michelle Covelle and Krysztof Fidkowski. The discrete inverse scattering problem. 2001.

[2] David Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Undergraduate Texts in Mathematics. Springer-Verlag, 1992.

[3] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks*, volume 13 of *Series on Applied Mathematics.* World Scientific, 2000.

[4] Richard Oberlin. Discrete inverse problems for schrodinger and resistor networks. 2000.

[5] Richard Oberlin. Notes on schrodinger and couductivity networks with tower of hanoi graphs. 2000.