A Method for Recovering Arbitrary Graphs

Nicolas Addington

January 4, 2005

Abstract

We develop a method for recovering arbitrary graphs, a generalization of the star–k method that avoids spurious parameters. They key is to recognize that the inverse problem amounts to undoing the Schur complement and to analyze the residue term $-BC^{-1}B^{\top}$.

Contents

1 Introduction

We assume that the reader is familiar with the basics of Curtis and Morrow's inverse problem [1]: graphs, Kirchhoff matrices, response matrices, and the Schur complement.

The forward problem is to take a Kirchhoff matrix K and find its response matrix Λ. This is easily done using the Schur complement: if we write K in the block form $\left(\begin{smallmatrix} A & B \\ B^\top & C \end{smallmatrix}\right)$ where C represents the interior-to-interior connections, $\Lambda = A - BC^{-1}B^\top = K/C$, the Schur complement of C in K. At times it will be convenient, albeit awkward-sounding, to say "K Schur C" rather than "the Schur complement of C in K." It will also be convenient to refer to $-BC^{-1}B^\top$ as the "residue term."

The inverse problem is to take a graph and its response matrix and find the Kirchhoff matrix, that is, knowing K Schur C and knowing which entries of K are zero, to recover the remaining entries of K. The inverse problem, then, amounts to undoing the Schur complement. This is difficult because the Schur complement destroys a lot of information: if a graph has eight boundary nodes and eight interior nodes, we are trying to recover a 16×16 matrix from an 8×8 with the help of a few zeros. The inverse problem has been solved for circular planar graphs [1] and annular graphs have been studied some [3]; in this paper we are interested in the recoverability of arbitrary graphs.

Our plan is to set up a sequence of intermediate matrices between the Kirchhoff matrix and the response matrix and pull the information in the response matrix back through these intermediate matrices to the Kirchhoff matrix. In §2 we study the simplest case, graphs with a single interior node. In §3 we describe the "star–k method," which amounts to iterating the one-node case. In §4 we develop a general recovery method, which is our main result. In §5 we suggest directions for future research.

2 Graphs with One Interior Node

2.1 Basic Recovery Technique

Consider a graph with n boundary nodes and one interior node. Fix an ordering of the nodes with the boundary nodes first. Let $\gamma_1, \ldots, \gamma_n$ be the conductances of the edges joining the interior node to the boundary nodes $1, \ldots, n$, let $\gamma = (\gamma_1 \ \cdots \ \gamma_n)^\top$ be the column vector of these conductances, and let $\sigma = \gamma_1 + \cdots + \gamma_n$ be their sum. Then when we write K in block form,

$$K = \begin{pmatrix} A & -\gamma \\ & \\ -\gamma^\top & \sigma \end{pmatrix}$$

so $\Lambda = A - \frac{\gamma\gamma^\top}{\sigma}$. We wish to view this in the following manner: Λ is just the upper left corner of K superimposed with a residue $R = -\frac{\gamma\gamma^\top}{\sigma}$ from the elimination of the interior node. Let us introduce the notation $\underline{K|}$ for the upper left corner of K (the size of which will always be clear from context), so $\Lambda = \underline{K|} + R$. If we imagine laying these matrices flat, we can write

$$\begin{array}{cccccccc} & R & - & - & - & - & & \\ + & K & - & - & - & - & - & - \\ \hline & \Lambda & - & - & - & - & & \end{array}. \tag{2.1.1}$$

Recall that K and Λ are symmetric, and observe that R must be as well.

We are given all of Λ and the position of the zeros in K, so wherever there is a zero in K, we can read the corresponding entry of R directly from Λ. Now

$$R = -\frac{\gamma\gamma^\top}{\sigma} = -\frac{1}{\sigma} \begin{pmatrix} {\gamma_1}^2 & \gamma_1\gamma_2 & & \gamma_1\gamma_3 & \cdots & \gamma_1\gamma_n \\ \gamma_1\gamma_2 & {\gamma_2}^2 & & \gamma_2\gamma_3 & \cdots & \gamma_2\gamma_n \\ \gamma_1\gamma_3 & \gamma_2\gamma_3 & & {\gamma_3}^2 & \cdots & \gamma_2\gamma_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \gamma_1\gamma_n & \gamma_2\gamma_n & & \gamma_3\gamma_n & \cdots & {\gamma_n}^2 \end{pmatrix}$$

so any 2×2 submatrix of R has determinant 0. Thus if we know three entries in some 2×2 submatrix of R, we can recover the fourth. If there are sufficiently many well-placed zeros in K, we may be able to recover all of R this way. Once we have R, we can get all but the last row and column of K since $\underline{K|} = \Lambda - R$. Since K is a Kirchhoff matrix, its rows and columns sum to zero, so we can recover the last row and column as well.

2.2 Example: The Kite

Consider the "kite" graph (Figure 1). We are given $\Lambda = (\lambda_{ij})$ and we wish to recover K. We know that k_{13}, k_{23}, k_{24}, and k_{34} are all zero since there are no edges between the corresponding pairs of nodes, so we know the following entries of R:

$$R = \begin{pmatrix} ? & ? & \lambda_{13} & ? \\ ? & ? & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & ? & \lambda_{34} \\ ? & \lambda_{24} & \lambda_{34} & ? \end{pmatrix}.$$

Since $r_{13}r_{24} - r_{14}r_{23} = 0$, we recover $r_{14} = \frac{\lambda_{13}\lambda_{24}}{\lambda_{23}}$. We can recover all of R similarly:

$$R = \begin{pmatrix} \frac{{\lambda_{13}}^2\lambda_{24}}{\lambda_{23}\lambda_{34}} & \frac{\lambda_{13}\lambda_{24}}{\lambda_{34}} & \lambda_{13} & \frac{\lambda_{13}\lambda_{24}}{\lambda_{23}} \\ \frac{\lambda_{13}\lambda_{24}}{\lambda_{34}} & \frac{\lambda_{23}\lambda_{24}}{\lambda_{34}} & \lambda_{23} & \lambda_{24} \\ \lambda_{13} & \lambda_{23} & \frac{\lambda_{23}\lambda_{34}}{\lambda_{24}} & \lambda_{34} \\ \frac{\lambda_{13}\lambda_{24}}{\lambda_{23}} & \lambda_{24} & \lambda_{34} & \frac{\lambda_{24}\lambda_{34}}{\lambda_{23}} \end{pmatrix}.$$

Observe that there are several ways to recover a given entry, for example $r_{11} = \frac{r_{13}r_{14}}{r_{34}} = \frac{r_{12}r_{13}}{r_{23}} = \frac{{r_{14}}^2}{r_{44}}$, but these all give the same result. (Observe also that "the indices cancel" in each expression.) It appears that in general, if we begin with a set of entries in R which is independent, that is, if no entry in our set can be gotten from the others, then no matter how we recover the remaining entries we get the same values.

Now that we have R, we can recover the upper left corner of K:

$$K = \begin{pmatrix} \lambda_{11} - \frac{{\lambda_{13}}^2\lambda_{24}}{\lambda_{23}\lambda_{34}} & \lambda_{12} - \frac{\lambda_{13}\lambda_{24}}{\lambda_{34}} & 0 & \lambda_{14} - \frac{\lambda_{13}\lambda_{24}}{\lambda_{23}} & ? \\ \lambda_{12} - \frac{\lambda_{13}\lambda_{24}}{\lambda_{34}} & \lambda_{22} - \frac{\lambda_{23}\lambda_{24}}{\lambda_{34}} & 0 & 0 & ? \\ 0 & 0 & \lambda_{33} - \frac{\lambda_{23}\lambda_{34}}{\lambda_{24}} & 0 & ? \\ \lambda_{14} - \frac{\lambda_{13}\lambda_{24}}{\lambda_{23}} & 0 & 0 & \lambda_{14} - \frac{\lambda_{24}\lambda_{34}}{\lambda_{23}} & ? \\ ? & ? & ? & ? & ? \end{pmatrix}$$

and since the rows and columns of K sum to 0, we can recover all of K.

Observe that

$$k_{12} = \lambda_{12} - \frac{\lambda_{13}\lambda_{24}}{\lambda_{34}} = \frac{\begin{vmatrix} \lambda_{12} & \lambda_{13} \\ \lambda_{24} & \lambda_{34} \end{vmatrix}}{\lambda_{34}}$$

and that the determinant in the last expression corresponds to the 2-connection from nodes 1 and 4 to 2 and 3. Since there is an edge between nodes 1 and 2, k_{12} is negative, so this determinant is positive, and the connection exists. Removing this edge would break this connection, but it would also make $k_{12} = 0$ and hence make this determinant 0. Thus we see the link between 2-connections and determinants very clearly by studying graphs with one interior node.

2.3 Quadrilaterals: A Pictorial Tool

We can identify a network (i.e. a graph with conductances) with its Kirchhoff matrix, and a response matrix is the Kirchhoff matrix of a complete graph, so we can understand taking the Schur complement of the Kirchhoff matrix as replacing the interior nodes and the edges incident to them with a response-equivalent complete graph (Figure 2).

For graphs with one interior node, we can identify the superimposed complete graph with the off-diagonal entries of the residue R, since $R = \Lambda$ except on the diagonal if there are no boundary-to-boundary connections. Thus our statement $\Lambda = \underline{K|} + R$ can be interpreted pictorially as in Figure 3. In this interpretation, a 2×2 submatrix of R

$$\begin{pmatrix} r_{ik} & r_{il} \\ r_{jk} & r_{jl} \end{pmatrix}$$

corresponds to a quadrilateral in the superimposed complete graph as in Figure 4, so if we know three sides of this quadrilateral we can recover a fourth.

Quadrilaterals provide a quick way to check the recoverability of a graph. To show that the kite is recoverable, we begin by drawing the picture in Figure 5. Where there are doubled edges, knowing the response matrix gives us their sum, so we need to separate that number into the part that came from the boundary-to-boundary connection and the part that came from the superimposed complete graph. For the edge 1–2, we know the other three sides of the quadrilateral 1–3–4–2, and the quadrilateral 1–3–2–4 gives us the edge 1–4. Now we know the boundary-to-boundary connections and the superimposed complete graph, i.e. the off-diagonal entries of R, so we know we can recover the diagonal R, so the kite is recoverable.

If instead we were trying to recover the "bowtie," we would draw the picture in Figure 6. Any quadrilateral with 1–4 as a side also has 2–3 as a side, so we cannot get three sides of any quadrilateral to recover 1–4 or 2–3, and the bowtie is not recoverable.

2.4 The Square Root Trick

In §2.1 we said that since any 2×2 submatrix of R is singular, we can recover an entry in a 2×2 submatrix given the other three. Since R is symmetric, however, more is true. If our 2×2 submatrix is principal and we know only the two diagonal entries, we can recover the off-diagonal entries:

$$0 = \begin{vmatrix} r_{ii} & r_{ij} \\ r_{ij} & r_{jj} \end{vmatrix} = r_{ii} r_{jj} - r_{ij}^2,$$

so $r_{ij} = -\sqrt{r_{ii} r_{jj}}$; the entries of R are necessarily negative. Observe again that "the indices cancel."

The "Star of David" graph (Figure 7) can be recovered using this square root trick but not without it.

3 The Star–K Method

3.1 With Matrices

In this section we shall assume that our graph has 12 boundary nodes and 4 interior nodes to avoid drowning the reader in a sea of index variables; what is meant in general should be clear.

Let $K_{16} = K$ be the Kirchhoff matrix of the original network. Let K_{15} equal K_{16} Schur its lower right 1×1 corner. Let R_{16} be the residue $-\frac{\gamma\gamma^\top}{\sigma}$ from this Schur complement, i.e. the residue from the elimination of node 16, so $K_{15} = \underline{K_{16}|} + R_{16}$. Similarly, let K_{14} equal K_{15} Schur its lower right 1×1 corner and R_{15} be the residue, so $K_{14} = \underline{K_{15}|} + R_{15}$. Continue in this fashion down to K_{12}.

Proposition 1. *K_{12} equals K_{16} Schur its lower right 4×4 corner, that is, $K_{12} = \Lambda$.*

Proof. In [2], Crabtree and Haynsworth show that if C is an invertible square submatrix of M and D is an invertible square submatrix of D then $M/D = (M/C)/(C/D)$. It is easy to check that the lower right 1×1 corner of K_{15} equals the lower right 2×2 corner of K_{16} Schur its lower right 1×1 corner, so K_{14} equals K_{16} Schur its lower right 2×2 corner. The desired result is obtained by induction. □

We are given the graph, or equivalently, the zeros of $K = K_{16}$. From this we can determine the zeros of R_{16} and hence the zeros of K_{15}, and so on down to K_{13}. Given $\Lambda = K_{12}$ and the zeros of K_{13} we may be able, using our one-interior-node method, to recover R_{14} and hence K_{14}. From this we may be able to recover K_{15}, and so on back up to K_{16}. If this process succeeds, we know that the graph is recoverable.

If at some point during this process we get stuck, that is, we know a few entries of some R_n but cannot recover any more using 2×2 submatrices, we can parametrize one of the unknown entries and continue. The number of parameters gives a measure of the "irrecoverability" of our graph. It may also happen that at one stage we do not have enough information to continue, so we introduce a parameter, but at a later stage we have more information than we need and are able to use 2×2 submatrices to eliminate the parameter. An example will be given in §3.3.

3.2 With Quadrilaterals

In §2.3, we replaced the interior node and the edges incident to it, or the "star" around the interior node, with the response-equivalent complete graph, or "k." Now that we are dealing with several interior nodes, we make this star–k replacement several times and obtain a sequence of intermediate graphs as in Figure 8. The K_n of the previous section are the Kirchhoff matrices of these intermediate graphs; specifically, K_n is the intermediate graph with n nodes. If each intermediate graph is recoverable, the original graph is recoverable.

Here again, quadrilaterals provide a quick check of recoverability. Consider the "hexagons on a tetrahedron" in Figure 9. With the quadrilateral 1–5–4–2 we can separate the two edges 1–2, and by symmetry all the doubled edges, so we can recover all four interior nodes. This example is interesting because it is a "flower:" it has neither a boundary-to-boundary connection nor a boundary spike (a boundary node connected to a single interior node). Curtis and Morrow have shown that no circular planar flower is recoverable; this flower is, of course, not circular planar. Jeff Russell and Tracy Lovejoy's "toy drum" (Figure 10) is another example of a recoverable flower.

3.3 Example: The Marshmallow

Consider the "marshmallow" graph (Figure 11), so called because the presentation in which the author introduced it involved a marshmallow model. For this graph,

$$K_7 = \begin{pmatrix} ? & 0 & 0 & 0 & 0 & ? & 0 \\ 0 & ? & 0 & 0 & 0 & ? & ? \\ 0 & 0 & ? & 0 & 0 & ? & ? \\ 0 & 0 & 0 & ? & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & 0 \\ 0 & ? & ? & ? & ? & 0 & ? \end{pmatrix} \qquad K_6 = \begin{pmatrix} ? & 0 & 0 & 0 & 0 & ? \\ 0 & ? & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? & ? \\ 0 & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{pmatrix} \qquad K_5 = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \lambda_{12} & \lambda_{22} & \lambda_{23} & \lambda_{24} & \lambda_{25} \\ \lambda_{13} & \lambda_{23} & \lambda_{33} & \lambda_{34} & \lambda_{35} \\ \lambda_{14} & \lambda_{24} & \lambda_{34} & \lambda_{44} & \lambda_{45} \\ \lambda_{15} & \lambda_{25} & \lambda_{35} & \lambda_{45} & \lambda_{55} \end{pmatrix}$$

so

$$R_6 = \begin{pmatrix} ? & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \lambda_{12} & ? & ? & ? & ? \\ \lambda_{13} & ? & ? & ? & ? \\ \lambda_{14} & ? & ? & ? & ? \\ \lambda_{15} & ? & ? & ? & ? \end{pmatrix}.$$

No further entries of R_6 can be recovered, but if we parametrize the $1, 1$ entry by $t < 0$, we can get everything else:

$$R_6 = \begin{pmatrix} t & \lambda_{12} & \lambda_{13} & \lambda_{14} & \lambda_{15} \\ \lambda_{12} & \frac{\lambda_{12}{}^2}{t} & \frac{\lambda_{12}\lambda_{13}}{t} & \frac{\lambda_{12}\lambda_{14}}{t} & \frac{\lambda_{12}\lambda_{15}}{t} \\ \lambda_{13} & \frac{\lambda_{12}\lambda_{13}}{t} & \frac{\lambda_{13}{}^2}{t} & \frac{\lambda_{13}\lambda_{14}}{t} & \frac{\lambda_{13}\lambda_{15}}{t} \\ \lambda_{14} & \frac{\lambda_{12}\lambda_{14}}{t} & \frac{\lambda_{13}\lambda_{14}}{t} & \frac{\lambda_{14}{}^2}{t} & \frac{\lambda_{14}\lambda_{15}}{t} \\ \lambda_{15} & \frac{\lambda_{12}\lambda_{15}}{t} & \frac{\lambda_{13}\lambda_{15}}{t} & \frac{\lambda_{14}\lambda_{15}}{t} & \frac{\lambda_{15}{}^2}{t} \end{pmatrix}.$$

¿From here we can get K_6 and complete the recovery. Along the way, however, an interesting thing happens. In R_5, rows 2 and 3 by columns 4 and 5 are

$$\begin{pmatrix} \lambda_{24} - \frac{\lambda_{12}\lambda_{14}}{t} & \lambda_{25} - \frac{\lambda_{12}\lambda_{15}}{t} \\ \lambda_{34} - \frac{\lambda_{13}\lambda_{14}}{t} & \lambda_{35} - \frac{\lambda_{13}\lambda_{15}}{t} \end{pmatrix} = \frac{1}{t^2} \begin{pmatrix} \begin{vmatrix} t & \lambda_{12} \\ \lambda_{14} & \lambda_{24} \end{vmatrix} & \begin{vmatrix} t & \lambda_{12} \\ \lambda_{15} & \lambda_{25} \end{vmatrix} \\ \\ \begin{vmatrix} t & \lambda_{13} \\ \lambda_{14} & \lambda_{34} \end{vmatrix} & \begin{vmatrix} t & \lambda_{13} \\ \lambda_{15} & \lambda_{35} \end{vmatrix} \end{pmatrix}$$

but since this is a 2×2 submatrix of an R_n, it is singular, so by the six-term identity,

$$0 = \frac{1}{t^2} \begin{vmatrix} \begin{vmatrix} t & \lambda_{12} \\ \lambda_{14} & \lambda_{24} \end{vmatrix} & \begin{vmatrix} t & \lambda_{12} \\ \lambda_{15} & \lambda_{25} \end{vmatrix} \\ \\ \begin{vmatrix} t & \lambda_{13} \\ \lambda_{14} & \lambda_{34} \end{vmatrix} & \begin{vmatrix} t & \lambda_{13} \\ \lambda_{15} & \lambda_{35} \end{vmatrix} \end{vmatrix} = \frac{1}{t} \begin{vmatrix} t & \lambda_{14} & \lambda_{15} \\ \lambda_{12} & \lambda_{24} & \lambda_{25} \\ \lambda_{13} & \lambda_{34} & \lambda_{35} \end{vmatrix}.$$

If $\left(\begin{smallmatrix} \lambda_{24} & \lambda_{25} \\ \lambda_{34} & \lambda_{35} \end{smallmatrix}\right)$ is invertible, this allows us to solve for t, that is, t was a spurious parameter.

Observe that $\left(\begin{smallmatrix} \lambda_{24} & \lambda_{25} \\ \lambda_{34} & \lambda_{35} \end{smallmatrix}\right)$ corresponds to the 2-connection from nodes 2 and 3 to 4 and 5. This connection exists, but the corresponding determinant may be zero because we can also connect 2 and 3 to 5 and 4 (the opposite permutation). For most graphs in which spurious parameters arise, it is possible to decide whether the necessary determinant is invertible or not by examining connections; the marshmallow is something of a pathalogical case in this respect.

If $\left(\begin{smallmatrix} \lambda_{24} & \lambda_{25} \\ \lambda_{34} & \lambda_{35} \end{smallmatrix}\right)$ is invertible, the marshmallow is recoverable. By symmetry, if $\left(\begin{smallmatrix} \lambda_{23} & \lambda_{25} \\ \lambda_{34} & \lambda_{45} \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \lambda_{23} & \lambda_{24} \\ \lambda_{35} & \lambda_{45} \end{smallmatrix}\right)$ is invertible, it is also recoverable. If all three of these are singular, it is possible to exhibit a one-parameter family of Kirchhoff matrices with the same response matrix. The marshmallow, then, is "almost always" recoverable. "Tracy's J" (Figure 12) has the same property. It is interesting to compare this sort of graph with the 2-to-1 graphs studied by Ernie Esser and Tracy Lovejoy [5], which are "almost never" recoverable—the former are recoverable unless certain submatrices of Λ are singular, whereas the latter are recoverable only if certain submatrices of Λ are singular.

4 A General Recovery Method

4.1 Shortcomings of the Star–K Method

The star–k method has two main shortcomings. First, it only works unequivocally when each of the intermediate graphs is recoverable. This is often true for small graphs, such as the tophat (Figure 8), and graphs with few connections, such as hexagons on a tetrahedron, but it fails for many denser graphs. There is no order of interior nodes for which the star–k method can recover Ernie Esser's "2 circles, 4 rays" (Figure 13a) or even the well-connected graph on 5 nodes shown in Figure 14a, which is circular planar. Both these graphs are known to be recoverable by other means. Second, the star–k method is very sensitive to the order in which the interior nodes are eliminated. Figure 15a is recoverable using the star–k method, but Figure 15b is not. In the previous section, the recovery of the marshmallow required one parameter, but if we switch the order of nodes 6 and 7, it requires four parameters (three of which end up being spurious).

Parameters which come and go provide a way around these shortcomings, and it may be that all recoverable graphs can be recovered using the star–k method and spurious parameters. This method is inelegant, however, and does not reflect what is really going on. If a spurious parameter arises in the star–k recovery of a graph, it is because one of the intermediate graphs was not recoverable but the original graph was; some information was present in the chain of intermediate graphs that could not be seen by looking only from one graph to the next. In the previous section we saw how when we look across several steps, 3×3 determinants arise from 2×2 determinants of 2×2 determinants by way of the six-term identity. In this section we will use 3×3 and larger determinants directly, without the help of spurious parameters.

4.2 Looking at Multiple Layers

In §2.1 we wrote $\Lambda = \underline{K|} + R$, or when we laid K and R flat,

$$\begin{array}{cccccccc} & R & - & - & - & - & & \\ + & K & - & - & - & - & - & \\ \hline & \Lambda & - & - & - & - & & \end{array}.$$

We were given certain entries of the matrices in this sum: the zeros of K and R and all the entries of Λ. We used vertical information—knowing k_{ij} and λ_{ij} we could find r_{ij}, and knowing r_{ij} and λ_{ij} we could find

k_{ij}—and horizontal information—the singularity of certain submatrices of R and the fact that the rows and columns of K summed to zero—to recover all the entries of everything, and particular to recover K.

In §3.1 we had

$$\begin{aligned} K_{12} &= \underline{K_{13}|} + R_{13} \\ &= \underline{K_{14}|} + \underline{R_{14}|} + R_{13} \\ &= \underline{K_{15}|} + \underline{R_{15}|} + \underline{R_{14}|} + R_{13} \\ &= \underline{K_{16}|} + \underline{R_{16}|} + \underline{R_{15}|} + \underline{R_{14}|} + R_{13} \end{aligned}$$

or if we lay the matrices flat,

$$\begin{array}{rrl} & R_{13} & - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\ & R_{14} & - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\ & R_{15} & - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\ & R_{16} & - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\ + & K_{16} & - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \\ \hline & K_{12} & - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \; - \end{array} . \tag{4.2.1}$$

Here we did the same thing: we began with the known zeros of K_{16} (and hence of the R_n) and the known entries of $\Lambda = K_{12}$ and used vertical and horizontal information to peel off the R_n one at a time and recover K_{16}.

The following sort of thing happens often: the i, j entries of K_{16}, R_{16}, and R_{15} are zero but the i, j entries of R_{13} and R_{14} are not, so we can recover the sum of these two entries, but we do not know what portion of the number comes from R_{13} and what portion from R_{14}. To make use of this information, we introduce a matrix $R_{14}^{13} = \underline{R_{14}|} + R_{13}$; we typically know more about this matrix than we do about R_{13} and R_{14} separately.

This is our plan: in addition to considering the single layers $R_{13}, \ldots, R_{16}$, we will also consider all sums of contiguous blocks of them, for example $R_{15}^{13} = \underline{R_{15}|} + \underline{R_{14}|} + R_{13}$ and $R_{16}^{15} = \underline{R_{16}|} + R_{15}$ and even $K_{14} = \underline{K_{16}|} + \underline{R_{16}|} + R_{15}$. These multi-layer gadgets add up nicely: $R_{16}^{13} = \underline{R_{16}^{14}|} + R_{13} = \underline{R_{16}^{15}|} + R_{14}^{13} = \underline{R_{16}|} + R_{15}^{13}$. Thus we understand the vertical information for the R_n^m.

To recover a graph, we wish to use horizontal information about the R_n^m to recover their entries from a few given entries. For single layers, we used the fact that 2×2 submatrices were singular. For multiple layers, we will use larger submatrices.

4.3 Recovery with Larger Submatrices

The following Lemma was used tacitly in §3.3 and will be used extensively in what follows.

Lemma 2. *If an $n \times n$ matrix M is singular, we know all but one entry m_{ij}, and the cofactor M_{ij} is invertible, we can recover the unknown entry.*

Proof. By cofactor expansion along the i^{th} row,

$$0 = \det M = (-1)^{i+1} m_{i1} \det M_{i1} + \cdots + (-1)^{i+j} m_{ij} \det M_{ij} + \cdots + (-1)^{i+1} m_{i1} \det M_{in},$$

so since $\det M_{ij} \neq 0$, we can solve for m_{ij}. $\square$

In §2.4, we used the symmetry of R to do slightly better for 2×2 matrices. It is unclear whether it is worth doing this for larger matrices. Suppose that an $n \times n$ matrix M is singular and symmetric, with the block form

$$M = \begin{pmatrix} a & B & x \\ B^\top & C & D \\ x & D^\top & e \end{pmatrix}$$

where the corners a, x, and e are 1×1. We wish to solve for x. By the six-term identity,

$$
\begin{aligned}
0 &= |M||C| \\
&= \begin{vmatrix} a & B \\ B^\top & C \end{vmatrix} \begin{vmatrix} C & D \\ D^\top & e \end{vmatrix} - \begin{vmatrix} B & x \\ C & D \end{vmatrix} \begin{vmatrix} B^\top & C \\ x & D^\top \end{vmatrix} \\
&= \begin{vmatrix} a & B \\ B^\top & C \end{vmatrix} \begin{vmatrix} C & D \\ D^\top & e \end{vmatrix} - \begin{vmatrix} B & x \\ C & D \end{vmatrix}^2 \\
&= \begin{vmatrix} a & B \\ B^\top & C \end{vmatrix} \begin{vmatrix} C & D \\ D^\top & e \end{vmatrix} - \left(\begin{vmatrix} B & 0 \\ C & D \end{vmatrix} + (-1)^n x|C| \right)^2
\end{aligned}
$$

so if C is invertible,

$$
x = \frac{(-1)^{n+1}}{|C|} \left(\begin{vmatrix} B & 0 \\ C & D \end{vmatrix} \pm \sqrt{\begin{vmatrix} a & B \\ B^\top & C \end{vmatrix} \begin{vmatrix} C & D \\ D^\top & e \end{vmatrix}} \right).
$$

If the plus-or-minus term is nonzero, we have two possible values for x. If one is positive and one negative, in our application we will know which one we want, but there is no obvious way of telling whether the two solutions are of the same or opposite sign. On the other hand, if the plus-or-minus term is zero, $\left|\begin{smallmatrix} B & x \\ C & D \end{smallmatrix}\right| = 0$, so we can find x from that.

It is not known whether any graphs can be recovered using this generalized square root trick but not without it. We guess that there are none, but suspect that this trick is at work in 2-to-1 graphs.

4.4 Testing for Singularity

To make use of Lemma 2, we need to be able to test submatrices of the R_n^m for singularity. We continue to assume that our graph has 16 nodes.

We defined R_{15} (for example) as the residue term $-\frac{\gamma\gamma^\top}{\sigma}$ from K_{15} Schur its lower right 1×1 corner. We wish to show that R_{15}^{14} (for example), which we defined as $\underline{R_{15}|} + R_{14}$, is the residue from K_{15} Schur its lower right 2×2 corner. Let C_{15}^{14} be that corner: rows 14 to 15 by columns 14 to 15 of K_{15}. Let $\underline{K_{15}|}$ be rows 1 to 13 by columns 1 to 13 and B be rows 1 to 13 by columns 14 to 15, so

$$
K_{15} = \begin{pmatrix} \underline{K_{15}|} & B \\ \\ B^\top & C_{15}^{14} \end{pmatrix}.
$$

Now $K_{13} = \underline{K_{15}|} + \underline{R_{15}|} + R_{14} = \underline{K_{15}|} + R_{15}^{14}$, but by Proposition 1, $K_{13} = \underline{K_{15}|} - B\ (C_{15}^{14})^{-1} B^\top$ as well, so $R_{15}^{14} = -B\ (C_{15}^{14})^{-1} B^\top$, i.e. the residue from the elimination of nodes 15 and 14.

K_{13} is the Schur complement of C_{15}^{14} in K_{15}. R_{15}^{14} is the Schur complement of C_{15}^{14} in a closely related matrix: with B as above, define

$$
Z_{15}^{14} = \begin{pmatrix} 0 & B \\ \\ B^\top & C_{15}^{14} \end{pmatrix},
$$

that is, K_{15} with the upper left 13×13 corner replaced with zeros. Now

$$
R_{14}^{15} = 0 - B\ (C_{15}^{14})^{-1} B^\top = Z_{15}^{14} / C_{15}^{14}.
$$

Proposition 3. *Let M be the submatrix of R_{15}^{14} consisting of rows $r_1, \ldots, r_n$ by columns $c_1, \ldots, c_n$. Let N be the submatrix of Z_{15}^{14} consisting of rows $r_1, \ldots, r_n, 14, 15$ by columns $c_1, \ldots, c_n, 14, 15$. Then M is singular if and only if N is.*

Proof. $M = N/C_{15}^{14}$, so $\det M = (\det N)/(\det C_{15}^{14})$. C_{15}^{14} is not singular since it is a principal proper submatrix of a Kirchhoff matrix. $\square$

In other words, to decide whether a square submatrix M of R_{15}^{14} is singular, we can take the same submatrix of Z_{15}^{14} (which is just K_{15} with the upper left corner suppressed), tack on the rows and columns that would be chopped off in taking the Schur complement of C_{15}^{14}, and decide if that is singular.

We know which entries of K_{15} are zero, which are positive (the diagonal entries), and which are negative (the off-diagonal entries that are not zero). We can try to decide whether a submatrix N of Z_{15}^{14} is singular as follows. Expand the determinant as a polynomial in the entries of N (hence of K_{15}). If the terms are all zero, N is singular. If the terms are all positive or all negative, N is invertible. If some of the terms are positive and some negative, the signs of the entries alone are not enough to decide.

4.5 Recovering an Arbitrary Graph

Now we are in a position to outline a method for recovering the Kirchoff matrix of an arbitrary graph from its response matrix. Suppose our graph has M boundary nodes and N nodes altogether.

1. Write down the signs of all the entries of K_N, which we know from the graph. From these, determine the signs of all the entries of $K_M, \ldots, K_{N-1}$. We will use these to test submatrices for singularity using Proposition 3.
2. Make empty matrices $K_M, \ldots K_N$, $R_{M+1}, \ldots, R_N$, and R_n^m, $M < m < n \leq N$ of the appropriate sizes: K_n is $n \times n$, and R_n and R_n^m are $n-1 \times n-1$. Fill in the zeros of all these, which can be derived from the zeros of K_N. Fill in the entries of K_M, the response matrix.
3. Whenever we know two of three entries from something of the form $R_{16}^{13} = \underline{R_{16}^{15}|} + R_{14}^{13}$ or $K_{13} = \underline{K_{15}|} + R_{15}^{14}$, recover the third.
4. Whenever we know all but one entry of a submatrix of any matrix, if the submatrix is singular and the cofactor of the unknown entry is invertible, recover the unknown entry using Lemma 2. Also use the basic square root trick from §2.4.
5. Whenever we know all but one entry in a row of a K_n, recover it using the fact that the rows of Kirchhoff matrices sum to zero.
6. If at any point no more entries can be recovered but some are still missing, parametrize an unknown entry. The first single layer R_n with unknown entries (first in the sense that n is least) seems to be the best place to parameterize.

Observe that if we restrict our recovery to the single layers R_n and omit the mutli-layers R_n^m, this is exactly the star–k method.

This method, while powerful (there is no known recoverable graph that it fails to recover), is impractical to work by hand for large graphs, as we shall see in the next section, and unnecessary for small graphs, where the star–k method usually works. It is ideally suited to a computer, however. The author has implemented the method as a C++ program, which is available at http://www.math.washington.edu/∼reu/.

4.6 Example: 2 Circles, 4 Rays

Consider 2 circles, 4 rays (Figure 13a), which has 8 boundary nodes and 8 interior nodes. An ad hoc proof of this graph's recoverability is given by Ernie Esser in [3]. Here we recover it using our general method.

These are the signs of the entries in K_{16}, the Kirchhoff matrix:

$$K_{16} = \begin{bmatrix}
+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\
0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 \\
0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \\
0 & 0 & 0 & 0 & + & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & - & 0 & 0 & 0 & + & - & 0 & - & - & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & + & - & 0 & 0 & - & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & + & - & 0 & 0 & - & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 0 & - & + & 0 & 0 & 0 & - \\
- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & + & - & 0 & - \\
0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & + & - & 0 \\
0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & + & - \\
0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 0 & - & +
\end{bmatrix}$$

We are given K_8, the response matrix. First we wish to recover R_9, the residue from the elimination of node 9. This is an 8×8 matrix, all of whose entries are negative. Now

$$R_{16}^{10} = \begin{bmatrix} - & - & - & - & 0 & - & - & - & - \\ - & - & - & - & 0 & - & - & - & - \\ - & - & - & - & 0 & - & - & - & - \\ - & - & - & - & 0 & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & - & - & - & 0 & - & - & - & - \\ - & - & - & - & 0 & - & - & - & - \\ - & - & - & - & 0 & - & - & - & - \\ - & - & - & - & 0 & - & - & - & - \end{bmatrix} \qquad K_9 = \underline{K_{16}|} + R_{16}^{10} = \begin{bmatrix} + & - & - & - & 0 & - & - & - & - \\ - & + & - & - & 0 & - & - & - & - \\ - & - & + & - & 0 & - & - & - & - \\ - & - & - & + & 0 & - & - & - & - \\ 0 & 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \\ - & - & - & - & 0 & + & - & - & - \\ - & - & - & - & 0 & - & + & - & - \\ - & - & - & - & 0 & - & - & + & - \\ - & - & - & - & 0 & - & - & - & + \end{bmatrix}$$

so we know the following entries of $R_9 = K_8 - \underline{K_9|}$:

$$R_9 = K_8 - \underline{K_9|} = \begin{pmatrix} ? & ? & ? & ? & \lambda_{15} & ? & ? & ? \\ ? & ? & ? & ? & \lambda_{25} & ? & ? & ? \\ ? & ? & ? & ? & \lambda_{35} & ? & ? & ? \\ ? & ? & ? & ? & \lambda_{45} & ? & ? & ? \\ \lambda_{15} & \lambda_{25} & \lambda_{35} & \lambda_{45} & ? & \lambda_{56} & \lambda_{57} & \lambda_{58} \\ ? & ? & ? & ? & \lambda_{56} & ? & ? & ? \\ ? & ? & ? & ? & \lambda_{57} & ? & ? & ? \\ ? & ? & ? & ? & \lambda_{58} & ? & ? & ? \end{pmatrix}.$$

To recover R_9, it suffices to recover the $5, 5$ entry, for then we can the rest using 2×2 submatrices as we did in §2.2. We will recover the $5, 5$ entry by way of R_{16}^9.

Consider the submatrix M of R_{16}^9 consisting of rows 1, 2, 4, 5 by columns 3, 5, 7, 8. Since $R_{16}^9 = K_8 - \underline{K_{16}|}$ and the submatrix $1, 2, 4, 5 \times 3, 5, 7, 8$ of K_{16} is zero except at $5, 5$, we have

$$M = \begin{pmatrix} \lambda_{13} & \lambda_{15} & \lambda_{17} & \lambda_{18} \\ \lambda_{23} & \lambda_{25} & \lambda_{27} & \lambda_{28} \\ \lambda_{34} & \lambda_{45} & \lambda_{47} & \lambda_{48} \\ \lambda_{35} & ? & \lambda_{57} & \lambda_{58} \end{pmatrix}$$

It suffices to show that $1, 2, 4, 5 \times 3, 5, 7, 8$ is singular but $1, 2, 4 \times 3, 7, 8$ is invertible, for then we can recover the $5, 5$ entry by Lemma 2; then since $R_9 = R_{16}^9 - \underline{R_{16}^{10}|}$, and the $5, 5$ entry of R_{16}^{10} is 0, we can recover the $5, 5$ entry of K_9.

¿From §4.4 we know that the submatrix $1, 2, 4 \times 3, 7, 8$ of R_{16}^9 is invertible if and only if the submatrix 1,2,4,9–16 $\times$ 3,7,8,9–16 of K_{16} with the upper left 3×3 corner changed to zeros is invertible:

$$\begin{array}{c|ccccccccccc} & 3 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - \\ 9 & 0 & 0 & 0 & + & - & 0 & - & - & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & - & + & - & 0 & 0 & - & 0 & 0 \\ 11 & 0 & - & 0 & 0 & - & + & - & 0 & 0 & - & 0 \\ 12 & 0 & 0 & - & - & 0 & - & + & 0 & 0 & 0 & - \\ 13 & 0 & 0 & 0 & - & 0 & 0 & 0 & + & - & 0 & - \\ 14 & 0 & 0 & 0 & 0 & - & 0 & 0 & - & + & - & 0 \\ 15 & - & 0 & 0 & 0 & 0 & - & 0 & 0 & - & + & - \\ 16 & 0 & 0 & 0 & 0 & 0 & 0 & - & - & 0 & - & + \end{array}.$$

By cofactor expansion along the top row, this submatrix is invertible if and only if the submatrix 2,4,9–16 $\times$ 3,7,8,9–12,14–16 is invertible; that is, since there is a single nonzero entry in the top row, we can cross out row 1 by column 13 (the reader is encouraged to get a pencil and do this). Similarly, we can cross out 2×14, 4×16, 15×3 (going down the left-hand column now), 11×7, 12×8, 13×9, and 10×11. This leaves us with the submatrix $9, 14, 16 \times 10, 12, 15$:

$$\begin{array}{c|ccc} & 10 & 12 & 15 \\ \hline 9 & - & - & 0 \\ 14 & - & 0 & - \\ 16 & 0 & - & - \end{array}$$

which is necessarily invertible, so the submatrix $1, 2, 4 \times 3, 7, 8$ of R_{16}^9 is invertible. For $1, 2, 4, 5 \times 3, 5, 7, 8$, we play the same game, but after crossing out a few things we find a row of all zeros, so that matrix is singular.

Thus R_9 is recoverable, so K_9 is recoverable. The recovery of $K_{10}, \ldots, K_{16}$ is similar.

Exercise. Show that the well-connected graph on 5 nodes (Figure 14a) is recoverable. (Hint: Recover the $1, 1$ entry of R_9^6 using the submatrix $1, 2, 3 \times 1, 4, 5$.)

4.7 Connections and Determinants

In the previous example, we saw that a certain submatrix of R_{16}^9 was invertible because a related submatrix of Z_{16}^9 was. Because of the placement of zeros in K_{16}, the submatrix of R_{16}^9 was equal to a submatrix of K_8 and the submatrix of Z_{16}^9 was equal to a submatrix of K_{16}. But we know that the subdeterminant $1, 2, 4 \times 3, 7, 8$ of K_8, the response matrix, corresponds to a 3-connection from nodes 1, 2, and 4 to 3, 7, and 8. The calculation we just made shows us how.

The reader is encouraged to mark Figure 13a with a pencil. First we crossed out row 1 by column 13. A 3-connection from 1, 2, 4 to 3, 7, 8, must include the edge from node 1 to node 13; from node 1, we cannot go anywhere else. Similarly, we must go from 2 to 14, 4 to 16, 15 to 3, 11 to 7, and 12 to 8. Now we could go from 13 to 14, 16, or 9, but if we went to 14 or 16 we would collide with the connections coming from 2 and 4, so we must in fact go to 9. Similarly, we must get to 11 from 10. Now there are only two ways to complete the connection: either 9 to 10, 14 to 15, and 16 to 12, or 9 to 12, 14 to 10, and 16 to 15; these correspond to the two non-zero terms in the determinant of

$$\begin{array}{c|ccc} & 10 & 12 & 15 \\ \hline 9 & - & - & 0 \\ 14 & - & 0 & - \\ 16 & 0 & - & - \end{array}.$$

How subdeterminants of R_n^m translate into connections like this has not been studied carefully. We can say a few things. The lower index n indicates the intermediate graph through which the connection is going; subdeterminants of R_{14}^9, for example, correspond to connections not through the original graph but through the intermediate graph with 14 nodes (where nodes 15 and 16 have been "star–k'd"). The upper index m indicates how many nodes are considered boundary nodes; in R_{16}^9, nodes 1–8 were boundary nodes, but for R_{16}^{11}, nodes 1–10 would act as boundary nodes. The fact that any 2×2 submatrix of a single layer, say R_{11}, is singular reflects the fact that there are no 2-connections through a star. Similarly, for an R_n^m comprising $k = n - m + 1$ layers, any $k + 1 \times k + 1$ submatrix is singular since there are no $k + 1$ connections through a graph with k interior nodes.

There are many questions, however. What does it mean to replace the upper right corner of a Kirchhoff matrix with zeros? If the zeros in K_{16} had not been so fortuitously placed, how would a subdeterminant of R_{16}^9 differ from a subdeterminant of K_8? What does it mean when a subdeterminant intersects the diagonal—does it make sense to speak of the connection from nodes 1, 2, and 3 to 1, 4, and 5, as we seem to have been doing in §3.3?

A circular planar graph is recoverable if and only if it is "critical," that is, if deleting or contracting any edge (replacing its conductivity with 0 or ∞) would break some connection. What is the analogous result for non-planar graphs? To recover a piece of information with our method, we need some determinant to be zero, but become nonzero when we crossed off the unknown entry's row and column (i.e. consider the unknown entry's cofactor); this must be closely related to criticality.

5 Directions for Future Research

Our study on undoing the Schur complement raises many questions.

1. If our method succeeds, the graph is recoverable. Is the converse true? Are there any recoverable graphs for which this method fails?

2. If we have to introduce a parameter, does it matter where we introduce it? Suppose we can recover some but not all entries of the K_n, R_n, and R_n^m. We parametrize an unknown entry α, which allows us to recover one or more additional entries β. We would hope that parametrizing any β would have allowed us to recover α; that is, any entry in the set we gain is as good as any other. This does not appear to be the case. If we parametrize entries of single layers at the top of the pile (4.2.1), say R_{14} if R_{13} is completely recovered, this seems to give us more than if we parametrize near the bottom, say in K_{16}. Things propogate down better than they propogate up. Thus we suspect that we have overlooked some information available to us, just as the star–k method overlooked the multi-layers R_n^m.

One solution may be to look at things like this. Using the notation of §2.1,

$$\begin{pmatrix} \frac{-\gamma_i\gamma_j}{\sigma} & \gamma_j \\ \frac{-\gamma_i\gamma_k}{\sigma} & \gamma_k \end{pmatrix}$$

is singular. Half of this matrix comes from R and half from K; it is as though sitting in R in (2.1.1), we have peered over the ledge, down into K. Now $-\gamma_j$ is the sum of the j^{th} row of R, so more generally, we can consider sums of entries in R: the matrix

$$\begin{pmatrix} r_{ik} + r_{il} & r_{im} \\ r_{jk} + r_{jl} & r_{jm} \end{pmatrix}$$

is singular, for example, and as we have seen, we often know more about the sum than about the summands, and investigating this can be fruitful. Some things can be said about the row sums of the R_n^m as well. We do not want to use these facts piecemeal, however. What is the general phenomenon that is at work here?

3. Is any of the information we are using redundant? Can our recovery method be simplified? In particular, is it necessary to consider submatrices of the intermediate K_n or do the the R_n^m suffice? Are there any graphs that can be recovered by looking at the K_n but not by looking at the R_n^m alone?

4. Is our recovery method independent of the order of interior nodes? That is, if it succeeds for one order, does it succeed for all possible orders? This appears to be true. Some orders are nicer than others, however. It is much easier to recover 2 circles, 4 rays with the order shown in Figure 13b (due to Jeff Russell) than with the usual order (Figure 13a). Using the star–k method, the former requires fewer spurious parameters. Figure 14b is similar. Why is this? What makes one order nicer than another? How should one choose an ordering of the interior nodes to make the recovery as fast as possible?

5. During the recovery process, most of the work is done at the surface, in the single-layer R_ns, but we usually need at least one determinant buried in the thickest R_n^ms. In §4.6, this was the submatrix $1, 2, 4, 5 \times 3, 5, 7, 8$ of R_{16}^9, which is a residue matrix eight layers thick. Currently, the only way to find these is for a human to have some intuition about the graph or for the computer to search exhaustively. Is there some way to make this search smarter? How can we know in advance which deeply buried submatrices will be helpful? Where should we look for them?

6. If we have to introduce parameters, it appears that the graph is not recoverable. However, for 2-to-1 graphs, which are almost recoverable, the parameter is almost spurious—presumably some determinant somewhere forces it to take only finitely many values. Where exactly does this 2-to-1–ness come from in the R_n^m? What role does the generalized square root trick of §4.3 play? Are there any 3-to-1 graphs?

7. The questions about determinants, connections, and critical graphs from the end of the previous section.

8. Aside from the square root trick, we did not use the symmetry of our matrices. How easily can our method be applied to directed graphs? How can we understand ordinary (undirected) graphs as directed graphs with some additional constraints, just like layered networks? What about vertex conductivity networks?

9. Michael Goff makes an extensive study of networks with negative conductivities in [4]. The Schur complement still plays a key role for such networks, but they can be quite pathalogical—principal proper submatrices of Kirchhoff need not be invertible, for example. Our method relied heavily on the signs of the Kirchhoff matrices being well-behaved. Can it be modified to work with negative conductivities?

10. What does our method allow us to say about special classes of graphs, such as three-dimensional lattices?

6 Acknowledgements

The work on the star–k method was done in collaboration with Jeff Russell, whose exposition can be found in [6]. The idea of introducing parameters is due to Jeff Giansiracusa. The author would also like to thank Sam Coskey and Jim Morrow for many helpful discussions and suggestions.

References

[1] E. B. Curtis and J. A. Morrow, *Inverse problems for electrical networks*, World Scientific, (2000).

[2] D. Crabtree and E. Haynsworth, *An identity for the Schur complement of a matrix*, Proc. Amer. Math Soc. **22** (1969), 364-366.

[3] Ernie Esser, *On solving the inverse conductivity problem for annular networks*, (2000).

[4] Michael Goff, *Recovering networks with signed conductivities*, (2003).

[5] Tracy Lovejoy, *Applications of the star–k tool*, (2003).

[6] Jeff Russell, *Star and K solve the inverse problem*, (2003).