Solving Inverse Problems for Resistor Networks by Zipping Graphs

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July 2002

Abstract

In this paper, we consider critical circular planar graphs and electrical networks that are associated with those graphs. The motivation of the work comes from [1] and [2]. A special type of amalgamation of graphs by a process called "zipping" is studied where all the interior nodes and boundary nodes stay as they are after combining two graphs.

1 Introduction

In [2], a genaralization of amalgamating two networks through a number of boundary nodes is given as well as formulas for computing the new response matrix based on two original response matrices of the subnetworks. However, connecting two networks through interior nodes has not been well-studied thus far. Here, we will restrict ourselves to the case of circular planar graphs and give an attempt on solving this problem with the tool of medial graphs. The goal is to show that any two critical circular planar networks can be amalgamated in a certain way to preserve the criticality in the new network. We will make use of the following theorem from [1].

Theorem 1.1 *If Γ is a circular planar network, then the following propositions are equivalent:*

(1) Γ is critical.

(2) Γ is recoverable.

(3) No geodesic in the medial graph for Γ crosses any other twice, i.e. the medial graph is lensless.

Separating a critical circular planar network will also be studied. An algorithm for computing the new response matrix from the old ones will be given in section 4.

2 The "Zipping" of Two Networks

2.1 Zip path

In this section, we will discuss a special amalgamation of two networks. All networks that are discussed in this section are circular planar.

Definition 2.1 *A* ***zip path*** *is a path $pr_1r_2 \ldots r_nq$ where p and q are boundary nodes and all of r_i's are interior nodes which lie consecutively on the geometric boundary of a network Γ.*

We could also define zip path in a different way based on properties of medial graphs and their duals. Given a medial graph of a circular planar network, we could get its dual by applying the 2-coloring theorem.

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Figure 1: medial graph of a network and its dual

In Figure 1, the dual graph obtained by connecting the black cells is $Y-\Delta$ equivalent [1] to the original graph. Note that every white cell on the boundary of the medial graph is surrounded by several black cells. Each of these black cells corresponds to an interior node in the original network, and thus every zip path is equivalent to a boundary white cell in the medial graph of a given network. The path pp_1p_2q in Figure 1 is a zip path. It is easy to observe that the black cells that correspond to interior nodes all lie on the geometric boundary. Thus in an ordinary zipping of two networks discussed later in this section, two boundary white cells with the same number of surrounding black cells are identified with each other and the two medial graphs are combined by identifying these white cells. It is easy to observe that none of the interior nodes on a zip path is connected to a boundary spike.

2.2 The zipping theorem

Theorem 2.1 *Given two critical circular planar networks Γ_1 and Γ_2, the amalgamated network $\Gamma = \Gamma_1 \vee \Gamma_2$ by attaching the same number of interior nodes and two boundary nodes from Γ_1 and Γ_2 along a zip path is a critical circular planar network.*

Proof: We will proceed by an induction on the number of interior nodes that is attached at each step. First, note that the new network Γ is circular planar because no interior nodes are converted to any boundary node and vice versa. Thus, it allows us to prove the theorem by looking at the geodesics in the medial graph of the combined network $M(\Gamma)$. Since all geodesics from $M(\Gamma_1)$ and $M(\Gamma_2)$ form no loops or self-intersections, there will be no loops or self-intersections for the geodesics in $M(\Gamma)$. That leaves us to check only for possible lenses formed by any two geodesics in $M(\Gamma)$.

Suppose that two circular planar networks are zipped together along a zip path, then the geodesics in one medial graph of the networks which join the geodesics in the other medial graph are called *affected geodesics*. We call the geodesics that are not joined *unaffected geodesics*.

Before we construct the inductive proof, an important observation must be made. Any possible zip path from a critical circular planar graph has the following structure for the medial graph.

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Figure 2: general structure of a medial graph near a zip path

In Figure 2, the solid dots indicate boundary nodes where as the empty dots indicate interior nodes. Note that the degrees for both interior and boundary nodes can vary. However, as it will be shown in the proof that the geodesics which do not cross the zip path but only the extra edges attached to the boundary nodes will never form a lens with any other geodesic in $M(\Gamma)$. In fact, those geodesics are not affected by the amalgamation. Therefore, we only need to consider those geodesics that cross the zip path.

step - (a): Attach one boundary node p.

We first attach a single boundary node on the designated zip path. Figure 2 gives a brief image for this procedure.

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Figure 3: attach one boundary node

Assume that there are m and n boundary nodes in Γ_1 and Γ_2, respectively. Then the combined graph at this stage, Γ_a, contains $m+n-1$ boundary nodes. Therefore, $M(\Gamma_a)$ has $m+n-1$ geodesics. Only 2 geodesics that originally cross the zip path are affected. We can assign each geodesic in a medial graph a proper notation so that we have $G = G_1 \cup G_3 \cup G_2$. Let $\mathcal{S}_1(G)$ and $\mathcal{S}_2(G)$ denote the sets of geodesics in $M(\Gamma_1)$ and $M(\Gamma_2)$, respectively. Now, $G_1 \cap \mathcal{S}_2(G) = \emptyset$, since they belong to different graphs before the boundary nodes are attached. Similarly, $G_2 \cap \mathcal{S}_1(G) = \emptyset$. Thus, G does not form a lens with any geodesic in $M(\Gamma) = M(\Gamma_1 \vee \Gamma_2)$ and therefore, $M(\Gamma_a)$ is lensless. Note, if the second node on the zip path is a boundary node, then we could skip the next step of attaching interior nodes. The graph is not shown, but is similar to Figure 3.

step - (b-i): Attach the first interior node r_1, assuming there are more nodes that need to be attached.

This is the base step for the induction. Note that since two boundary-to-interior edges are zipped and become a single edge in the new graph, identifying the two interior nodes is equivalent to identifying the corresponding two intersections of the geodescis in the original medial graph. Figure 4 shows how this is done.

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Figure 4: attach one interior node

WLOG, we give labels to $G_{L_1}, G_{L_2}, G_{R_1}$, and G_{R_2} such that $G_{L_1} \cap G_{L_2} = \emptyset$ and $G_{R_1} \cap G_{R_2} = \emptyset$. These segments of geodesics are broken and joined with other segments of geodesics from the other medial graph. In Figure 4, we have $G\prime_L = G_{L_1} \cup G_2$, $G\prime_R = G_{R_1} \cup G_1$, and $G\prime = G_{L_2} \cup G_{R_2}$. The portion that connects G_1 and G_2, i.e.G_3, is gone in the new medial graph. We will look at all possible cases to show that there are no lenses in $M(\Gamma)$.

1. $G\prime_L$ and $G\prime_R$

It is clear that $G_{L_1} \cap G_2 = G_{R_2} \cap G_1 = \emptyset$. G_{L_1} intersects G_1 only at a single point a and G_{R_1} only intersects G_2 at point b. Thus, $G\prime_L$ only crosses $G\prime_R$ once, at c.

2. $G\prime_L$ and $G\prime$

$G_{L_1} \cap G_{R_2} = \emptyset$ since they belong to two different medial graphs in Figure 3a, and since $G_{L_1} \cap G_{L_2} = \emptyset$ by our assumption, thus $G\prime$ does not intersect $G\prime_L$ to the left of the zip path. $G_2 \cap G_{L_2} = \emptyset$ in Figure 3a where they are in two separate portions of the medial graph. $G_2 \cap G_{R_2} = \emptyset$ bacause $G_2 \cap (G_{R_1} \cup G_{R_2}) = \{b\} = G_2 \cap G_{R_1}$ and $G_{R_1} \cap G_{R_2} = \emptyset$. Therefore $G\prime$ does not intersect $G\prime_L$ to the right of the zip path, and thus $G\prime_L \cap G\prime = \emptyset$.

3. $G\prime_R$ and $G\prime$

The argument is almost the same as in 2. We could switch the subscripts L and R. Thus we have $G\prime_R \cap G\prime = \emptyset$.

4. $G\prime_L$ and $\mathcal{S}(G) \setminus \{G\prime_R, G\prime\}$

Note that $G\prime_L$ only intersect $G\prime_R$ on the zip path, and therefore for $G\prime_L$ to form a lens with another geodesic, not $G\prime_R$ or $G\prime$, it has to intersect some geodesic twice in either the portion to the left or the one to the right of zip path. Neither of these can happen since G_{L_1} does not cross any geodesic in $\mathcal{S}_1(G)$ twice and since G_2 never crosses any geodesic in $\mathcal{S}_2(G)$ twice. Thus, $G\prime_L$ does not from a lens with any geodesic in $M(\Gamma)$.

5. $G\prime_R$ and $\mathcal{S}(G) \setminus \{G\prime_L, G\prime\}$

Again, the agrument here is similar to the one above.

6. $G\prime$ and $\mathcal{S}(G) \setminus \{G\prime_L, G\prime_R\}$

All geodesics that $G\prime$ intersects except $G\prime_L$ and $G\prime_R$ are unaffected. $G\prime$ can not cross any geodesic in $\mathcal{S}_1(G)$ twice since G_{L_2} never does and $G\prime$ can not cross any geodesic in $\mathcal{S}_2(G)$ twice since G_{R_2} never does. Therefore, $G\prime$ does not form a lens with any geodesic in $M(\Gamma)$.

Thus far, we have shown that the $M(\Gamma)$ after an interior node is attached is still lensless.

step - (b-ii): Suppose that k interior nodes have been attached, we need to attach the $(k+1)$st interior node, assuming there is at least one more node that will be attached.

This is the inductive step. Figure 5 shows the changes that are made when we identify one more interior node, or in other words, zip together one more interior edge.

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Figure 5: attach more interior nodes

Basically, the proof here will be similar to the one in the base step of the induction. It may be useful to point out several important observations. Throughout step (b), the number of boundary nodes in Γ stays the same, and so does the number of geodesics in $M(\Gamma)$. Also, none of the geodesics that were affected in a previous stage is ever affected again, so we are constantly moving the same struture of geodesics down by an edge as we zip one more interior edge. By looking at all 6 cases, it can be easily shown that at each stage, the new medial graph does not contain any lens.

step - (c): Attach the second boundary node, finishing the zipping process.

For this step, we only need to check for three cases.

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Figure 6: attach the second boundary node

Refer to Figure 6, we have $G\prime_L = G_L \cup G_2$ and $G\prime_R = G_R \cup G_1$. In the new medial graph, there are only two geodesics bacause the number of boundary nodes is decreased by 1. We first check $G\prime_L$ and $G\prime_R$. They will only cross at a single point. The argument is similar to case 1 in step - (b-i). Next, we check $G\prime_L$ and $\mathcal{S}(G) \setminus \{G\prime_R\}$. Again, this is similar to case 4 in the base step of the induction. With $G\prime$ gone, this case will only be easier to check. And similarly, $G\prime_R$ does not form a lens with any geodesic in the set $\mathcal{S}(G) \setminus \{G\prime_L\}$.

Therefore, $M(\Gamma)$ is lensless after the zipping. And equivalently, Γ is critical. $\square$

A useful application of the above theorem is the **doubling** of a circular planar network.

Definition 2.2 *Suppose we have a circular planar network Γ which contains a potential zip path, then to* ***double*** *Γ is to zip with Γ its mirror image along that zip path. We call this process* ***doubling*** *of a network.*

Corollary 2.1 *Any network obtained by* ***doubling*** *a critical circular planar network is still a critical circular planar network, and thus recoverable.*

An example is shown in Figure 7.

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Figure 7: doubling a graph

It is also worthwhile to mention that in each stage of the proof for the previous theorem, the combined medial graph contains no lenses. Thus, we can generalize our theorem and introduce the notation of **partial zipping**.

Definition 2.3 *Suppose we have two circular planar networks which contain potential zip paths, a **partial zipping** is a zipping of those networks such that starting with attaching one boundary node from both networks on a zip path, we could zip any number of consecutive interior nodes along that path and without ever attaching the second boundary nodes.*

Here is an example of partial zipping. (see Figure 8)

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Figure 8: partial zipping

Theorem 2.2 *Any network obtained by a partial zipping of two critical planar networks is still a critical planar network, and thus recoverable.*

Proof: We only need step (a) and step (b) in the proof of Theorem 4.1. □

2.3 A special zipping

A *special* type of zipping is also studied where we do not necessarily zip the adjacent interior nodes on a zip path. This allows us to skip several interior nodes on a zip path of a network and continue the zipping or partial zipping process on the next desirable node. It is clear that given two circular planar networks that may have different numbers of interior nodes on their paths, a zipping or partial zipping yields a new circular planar network. An example is shown in Figure 9.

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Figure 9: special zipping

Conjecture 2.1 *A special zipping of two critical circular planar networks based on the above method produces a new critical circular planar network.*

When a single edge is merged with multiple edges that contain interior nodes on them, it seems that the single edge does not affect anything in the combined network and the geodesics that go through that same edge make no contribution in the medial graph of the new network.

It is hard to tell what could happen to the conductances on those edges that are attached when a special zipping is done, although the new network may be recoverable. However, many problems involving circular planar networks could become easier to solve when the conductances of this kind of zipping are well-defined.

3 Some Notes on the Inverse Procedure

3.1 Symmetric unzipping

It seems natural to ask questions about the possibility of unzipping a critical circular planar network after what has been done in the previous section. The goal is to get two critical circular planar subnetwork when we unzip along a boundary-to-boundary path in a critical circular planar network. It is easy to show that the first step of such unzipping satisfies all the conditions.

Theorem 3.1 *Given any critical circular planar network, unzipping any boundary-to-interior edge so that an extra edge is inserted to connect that interior node and a new boundary node yields a new critical circular planar network.*

Proof: Starting with a critical circular planar network Γ, we will proceed as is shown in Figure 10. It is clear that the new network with one more boundary node is still a circular planar network. Since all geodesics except the ones that cross pq are unaffected, we need only investigate the changes made on pq, instead of all edges that are connected these nodes.

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Figure 10: unzip a single edge

Since the edge in Figure 10a will be broken, we could define segments of geodesics $G_{L_1}, G_{L_2}, G_{R_1}, G_{R_2}$ so that $G_{L_1} \cap G_{L_2} = \emptyset$ and $G_{R_1} \cap G_{R_2} = \emptyset$. We also define temporarily the left and right hand side of the network and denote them Γ_L and Γ_R, where the the boundary-to-interior edge pq belongs to both subnetworks. And as usual, let $\mathcal{S}_L(G), \mathcal{S}_R(G)$ and $\mathcal{S}(G)$ denote the set of all geodesics in $M(\Gamma_L), M(\Gamma_R)$, and $M(\Gamma)$, respectively. Thus, in Figure 10b, we have $G\prime = G_{L_1} \vee G * \vee G_{R_1}$, where $G*$ is the new segment in the middle of $G\prime$. Also, G_{R_2} and G_{L_2} extend to $G\prime_L$ and $G\prime_R$, respectively. The increase in the number of geodesics is consistent to the increase in the number of boundary nodes. Since the medial graph of original network does not contain any loop or self-intersection of geodesics, the new medial graph does not have any of those, either. However, we need to check for several cases where geodesics may possibly form a lens.

1. $G\prime_L$ and $G\prime$

G_{R_2} does not cross G_{L_1} twice in $M(\Gamma)$, so $G\prime_L$ and $G\prime$ do not cross twice in $M(\Gamma_L)$. They do not cross in $M(\Gamma_R)$ since otherwise G_{R_2} is connected to another geodesic in $M(\Gamma)$ that crosses the zip path. This cannot happen because a lens is form that way and it contradicts that $M(\Gamma)$ is lensless. Therefore, no lens is fomred by $G\prime_L$ and $G\prime$.

2. $G\prime_R$ and $G\prime$

The argument is similar to the one above.

3. $G\prime_L$ and $\mathcal{S}(G) \setminus G\prime$

$G\prime_L$ does not cross any geodesic twice in $M(\Gamma_L)$ since G_{R_2} does not intersect any geodesic in $\mathcal{S}_L(G)$ for more than once. $G\prime_L$ does not cross any geodesic in $M(\Gamma_R)$ since G_{R_2} is not connected to any other geodesic in $\mathcal{S}(G)$ that crosses pq.

4. $G\prime$ and $\mathcal{S}(G) \setminus \{G\prime_L, G\prime_R\}$

There are no lenses in the left half of the network simply because G_{L_1} does not cross any geodesic twice other than the geodesic which contains segments G_{R_1} and G_{R_2} in $\mathcal{S}(G)$. And similarly, there are no lenses in the right half of the network, either.

5. $G\prime_R$ and $\mathcal{S}(G) \setminus G\prime$

The argument is obtained by replacing G_{R_2} with G_{L_2} in case 3.

Hence, the partially unzipped network is still lensless. Since it is still circular planar, the network is critical. □

Note that there is an alternative proof for this theorem in the case of unzipping a boundary spike.

The theorem 3.1 seems to give us a start on the process of unzipping a critical circular planar network, but unfortunately, not all networks of this type can be unzipped completely into two smaller circular planar networks and preserve the criticality. An example is shown in Figure 11, where we have a critical network to begin with and we will obtain two noncritical networks.

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Figure 11: unzip a critical network to obtain two noncritical networks

Note that the network in Figure 11 is symmetric. Thus it appears that we need more restrictions in order to be able to unzip a circular planar network and obtain in the desirable way. We introduce the notation of **symmetric unzipping** of a network.

Definition 3.1 *A **symmetric unzipping** is to unzip a symmetric circular planar network along its axis of symmetry so that at each step, the network remains symmetric.*

It appears that the problem with the network in Figure 11 is that it is symmetric in the beginning, but not symmetric during the process of unzipping. It is easy to observe that a symmetric unzipping of two circular planar network yields two identical circular planar networks. And at each step, the partially unzipped network is symmetric. The symmetric unzipping is the inverse procedure of doubling a network. However, even with the help from this assumption, there are still counterexamples where the subnetworks after unzipping are not critical. We give one of them here.

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Figure 12: symmetric unzipping, a counterexample for a network and its medial graph

In Figure 12, neither one of the two subnetworks is critical. In fact, the network fails to remain critical when the first boundary-to-interior edge is separated because of the re-entrant geodesics in its medial graph. Thus, unzipping a critical network is in general a harder problem than zipping. The fact that we lack the knowledge of interior nodes prevents us from obtaining more useful results.

3.2 Unzip well-connected networks

A circular planar network is called well-connected if for every circular pair $(P, Q) = (p_1, ..., p_k; q_1, ..., q_k)$ of sequences of boundary nodes, there is a k-connection from P to Q in G. Any network that is well-connected with n boundary nodes contains $\begin{pmatrix} n \\ 2 \end{pmatrix}$ edges. This is also the maximum number of edges for a planar network to be critical. For a given well-connected network, we have found that it cannot be unzipped in a certain way to yield two critical subnetworks.

Theorem 3.2 *Given a well-connected network that contains an even number of boundary nodes with more than one edge and suppose it is unzipped in a fashion so that both subnetworks have the same number of boundary nodes, then neither of the subnetworks is critical.*

Note first that two networks generated by unzipping a well-connected network are not well-connected except for a *trivial unzipping*, where only a boundary-to-boundary edge is separated.

Proof: Suppose a well-connected network Γ contains n boundary nodes and more than one edge. Let $n = 2k$, where $k \in \mathbb{Z}^+$. Indeed, $k > 1$. Suppose we could unzip Γ along a path pq that contains x edges, resulting in two subnetworks Γ_1 and Γ_2, that both have pq as part of their geometric boundaries. WLOG, suppose Γ_1 and Γ_2 contain a and b edges, respectively, besides the x edges on pq. Thus, in Γ, we have $a + b + x$ edges.

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Figure 13: unzip a well-connected network

In Figure 13, after the unzipping, both Γ_1 and Γ_2 contain $k+1$ boundary nodes. Since the maximum number of edges for either subnetwork to be critical is $\left(\begin{array}{c} k+1 \\ 2 \end{array} \right)$, we have the following inequalities:

$$a + x \leq k(k+1)/2 \tag{1}$$

$$b + x \leq k(k+1)/2 \tag{2}$$

And we also have:

$$a + b + x = \left(\begin{array}{c} 2k \\ 2 \end{array} \right) = k(2k-1) \tag{3}$$

The above relations simplify to:

$$x \leq -(k-1)^2 + 1 \tag{4}$$

Since both x and k are positive integers, i.e. $x > 0$, the only solution to (4) is $x = k = 1$. Note that this is the case where a single edge connecting two boundary nodes is separated into two identical pieces. Since $k > 1$ by our assumption, (4) has no solution. Hence, this completes the proof. □

The above result could be generalized.

4 The Inverse Problem

4.1 For ordinary zipping

Given a response map Λ, the *inverse problem* is to find the conductances of each edge in G. If the response matrices for two critical circular planar networks are given, we would like to recover the conductances for the zipped network using the given matrices. For ordinary zipping, an algorithm is devised to compute the new response matrix.

4.1.1 attach a boundary-to-interior edge

Suppose Λ_1 and Λ_2 are two response matrices associated with two critical circular planar networks Γ_1 and Γ_2, respectively. We need first compute the new response matrix when two boundary-to-interior edges are attached. In the case of attaching two boundary spikes, the following algorithm gives us the new response matrix:

1. Compute the conductances for two boundary spikes.

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Figure 14: attach two boundary spikes

In [1], a *boundary spike formula* is given to compute the conductance of a boundary spike from a given response matrix. Thus, we could use it to compute the conductances for ξ_1 and ξ_2. (see Figure 14a)

2. Compute the response matrices $\widetilde{\Lambda}_1$ and $\widetilde{\Lambda}_2$.

When we compute the conductances for ξ_1 and ξ_2, we contract the corresponding two boundary spikes so that the new networks have one fewer edge and interior node. This gives two new networks Γ_1' and Γ_2'. We could use another formula in [1] to get two new response matrices at this step, namely, $\widetilde{\Lambda}_1$ and $\widetilde{\Lambda}_2$. It is also possible to obtain $\widetilde{\Lambda}_1$ by amalgamating a boundary-to-boundary edge with conductances $-\xi_1$ to Γ_1 and use the formula in step 3 to calculate the response matrix for the new network. Applying the same method to Γ_2 gives $\widetilde{\Lambda}_2$.

3. Amalgamate Γ_1' and Γ_2', and compute the response matrix for the new network.

We will use the formula in [2] for network amlgamation. In this step, Γ_1' and Γ_2' are attached at a single boundary node p without it being internalized. (see Figure 14b)

Suppose we have:

$$\widetilde{\Lambda}_1 = \begin{pmatrix} A_1 & B_1 \\ B_1^T & C_1 \end{pmatrix} \qquad \text{and} \qquad \widetilde{\Lambda}_2 = \begin{pmatrix} A_2 & B_2 \\ B_2^T & C_2 \end{pmatrix}.$$

Then the response matrix for the amalgamated network $\Gamma *$ is given by:

$$\begin{aligned} \Lambda^* &= \Lambda^*/(C_1 + C_2) \\ &= \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} - \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} (C_1 + C_2)^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \end{aligned}$$

4. Adjoin a boundary spike.

This is the last step for this algorithm. We need to insert the boundary spike back into the network. It seems natural to sum the conductances ξ_1 and ξ_2 and make it the conductance for the combined boundary spike in the new network. Using the formula for adjoining a boundary spike with a given conductance in [1], we could eventually get the response matrix for the network after attaching two boundary spikes. (see Figure 14c)

The above algorithm also works when the boundary nodes that we are attaching have degrees higher than one. We need only to compute the conductances for all the edges that are connected to that boundary node as well as the new response matrices after contracting or deleting those edges. The formulas for deleting and adjoining a boundary-to-boundary edge which are given in [1] may be needed in step 1, 2 and 4. The formula for the amalgamation remains the same and we will obtain the new response matrix after we add back all the edges that are previously computed and deleted.

4.1.2 attach interior edges

It appears that when two interior-to-interior edges are combined, the computation gets more complicated. We will present a formula to compute the new response matrix in this case. However, we would have to go back to the Kirchhoff matrices of the old network to retrieve more information before carrying out this procedure.

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Figure 15: attach two interior edges

Suppose we have the same labelling as in Figure 15, we could partition the blocks in the Kirchhoff matrix to be:

$$K = \begin{pmatrix} A & B & C_1 & C_2 \\ B^T & D & E_1 & E_2 \\ C_1^T & E_1^T & f & 0 \\ C_2^T & E_2^T & 0 & g \end{pmatrix} \tag{5}$$

The response matrix of network in Figure 15 is the Schur complement of K, which could be computed using the previous algorithm. It is given by:

$$\Lambda = A - \begin{pmatrix} B & C_1 & C_2 \end{pmatrix} \begin{pmatrix} D & E_1 & E_2 \\ E_1^T & f & 0 \\ E_2^T & 0 & g \end{pmatrix}^{-1} \begin{pmatrix} B^T \\ C_1^T \\ C_2^T \end{pmatrix}$$

When two edges pq_1 and pq_2 are attached, we simply sum the columns and rows in K and thus obtain the new Kirchhoff matrix.

$$K^\star = \begin{pmatrix} A & B & C_1 + C_2 \\ B^T & D & E_1 + E_2 \\ C_1^T + C_2^T & E_1^T + E_2^T & f + g \end{pmatrix} \tag{6}$$

Again, by taking the Schur complement of the new Kirchhoff matrix, we will get the new response matrix:

$$\Lambda^\star = A - \begin{pmatrix} B & C_1 + C_2 \end{pmatrix} \begin{pmatrix} D & E_1 + E_2 \\ E_1^T + E_2^T & f + g \end{pmatrix}^{-1} \begin{pmatrix} B^T \\ C_1^T + C_2^T \end{pmatrix}$$

Both Λ and $\Lambda^\star$ are symmetric, and so is the difference between them.

The difference d_Λ remains the same based on a given Λ at each step when one more interior edge is zipped with another. Thus, we could obtain the response matrix for zipping all except the last boundary-to-interior edge by continuously applying this formula. However, we have to take steps almost back to the original Kirchhoff matrix to compute the new response matrix. In cases of large networks, retrieving information from Kirchhoff matrix in this way could be time-consuming.

4.1.3 attach the last edge

The above calculation leaves us only one more step – to close up the zip.

This is the most trivial step, where we are only required to adjoin two boundary-to-interior edges in a network. The new response matrix can be simply obtained by taking the row and column sums of the corresponding entries.

Therefore, this completes the algorithm for computing the new response matrix for a zipped network with given response matrices for the subnetworks.

References

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