The Dirichlet Problem for Infinite Networks

Nitin Saksena

Summer 2002

Abstract

This paper concerns the existence and uniqueness of solutions to the Dirichlet problem for infinite networks. We formulate a Dirichlet problem for infinite networks. Given that a solution to this problem exists, uniqueness is then shown. Existence cannot be proven for a general infinite network, since it depends on conditions imposed on the conductances. Consequently, it is pursued only for specific networks—for a line, a fishbone, and a ladder.

1 Introduction[1]

A *graph with boundary* is a triple $G = (V, \partial V, E)$, where V is the set of nodes and E is the set of edges for a finite graph, and ∂V is a nonempty subset of V called the set of *boundary* nodes. The set $int\, G = V - \partial V$ is called the set of *interior* nodes.

A *resistor network* $\Gamma = (G, \gamma)$ is a graph G together with a conductivity function γ that assigns to each edge e in G a positive real number $\gamma(e)$, called the *conductance* of the edge e.

A function v defined on the nodes of G is said to be γ-*harmonic* at node p if the sum of the currents from p to the neighboring nodes is 0. That is,

$$\sum_{q\sim p} \gamma(p, q)\,[v(p) - v(q)] = 0. \tag{1}$$

A *Dirichlet problem* for a *finite* resistor network can be stated as follows: Assume that the graph G and the conductance $\gamma(pq)$ of each edge pq in G are known. If a voltage φ is imposed at the boundary nodes, is there a unique function v defined throughout the network such that v is γ-harmonic on $int\, G$ and $v(p) = \varphi(p)$ for all $p \in \partial V$?

[1]Definitions are taken from [CM-2].

It is well-known that a unique solution exists for this problem (see [CM-1] for example). The goal of this paper is to formulate an analogous Dirichlet problem for infinite resistor networks and examine the associated notions of existence and uniqueness of solutions.

2 The Dirichlet Problem

2.1 Formulation

For an infinite network $\Gamma = (G, \gamma)$, we modify the above graph G so that $|V| = \infty$ and $|E| = \infty$. Unless otherwise stated, we assume the set ∂V of boundary nodes is finite. This set ∂V includes a boundary node p_∞ at infinity. If we think of the infinite network as the stereographic projection of the 2-sphere onto the plane, then the boundary node at infinity is the North Pole of the sphere. We also assume that the graph is locally finite, that is, every node has finite valence.

For some node $p_0 \in int\, G$ and some $q \in \partial V$, let $p_0 p_1 p_2 \cdots q$ denote a path connecting the two; there are infinitely many paths to consider. Let $\ell(p_0 q) \in \mathbb{R}$ denote the number of edges contained in such a path, and let

$$dist(p_0, q) = \min \{\ell(p_0 q)\} . \tag{2}$$

For a function v defined on nodes V, we define the following notion of a limit:

Definition 2.1 *For a path $p_0 p_1 p_2 \cdots p_k$ between $p_0 \in int\, G$ and $p_k \in \partial V$ (where p_k could be p_∞), a sequence $v(p_i)$ is said to converge to a limit ℓ if for every $\epsilon > 0$, there is an integer N such that $|v(p_i) - \ell| < \epsilon$ whenever $dist(p_0, p_i) \geq N$. In this case we write $\lim_{i \to k} v(p_i) = \ell$.*

We now state the Dirichlet problem for infinite networks. If a voltage φ is imposed at the boundary nodes ∂V, is there a unique function v defined throughout the network such that $v(p)$ is γ-harmonic on $int\, G$ and $v(p) = \varphi(p)$ for all $p \in \partial V$ (where $v(p_\infty) = \varphi(p_\infty)$ means $\lim_{i \to \infty} v(p_i) = \varphi(p_\infty)$ in the above sense)?

2.2 Uniqueness

Given that a solution exists to the above Dirichlet problem, uniqueness is straightforward to show. We will make use of the following result:

Theorem 2.2 (Finite Maximum Principle) *Suppose v is a γ-harmonic function on a (finite) resistor network Γ with boundary. Then the maximum and minimum values of v occur on the boundary of Γ.*

Proof. The proof is contained in [CM-1]. $\square$
The uniqueness result follows.

Theorem 2.3 *Suppose a solution $v\,(p)$ exists for the Dirichlet problem on an infinite network $\Gamma = (G, \gamma)$. Then v is unique.*

Proof. The difference $w \equiv v_1 - v_2$ of two solutions is γ-harmonic on $int\,G$ and satisfies $w = 0$ on ∂V. Let p_0 be an arbitrary point in G and q an arbitrary point in ∂V. Given $\epsilon > 0$, N can be chosen such that $|w(p_0) - 0| < \epsilon$ for $dist(p_0, q) \geq N$. Consider a finite subgraph $G_\epsilon = (V_\epsilon, \partial V_\epsilon, E_\epsilon)$, where $V_\epsilon = \{p : dist(p, q) \leq N\}$, $\partial V_\epsilon = \{p : dist(p, q) = N\}$, and E_ϵ is the appropriate subset of edges E. Since $w|_{\partial V_\epsilon} = |w(p_0)| < \epsilon$, the Finite Maximum Principle implies $|w(p)| < \epsilon$ on G_ϵ. Thus $|w(p)| < \epsilon$ on the entire graph G. But since ϵ can be made arbitrarily small, $w(p) \equiv 0$ and thus $v_1 = v_2$ on Γ. $\square$

2.3 Existence

The question of existence is a rather more difficult problem. In the above proof of uniqueness, nothing was specified about the conductances γ, but without certain conditions on the conductances, it is hopeless to show existence in general. As evidence of this, consider the following simple example:

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Figure 1: Infinite Line Network.

Example 2.4 *Consider the infinite line network depicted above in figure 1, with nodes $\{p_j\}_{j=0}^{\infty}$ and conductances $\{\gamma_i\}_{i=1}^{\infty}$. Let the set of boundary nodes be given by $\{p_0, p_\infty\}$, where p_∞ represents the boundary node at infinity. At the boundary nodes, we impose the voltages $\varphi\,(p_0) = 1$ and $\varphi\,(p_\infty) = 0$. Is there a solution to this Dirichlet problem if no conditions are imposed on the γ_i?*

Solution Let voltages v_n be defined on the nodes of the line such that $v_n = v(p_n)$. The current I between nodes p_0 and p_1 satisfies Ohm's Law:

$$I = (v_0 - v_1)\,\gamma_1, \tag{3}$$

and since the function v must be γ-harmonic, it follows that

$$I = (v_0 - v_1)\,\gamma_1 = (v_1 - v_2)\,\gamma_1 = \cdots = (v_{n-1} - v_n)\,\gamma_1. \tag{4}$$

This establishes the recursion

$$v_n = v_{n-1} - \frac{I}{\gamma_n}, \text{ with } v_0 = 1. \tag{5}$$

Expanding this yields

$$\begin{aligned} v_1 &= v_0 - \frac{I}{\gamma_1} = 1 - \frac{I}{\gamma_1} \\ v_2 &= v_1 - \frac{I}{\gamma_2} = 1 - \frac{I}{\gamma_1} - \frac{I}{\gamma_2} \\ &\vdots \\ v_n &= 1 - I\sum_{i=1}^{n}\frac{1}{\gamma_i} \end{aligned} \tag{6}$$

Thus as $n \to \infty$, $\lim_{n\to\infty} v_n$ does not even exist (much less approach $\varphi(p_\infty) = 0$) unless the sum $\sum_{i=1}^{\infty}\frac{1}{\gamma_i}$ converges. For example, the conditions $\gamma_i = 1$ for all i, or $\gamma_i = i$ for all i, would cause the sequence $\{v_n\}_{n=0}^{\infty}$ to diverge, implying no solution to the Dirichlet problem. □

The best we can do then is establish convergence criteria for specific infinite networks. We begin by revisiting the line.

3 Infinite Line Network

3.1 Effective Conductances

An alternative method for establishing the existence condition for the solution of a Dirichlet problem on the infinite line network is to use effective conductances. We present this method here because it will be an extremely useful tool in more complicated networks.

Assume the same formulation of the Dirichlet problem as in the above example. Consider the subset of the infinite line network given by the first $n+1$ nodes $\{p_0, \ldots, p_n\}$. Between nodes p_0 and p_n, we seek the single conductance γ_{0n} that is equivalent to the n-conductance series combination. By equivalent, we mean that γ_{0n} can replace the combination without changing the current through the combination or the potential difference between nodes p_0 and p_n. Since conductances add in series as resistors do in parallel, we obtain

$$\gamma_{0n} = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{\gamma_i}}. \tag{7}$$

Therefore the infinite line network depicted in figure 1 is equivalent to the following (figure 2):

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Figure 2: Effective Conductance for the Infinite Line Network.

The current I between nodes p_0 and p_n satisfies Ohm's Law:

$$I = (v_0 - v_n)\gamma_{0n} = (1 - v_n)\,\gamma_{0n} \tag{8}$$

or

$$v_n = 1 - \frac{I}{\gamma_{0n}}, \tag{9}$$

which gives

$$v_n = 1 - I\sum_{i=1}^{n} \frac{1}{\gamma_i}. \tag{10}$$

As we let $n \to \infty$, we again arrive at the existence condition $\displaystyle\sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty$.

4 Fishbone Network

4.1 Description

A *fishbone network* is obtained by taking the infinite line network and adding two boundary spikes on opposite sides of each interior node, as depicted in the figure 3 below. The resulting network has an infinite number of boundary nodes.

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Figure 3: Infinite Fishbone Network.

4.2 Existence Condition

Recall that the *Dirichlet-to-Neumann map* (or *response matrix*) Λ maps boundary value potentials $\varphi(p), p \in \partial V$ to the current function $I_{\varphi}(p), p \in \partial V$, which is determined by the solution to the Dirichlet problem with boundary values φ. Thus finding a condition for the existence of a solution to the Dirichlet problem is equivalent to writing down the response matrix Λ.

Recall that $\{\Lambda\}_{ij} = \lambda_{ij}$ is interpreted as the current at node i due to a potential of 1 at boundary node j and 0 at all other boundary nodes. The graph of the fishbone network is a tree and so there is at most one path connecting any two nodes. In particular, there is only one path β_{ij} between boundary nodes i and j. β_{ij} will resemble one of the three paths indicated in figure 4. Equivalently, β_{ij} can be viewed as a line with the same

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Figure 4: Three Types of Paths on Infinite Fishbone Network.

conductances in series and with boundary nodes i and j (figure 5). Thus, using Ohm's

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Figure 5: Paths Equivalent to Lines.

Law, and the expression for effective conductance on a line from §3.1, we obtain

$$\lambda_{ij} = (v_i - v_j)\,\gamma_{ij} = -\frac{1}{\displaystyle\sum_{\gamma \in \beta_{ij}} \frac{1}{\gamma}}. \tag{11}$$

Thus in order to write down the response matrix, we require that

$$\frac{1}{\sum_{\gamma \in \beta_{ij}} \frac{1}{\gamma}} < \infty \tag{12}$$

for each path β_{ij} between arbitrary boundary nodes i and j.

4.3 Inverse Problem

If the response map Λ is given, but the conductivity function γ is unknown, the inverse problem is to use Λ to calculate the conductance of each edge in G. The fishbone network, unlike other infinite networks we consider, has an infinite number of boundary nodes, which bodes well for finding a solution to this inverse problem. Here, we present an inductive procedure for doing so, recovering conductances from left to right.

Suppose we set a potential of 1 at node 1 and potential 0 at all other boundary nodes (figure 6). Then the current out of node 2 is given by λ_{21}, which satisfies

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Figure 6: Voltage Pattern to Recover γ_{12}.

$$\lambda_{21} = (v_2 - v_1)\,\gamma_{12} = \gamma_{12}. \tag{13}$$

In similar fashion, the conductance γ_{13} can also be recovered.

Suppose we now delete the boundary pendant 12. This is equivalent to adjoining a boundary pendant at node 1, with conductance $-\gamma_{12}$, without internalizing node 1. The effect of this on the response matrix Λ is documented in [CM-2] (pages 104-105) and is not reproduced here. Hence, if G' denotes the graph with pendant 12 deleted, the response matrix Λ' for G' can be expressed in terms of Λ. Similarly, we can then form the graph G'' with the pendant 13 deleted and obtain the response matrix Λ'' for G''.

Since edge 14 has now become a boundary spike, we can recover the conductance γ_{14}. To do so, we can impose a voltage of 1 at node 1 and voltage 0, as well as current 0, at node 6. These conditions imply voltage 0 at node 4 and some voltage α at node 5 (figure 7). Then the current at node 6 must satisfy

$$\lambda_{61} + \alpha\lambda_{65} = 0, \tag{14}$$

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Figure 7: Voltage and Current Pattern to Recover γ_{14}.

which implies $\alpha = -\lambda_{61}/\lambda_{65}$. Note that $\lambda_{65} \neq 0$, since there is a connection between nodes 5 and 6. Now let x denote the current at node 1. Then x must satisfy

$$\begin{aligned} x &= \lambda_{11} + \alpha\lambda_{15} = \lambda_{11} - \frac{\lambda_{61}\lambda_{15}}{\lambda_{65}} \\ &= \frac{\lambda_{11}\lambda_{65} - \lambda_{61}\lambda_{15}}{\lambda_{65}} = \gamma_{14}(1-0) = \gamma_{14}, \end{aligned} \tag{15}$$

and thus γ_{14} has been determined.

Finally, suppose we contract the boundary spike 14. This is equivalent to adjoining a boundary spike at node 4, with conductance $-\gamma_{14}$. The effect of this on the response matrix Λ'' is documented in [CM-2] (pages 106-107) and is not reproduced here. Hence, if G''' denotes the graph with spike 14 deleted, then we can obtain a response matrix Λ''' for G'''.

At this point, the remaining fishbone network resembles the original, modulo the edges $12, 13, 14$ (figure 8). By induction then, the above algorithm results in the recovery

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Figure 8: Fishbone Network after First Application of Algorithm.

of all conductances on the fishbone network.

5 Infinite Ladder Network

5.1 Description of the Network

We consider a ladder network that extends infinitely in the eastward direction (figure 9). The network is considered to have four boundary nodes, two on the West end, and *two* at infinity. The vertical conductances are designated $\{d_i\}$, those on the top are designated $\{b_i\}$, and those on the bottom are designated $\{c_i\}$, where $i \geq 1$. Analogous to the fishbone network, we seek conditions on the b_i, c_i, d_i that allow us to define a response matrix for the infinite ladder.

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Figure 9: Infinite Ladder Network.

5.2 Inverse Response Submatrices

Let V denote a vector of boundary potentials, I a vector of boundary currents, and Λ a response matrix. Since Λ maps from boundary voltages to boundary currents by definition, we can write $\Lambda V = I$. Since the row sums of the response matrix are zero, any constant vector V lies in the null space of Λ. Thus Λ is not invertible.

Consider, however, the matrix $\tilde{\Lambda}$, obtained from Λ by deleting the n^{th} row and n^{th} column. Since $\tilde{\Lambda}$ is a proper submatrix of the positive semi-definite matrix Λ, $\tilde{\Lambda}$ is positive definite and hence invertible. And since Λ has row and column sums equal to zero, Λ is completely determined by the $(n-1)\times(n-1)$ submatrix $\tilde{\Lambda}$. In order to have a legitimate Dirichlet problem, we can specify either potential or current at every boundary node. Let us prescribe values for $i_1,\ldots,i_{n-1}$ and set $v_n = 0$. Define the vectors $\tilde{V} = (v_1,\ldots,v_{n-1})^T$, which contains the unknown potentials, and $\tilde{I} = (i_1,\ldots,i_{n-1})^T$. Then we can write $\tilde{\Lambda}\tilde{V} = \tilde{I}$, which implies $\tilde{V} = \tilde{\Lambda}^{-1}\tilde{I}$. For the infinite ladder (where $n = 4$), we can specify the boundary conditions as in figure 10.

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Figure 10: Boundary Conditions at n^{th} Stage.

Our goal is to find $\tilde{\Lambda}_n^{-1}$ at every stage (i.e. after having added a new rung to the ladder) and obtain a recursion formula relating $\tilde{\Lambda}_{n+1}^{-1}$ to $\tilde{\Lambda}_n^{-1}$. We would then like to establish convergence of $\tilde{\Lambda}_n^{-1}$ to some limit $\tilde{\Lambda}^{-1}$ and show that this matrix is invertible. From this we can obtain $\tilde{\Lambda}$ and thus the limiting response matrix Λ.

Let us assume that we have $\tilde{\Lambda}_n^{-1}$ (satisfying $\tilde{V} = \tilde{\Lambda}_n^{-1}\tilde{I}$) at the n^{th} stage. Suppose we add a boundary spike at node 4; we want to determine how $\tilde{\Lambda}_n^{-1}$ is affected by this first update step.

Claim 5.1 *Let $\tilde{\Lambda}_{n+1/3}^{-1}$ denote the inverse response submatrix after the first update step. Then $\tilde{\Lambda}_{n+1/3}^{-1}$ is related to $\tilde{\Lambda}_n^{-1}$ by*

$$\tilde{\Lambda}_{n+1/3}^{-1} = \tilde{\Lambda}_n^{-1} + \frac{1}{c_n}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{16}$$

Proof. At the n^{th} stage, we specify the boundary conditions as in figure 10: current 1 at node 1, current 0 at nodes 2 and 3, and voltage 0 at node 4. Adjoin a boundary spike with conductance c_n to node 4. The node that was previously node 4 is now an interior node, and the endpoint of the boundary spike is now the new boundary node 4 (figure 11). Before adjoining the spike, current could only exit the network at old boundary node 4, which had potential 0 (figure 10). The same must be true now for the new boundary node 4, implying that the potential at 4 is some negative voltage v_4 (figure 11). Applying Ohm's Law to the boundary spike, we have

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Figure 11: Voltage at New Boundary Node 4.

$$c_n\,(0 - v_4) = -1, \tag{17}$$

or

$$v_4 = -\frac{1}{c_n}. \tag{18}$$

We want the same boundary conditions after the first update step as before. To achieve potential 0 at node 4, while maintaining current 1 at node 1 as well as current 0 at nodes 2 and 3, we subtract v_4 from each boundary node. This identical potential change at each boundary node does not change potential differences between boundary nodes, and thus does not change current flow out of the network. Therefore, the inverse response submatrix $\tilde{\Lambda}^{-1}_{n+1/3}$ corresponding to the network after the first update step must satisfy

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - \begin{pmatrix} v_4 \\ v_4 \\ v_4 \end{pmatrix} = \tilde{\Lambda}^{-1}_{n+1/3} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}, \tag{19}$$

where $i_1 = 1, i_2 = 0, i_3 = 0$. From this, it is straightforward to verify the claim:

$$\begin{aligned}
\tilde{\Lambda}^{-1}_{n+1/3} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} &= \left(\tilde{\Lambda}^{-1}_n + \frac{1}{c_n} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \right) \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} \\
&= \tilde{\Lambda}^{-1}_n \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} + \frac{1}{c_n} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} - \begin{pmatrix} v_4 \\ v_4 \\ v_4 \end{pmatrix}. \qquad (20)
\end{aligned}$$

□

We can now invert $\tilde{\Lambda}^{-1}_{n+1/3}$ to get $\tilde{\Lambda}_{n+1/3}$, and fill in the missing row and column to obtain $\Lambda_{n+1/3}$. Suppose we add a boundary spike at node 3 now; we want to determine how $\tilde{\Lambda}^{-1}_{n+1/3}$ is affected by this second step. A difficulty now arises in that we want to impose the following boundary conditions at nodes $1, 2, 3, 4$: current 1, current 0, voltage 0, and current 0, respectively. Since node 3 now has zero potential, $\tilde{\Lambda}$ is not the appropriate tool to use. Rather, we are interested in the inverse response submatrix formed by removing the third row and third column of Λ. We will denote this submatrix by $\tilde{\tilde{\Lambda}}$.

Thus, after adding a boundary spike at node 3, we want to determine how $\tilde{\tilde{\Lambda}}^{-1}_{n+1/3}$ is affected.

Claim 5.2 *Let $\tilde{\tilde{\Lambda}}^{-1}_{n+2/3}$ denote the inverse response submatrix after the second update step. Then $\tilde{\tilde{\Lambda}}^{-1}_{n+2/3}$ is related to $\tilde{\tilde{\Lambda}}^{-1}_{n+1/3}$ by*

$$\tilde{\tilde{\Lambda}}^{-1}_{n+2/3} = \tilde{\tilde{\Lambda}}^{-1}_{n+1/3} + \frac{1}{b_n} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \qquad (21)$$

Proof. The proof is analagous to that for the above claim. □

We can now invert $\tilde{\tilde{\Lambda}}^{-1}_{n+2/3}$ to get $\tilde{\tilde{\Lambda}}_{n+2/3}$ and fill in the missing row and column to obtain $\Lambda_{n+2/3}$. The final step in completing the next rung of the ladder is to adjoin a boundary edge between nodes 3 and 4. It is rather simple to determine how $\Lambda_{n+2/3}$ is affected.

Claim 5.3 *Let Λ_{n+1} denote the response matrix after the third and final update step. Then Λ_{n+1} is related to $\Lambda_{n+2/3}$ by*

$$\Lambda_{n+1} = \Lambda_{n+2/3} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & d_{n+1} & -d_{n+1} \\ 0 & 0 & -d_{n+1} & d_{n+1} \end{pmatrix} \tag{22}$$

Proof. The claim follows from arguments laid out on page 100 of [CM-2]. □

Finally, we can remove the last row and last column of Λ_{n+1} to obtain $\tilde{\Lambda}_{n+1}$, which we can invert to get $\tilde{\Lambda}_{n+1}^{-1}$. The schematic at the end of this paper summarizes the steps of the above algorithm (figure 18).

Remark 5.4 *The relationship between $\tilde{\Lambda}_{n+1}^{-1}$ and $\tilde{\Lambda}_{n}^{-1}$ has thus been established via the three recursions (16),(21), and (22) above. Taken alone, each step is remarkably simple. But we do not know how to combine these three recursions into one relationship between $\tilde{\Lambda}_{n+1}^{-1}$ and $\tilde{\Lambda}_{n}^{-1}$, which we ultimately need in order to address the convergence of $\tilde{\Lambda}_{n}^{-1}$. With the schematic in figure 18 as a guide, we wrote a short Matlab routine to combine these three recursions into one, but the ouput was extremely complicated and a relationship was not discernable. If such a relationship can be found, the algorithm detailed in this section would be an elegant way to address the issue of the existence of a limiting response matrix for infinite networks in general.*

5.3 From Effective Conductances to a Response Matrix

In this section, we present a method for constructing the response matrix given the effective conductances for a network. This method differs from the usual procedure of constructing the response matrix as the Schur complement of the Kirchhoff matrix. Moreover, we will see that our method motivates a condition under which the response matrix has finite entries. Suppose we are given all of the effective conductances between boundary nodes for a network (we will show, in the next section, how to derive two of the six equivalent conductances for the ladder network). Then we show that it is possible to reconstruct the response matrix.

Consider a generic electrical network with n boundary nodes. Recall that the effective conductance between boundary nodes i and j is obtained by setting a potential of 1 at node i and 0 at node j, and by insulating all other boundary nodes (i.e. imposing 0 current at these other boundary nodes). Then the current I emanating from node i

must exit the network at node j. We can solve for σ_{ij}, the effective conductance between nodes i and j, from the equation

$$(v_i - v_j)\,\sigma_{ij} = I. \tag{23}$$

From this equation, it is clear that we can normalize the current I to 1, by dividing both sides by I, and in what follows, we will always take $I = 1$.

Notation 5.5 *Let $\Delta_{k\ell}^{ij} = v_\ell^{ij} - v_k^{ij}$, which represents the potential difference between nodes ℓ and k, given that there is current 1 emanating from node i and current -1 exiting at node j.*

Then the above equation can be rewritten as

$$\Delta_{ij}^{ij}\sigma_{ij} = 1 \Rightarrow \sigma_{ij} = \frac{1}{\Delta_{ij}^{ij}}. \tag{24}$$

Since we are given σ_{ij}, the quantities Δ_{ij}^{ij} are always known and can be thought of as effective resistances. We now state and prove an important claim about the symmetry of the quantities $\Delta_{k\ell}^{ij}$:

Claim 5.6 *For $i, j, k, \ell \in [1, n]$, $i \neq k$ and $j \neq \ell$,*

$$\Delta_{k\ell}^{ij} = \Delta_{ij}^{k\ell}. \tag{25}$$

Proof. Suppose f and g are functions defined on the boundary nodes. For each boundary node p, let $\varphi_f(p)$ be the boundary current due to the function f. As defined in [CM-2], the bilinear form $\langle g, \Lambda f\rangle$ defined by the response matrix Λ satisfies

$$\langle g, \Lambda f\rangle = \sum_{p\in\partial V} g(p)\,\varphi_f(p)\,. \tag{26}$$

Furthermore, as shown in [CM-1],

$$\langle g, \Lambda f\rangle = \sum_{p\in\partial V} g(p)\,\varphi_f(p) = \sum_{p\in\partial V} f(p)\,\varphi_g(p) = \langle f, \Lambda g\rangle\,. \tag{27}$$

We define the functions ψ_1 and ψ_2 (which can be thought of as boundary currents) as follows:

$$\psi_1 = \begin{cases} 1 & \text{at node } k \\ -1 & \text{at node } \ell \\ 0 & \text{elsewhere} \end{cases},\ \psi_2 = \begin{cases} 1 & \text{at node } i \\ -1 & \text{at node } j \\ 0 & \text{elsewhere} \end{cases}. \tag{28}$$

Also let φ_1 and φ_2 (which can be thought of as boundary voltages) satisfy the relations

$$\Lambda\varphi_1 = \psi_1,\ \Lambda\varphi_2 = \psi_2. \tag{29}$$

Now consider the expression $\langle\psi_1, \Lambda^{-1}\psi_2\rangle$. We know that Λ^{-1} is not well-defined and so there is not a unique φ corresponding to $\Lambda^{-1}\psi_2$. Despite this ambiguity, however, the expression $\langle\psi_1, \Lambda^{-1}\psi_2\rangle$ is well-defined, since it represents the difference between the k and ℓ entries of some potential function φ corresponding to $\Lambda^{-1}\psi_2$. Regardless of which φ we choose, this potential difference must remain the same (since the associated boundary currents ψ_2 have not changed), and so we have attributed meaning to the expression $\langle\psi_1, \Lambda^{-1}\psi_2\rangle$. Moreover, we can write

$$\left\langle\psi_1, \Lambda^{-1}\psi_2\right\rangle = \langle\Lambda\varphi_1, \varphi_2\rangle\,. \tag{30}$$

Using equation (27), we obtain the equality

$$\langle\Lambda\varphi_1, \varphi_2\rangle = \langle\varphi_1, \Lambda\varphi_2\rangle\,. \tag{31}$$

Since $\Lambda\varphi_2 = \psi_2$ and since φ_1 is a potential function corresponding to $\Lambda^{-1}\psi_1$, we arrive at

$$\langle\varphi_1, \Lambda\varphi_2\rangle = \left\langle\Lambda^{-1}\psi_1, \psi_2\right\rangle, \tag{32}$$

and so we have shown that

$$\left\langle\psi_1, \Lambda^{-1}\psi_2\right\rangle = \left\langle\Lambda^{-1}\psi_1, \psi_2\right\rangle. \tag{33}$$

But since $\langle\psi_1, \Lambda^{-1}\psi_2\rangle = \Delta^{ij}_{k\ell}$ and $\langle\Lambda^{-1}\psi_1, \psi_2\rangle = \Delta^{k\ell}_{ij}$, we have established the claim. □

We now show that, given Δ^{ij}_{ij}, and using the symmetry condition (25), we can find all of the $\Delta^{ij}_{k\ell}$. Suppose we have a current of 1 emanating from node 1 and a current of -1 exiting at node 2, and 0 current at other boundary nodes (figure 12a). Then the

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Figure 12: Current Patterns on an Arbitrary Network.

voltage drop between nodes 1 and 3 is the sum of the drops between nodes 1 and 2, and between nodes 2 and 3. Symbolically,

$$\Delta^{12}_{13} = \Delta^{12}_{12} + \Delta^{12}_{23}. \tag{34}$$

Likewise, if we have current 1 at node 2 and -1 at node 3, and 0 current at other boundary nodes (figure 12b), we obtain the identity

$$\Delta_{13}^{23} = \Delta_{12}^{23} + \Delta_{23}^{23}. \tag{35}$$

A superposition of these two sets of boundary current patterns yields a current of 1 at node 1, -1 at node 3, and 0 at node 2. This implies that

$$\Delta_{13}^{13} = \Delta_{13}^{12} + \Delta_{13}^{23}, \tag{36}$$

or,

$$\Delta_{13}^{13} = \Delta_{12}^{12} + \Delta_{23}^{12} + \Delta_{12}^{23} + \Delta_{23}^{23}. \tag{37}$$

But since Δ_{13}^{13}, Δ_{12}^{12}, and Δ_{23}^{23} are known, and since $\Delta_{23}^{12} = \Delta_{12}^{23}$, we can solve for Δ_{23}^{12}. Similarly, focusing on the set of boundary nodes $\{2, 3, 4\}$ instead of $\{1, 2, 3\}$, we can write

$$\Delta_{24}^{24} = \Delta_{23}^{23} + \Delta_{34}^{23} + \Delta_{23}^{34} + \Delta_{34}^{34}, \tag{38}$$

and since Δ_{24}^{24}, Δ_{23}^{23}, and Δ_{34}^{34} are known, and since $\Delta_{23}^{34} = \Delta_{34}^{23}$, we can solve for Δ_{23}^{34}.

Having solved for Δ_{23}^{12} and Δ_{23}^{34}, we can broaden our scope to nodes $\{1, 2, 3, 4\}$ and obtain the equation

$$\Delta_{14}^{14} = \Delta_{12}^{12} + \Delta_{23}^{12} + \Delta_{34}^{12} + \Delta_{12}^{23} + \Delta_{23}^{23} + \Delta_{34}^{23} + \Delta_{12}^{34} + \Delta_{23}^{34} + \Delta_{34}^{34}, \tag{39}$$

from which we can solve for Δ_{34}^{12}. Continuing in such a manner, we can determine all of the $\Delta_{k\ell}^{ij}$.

We now reconstruct the response matrix column by column. From the above procedure, we know that we can determine the potential differences

$$\Delta_{12}^{1n}, \Delta_{23}^{1n}, \ldots \Delta_{n-1,n}^{1n}. \tag{40}$$

From these differences, it is straightforward to determine the potentials $v_1^{1n}, v_2^{1n}, \ldots v_n^{1n}$: WLOG, we can set $v_n^{1n} = 0$ and use the differences $\Delta_{k\ell}^{1n}$ to determine the potentials themselves. It follows from the relation $\Lambda V = I$ that

$$\Lambda \begin{pmatrix} v_1^{1n} \\ v_2^{1n} \\ \vdots \\ v_{n-1}^{1n} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}. \tag{41}$$

From arguments laid out in §5.2, we can rewrite this as

$$\tilde{\Lambda}\begin{pmatrix} v_1^{1n} \\ v_2^{1n} \\ \vdots \\ v_{n-1}^{1n} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{42}$$

where we have omitted the current $i_n = -1$. If we invert $\tilde{\Lambda}$ we can determine the first column of $\tilde{\Lambda}^{-1}$: $\tilde{\Lambda}^{-1}_{*1} = \left(v_1^{1n}, v_2^{1n}, \ldots, v_{n-1}^{1n}\right)^T$.

If we then repeat the above procedure with differences $\Delta_{k\ell}^{2n}$ instead of $\Delta_{k\ell}^{1n}$, we can obtain the second column $\tilde{\Lambda}^{-1}_{*2}$. Continuing for $\Delta_{k\ell}^{3n}, \ldots, \Delta_{k\ell}^{n-1,n}$, we have a procedure for constructing the inverse response submatrix $\tilde{\Lambda}^{-1}$ which we can invert to obtain $\tilde{\Lambda}$. Since row and column sums of the response matrix Λ are zero, $\tilde{\Lambda}$ uniquely determines Λ.

5.4 Effective Conductances

From the above method, we see that elements of the response matrix are finite only if the potential differences $\Delta_{k\ell}^{ij}$ are finite. This is equivalent to requiring all effective conductances σ_{ij} to be finite.

5.4.1 The Conductance σ_{12}

Let us first take up the issue of finding the effective conductance between nodes 1 and 2. To find σ_{12}, we can set the voltages at nodes 1 and 2 to 1 and 0, respectively, and the currents at nodes 3 and 4 to zero (figure 13). The zero current condition at boundary

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Figure 13: Voltage and Current Pattern to Find σ_{12}.

nodes 3 and 4 implies that there is no current flowing through the edges marked with an X, in figure 13. Consider the ladder subnetwork that starts at nodes 5 and 6 (which are boundary nodes for the subnetwork) and extends eastward to infinity. Suppose for the moment that we know the effective conductance between nodes 5 and 6; call it σ_{56}. Then the ladder network of figure 13 can be redrawn as in figure 14.

12cm3.5cmA:/Pic16.bmp

Figure 14: An Equivalent Ladder Network.

From this diagram, it is clear that d_1 is in parallel with the conductances b_1, c_1, σ_{56},which are in series themselves. It follows that

$$\sigma_{12} = d_1 + \frac{1}{\left(\frac{1}{b_1} + \frac{1}{c_1}\right) + \frac{1}{\sigma_{56}}}. \tag{43}$$

A similar argument can be made to find σ_{56}, which is given by

$$\sigma_{56} = d_2 + \frac{1}{\left(\frac{1}{b_2} + \frac{1}{c_2}\right) + \frac{1}{\sigma_{78}}}. \tag{44}$$

This implies that

$$\sigma_{12} = d_1 + \cfrac{1}{\left(\frac{1}{b_1} + \frac{1}{c_1}\right) + \cfrac{1}{d_2 + \cfrac{1}{\left(\frac{1}{b_2} + \frac{1}{c_2}\right) + \frac{1}{\sigma_{78}}}}}. \tag{45}$$

Proceeding in this manner, we generate an infinite, simple continued fraction for the effective conductance between nodes 1 and 2 of the ladder network that is of the form

$$\sigma_{12} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \tag{46}$$

where the a_n are given by

$$a_{2n} = d_{n+1}, \; n \geq 0 \tag{47}$$

$$a_{2n+1} = \left(\frac{1}{b_{n+1}} + \frac{1}{c_{n+1}}\right), \; n \geq 0. \tag{48}$$

Remark 5.7 *All partial denominators of the continued fraction are positive, real numbers, since they arise from conductances on the network.*

We now see that the effective conductance σ_{12} is finite if and only if the continued fraction (46) converges. The following theorem, taken from [W], provides us with conditions on the elements a_p of the continued fraction that are *sufficient* to ensure its convergence.

Theorem 5.8 *If the series*

$$\sum a_{2p+1} \tag{49}$$

and

$$\sum a_{2p+1} {s_p}^2, \text{ where } s_p = a_2 + a_4 + \cdots + a_{2p}, \tag{50}$$

converge, and

$$\lim s_p = \infty, \tag{51}$$

then the continued fraction (46) converges.

Proof. The proof is contained in [W]. The curious reader will soon be satisfied, for we will supply a non-trivial adaptation of Wall's proof in the sequel. □

For the ladder network then, these convergence conditions translate as follows:

$$\sum \left(\frac{1}{b_{i+1}} + \frac{1}{c_{i+1}} \right) < \infty \text{ and} \tag{52}$$

$$\sum \left(\frac{1}{b_{i+1}} + \frac{1}{c_{i+1}} \right) (d_2 + d_3 + \cdots + d_{i+1})^2 < \infty, \text{ while} \tag{53}$$

$$\lim_{i \to \infty} (d_2 + d_3 + \cdots + d_{i+1})^2 = \infty. \tag{54}$$

Example 5.9 *We present this brief example to emphasize that these convergence conditions for σ_{12} are sufficient but not necessary. Consider the infinite ladder network with the conductance pattern indicated in figure 15. Then*

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Figure 15: Conductances on the Ladder.

$$\sigma_{12} = 1 + \cfrac{1}{\left(\frac{1}{2} + \frac{1}{2}\right) + \cfrac{1}{1 + \cfrac{1}{\left(\frac{1}{2} + \frac{1}{2}\right) + \ddots}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}, \tag{55}$$

which converges to the golden ratio. However,

$$\sum_{i=1}^{\infty}\left(\frac{1}{b_{i+1}}+\frac{1}{c_{i+1}}\right)=\sum_{i=1}^{\infty}1=\infty. \tag{56}$$

5.4.2 The Conductance σ_{34}

Let us now take up the issue of finding the effective conductance between nodes 3 and 4. To find σ_{34}, we can set the voltages at nodes 3 and 4 to 1 and 0, respectively, and the currents at nodes 1 and 2 to zero. The zero current condition at boundary nodes 1 and 2 implies that there is no current flow across b_1, c_1, d_1 (figure 16). Analogous to the

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Figure 16: Voltage and Current Pattern to Find σ_{34}.

σ_{12} case, we can obtain what we will call a *reverse continued fraction* (RCF) expression for σ_{34}, given by

$$\sigma_{34}=\ddots+\cfrac{1}{d_n+\cdots+\cfrac{1}{d_3+\cfrac{1}{\left(\cfrac{1}{b_2}+\cfrac{1}{c_2}\right)+\cfrac{1}{d_2}}}}, \tag{57}$$

which can be put in the form

$$\ddots+\cfrac{1}{a_n+\ddots\cfrac{1}{a_3+\cfrac{1}{a_2+\cfrac{1}{a_1}}}}, \tag{58}$$

if we let

$$a_{2n+1} = d_{n+2},\ n\geq 0 \tag{59}$$

$$a_{2n+2} = \left(\frac{1}{b_{n+2}}+\frac{1}{c_{n+2}}\right),\ n\geq 0. \tag{60}$$

Remark 5.10 *The RCF is different from the standard continued fractions found in the literature. There is no good intuition, at the present time, for how to rearrange the terms of a standard continued fraction in order to show convergence of an RCF. Consequently, we will develop convergence theory for the RCF from scratch, paralleling Wall's [W] development for standard continued fractions.*

We are concerned with the problem of finding conditions on the elements a_p of the RCF that are sufficient to ensure its convergence.

Notation and Machinery Consider the transformations of the variable w given by

$$F_1(w) = a_1 + w,\ F_p(w) = a_p + \frac{1}{w},\ p > 1. \tag{61}$$

Evaluating n compositions of this transformation at zero gives

$$(F_n \circ F_{n-1} \circ \cdots \circ F_1)(0) = a_n + \ddots \frac{1}{a_3 + \cfrac{1}{a_2 + \cfrac{1}{a_1}}} \tag{62}$$

which we call the n^{th} *convergent* of an RCF.

Let $T_n(w) = (F_n \circ F_{n-1} \circ \cdots \circ F_1)(w)$. By mathematical induction, we can rewrite the $T_n(w)$ as the linear fractional transformation

$$T_n(w) = \frac{\alpha_n w + \beta_n}{\alpha_{n-1} w + \beta_{n-1}},\ n > 0, \tag{63}$$

where the quantities $\alpha_n, \beta_n, \alpha_{n-1}, \beta_{n-1}$ are independent of w and can be determined from the following recurrence relations and initial conditions:

$$\begin{aligned} \alpha_0 &= 0, \beta_0 = 1, \alpha_1 = 1, \beta_1 = a_1; \\ \alpha_{p+1} &= a_{p+1}\alpha_p + \alpha_{p-1},\ p > 0, && (64) \\ \beta_{p+1} &= a_{p+1}\beta_p + \beta_{p-1},\ p > 0. && (65) \end{aligned}$$

For the base case, take $n = 1$. We see that

$$T_1(w) = \frac{\alpha_1 w + \beta_1}{\alpha_0 w + \beta_0} = a_1 + w = F_1(w), \tag{66}$$

as desired. Now assume that (63) is true for $n = k$. Then

$$\begin{aligned} T_{k+1}(w) &= F_{k+1}(T_k(w)) = a_{k+1} + \frac{\alpha_{k-1}w + \beta_{k-1}}{\alpha_k w + \beta_k} \\ &= \frac{(a_{k+1}\alpha_k + \alpha_{k-1})\, w + (a_{k+1}\beta_k + \beta_{k-1})}{\alpha_k w + \beta_k} \\ &= \frac{\alpha_{k+1}w + \beta_{k+1}}{\alpha_k w + \beta_k}, \end{aligned} \tag{67}$$

and so (63) is true for $n = k+1$ and thus for all n.

From this, it follows that the n^{th} convergent can be written as

$$T_n(0) = \frac{\beta_n}{\beta_{n-1}}. \tag{68}$$

We also define the n^{th} *even convergent* to be

$$T_{2n}(0) = \frac{\beta_{2n}}{\beta_{2n-1}} \tag{69}$$

and the n^{th} *odd convergent* to be

$$T_{2n+1}(0) = \frac{\beta_{2n+1}}{\beta_{2n}}. \tag{70}$$

Sufficient Condition for Convergence Although the requirement that the RCF converge is rather opaque, it can be better understood in terms of conditions on an infinite series whose terms are the partial denominators of the continued fraction. The following theorem is the first step in making this relationship precise.

Theorem 5.11 *If the series $\sum a_p$ converges, then the sequences of even and odd numerators and denominators, $\{\beta_{2p}\}, \{\beta_{2p+1}\}$ of the reverse continued fraction*

$$\ddots + \cfrac{1}{a_3 + \cfrac{1}{a_2 + \cfrac{1}{a_1}}} \tag{71}$$

converge to finite, non-zero limits F_0, F_1, respectively.

Proof. We prove that the sequence $\{\beta_{2p}\}$ converges; the proof that the other sequence converges can be made in the same way. From the recurrence relation (65), we have

$$\begin{aligned}\beta_{2p} &= a_{2p}\beta_{2p-1}+\beta_{2p-2} \\ &= a_{2p}\beta_{2p-1}+a_{2p-2}\beta_{2p-3}+\beta_{2p-4} \\ &= a_{2p}\beta_{2p-1}+a_{2p-2}\beta_{2p-3}+\cdots+a_2\beta_1. \end{aligned} \tag{72}$$

Thus each β_{2p} is the sum of the first p terms of the infinite series $\sum a_{2r}\beta_{2r-1}$. Since the series $\sum a_r$ was convergent by hypothesis, it follows that $\sum a_{2r}$ is convergent. If we can now show that $\beta_{2p-1} \leq C$, where C is a constant independent of p, we can use the comparison test to show that the series $\sum a_{2r}\beta_{2r-1}$, and hence the sequence $\{\beta_{2p}\}$, converges.

We use induction to find the value of C. Let $M = \max\{\beta_0, \beta_1\}$. Then, we can rewrite (65) to obtain

$$\begin{aligned}\beta_2 &\leq a_2 M + M = M(1+a_2) \\ \beta_3 &\leq a_3\beta_2 + M \leq M a_3 (1+a_2) + M \\ &= M(1+a_2)(1+a_3). \end{aligned} \tag{73}$$

By induction then,

$$\beta_n \leq M(1+a_2)(1+a_3)\cdots(1+a_n), \tag{74}$$

for $n \geq 2$. We may then take

$$C = M\prod_{p=1}^{\infty}(1+a_p). \tag{75}$$

This infinite product converges because the series $\sum a_p$ converges by hypothesis (this is a standard but non-trivial result of complex analysis; see [Cop] for a proof). Thus $\sum a_{2r}\beta_{2r-1} < C\sum a_{2r} < \infty$, and so convergence of $\{\beta_{2p}\}$ is established. Since $a_p > 0$ for all p, $\sum a_{2r}\beta_{2r-1} > 0$, and so $\{\beta_{2p}\}$ converges to some finite, non-zero limit F_0.

As indicated earlier, convergence of $\{\beta_{2p+1}\}$ to the finite, non-zero limit F_1 can be shown similarly. □

Now that we have shown that the even and odd numerators and denominators converge, we want to show that the even and odd convergents of the RCF converge to the same value. This would establish the analog of Theorem 5.8 for the RCF.

Theorem 5.12 *If the series*

$$\sum a_{2p+1} \tag{76}$$

and

$$\sum a_{2p+1}s_p^2, \text{ where } s_p = a_2 + a_4 + \cdots + a_{2p}, \tag{77}$$

converge, and if $\lim s_p = \infty$ *for a certain subsequence of the indices* p*, then*

$$\lim_{p\to\infty} \frac{\beta_{2p}}{\beta_{2p-1}} = \frac{F_0}{F_1} = \lim_{p\to\infty} \frac{\beta_{2p+1}}{\beta_{2p}} = \frac{F_1}{F_0} = 1, \tag{78}$$

and thus the continued fraction (71) converges.

Proof. Since the series $\sum a_{2p+1}$ and $\sum a_{2p+1}s_p^2$ converge, it follows that the series

$$\sum a_{2p+1}s_p \tag{79}$$

converges. Therefore, there exists an index $n \geq 1$ such that

$$a_{2p+1}s_p < 1 \text{ for } p \geq n. \tag{80}$$

Hence the quantities

$$\pi_k = \prod_{p=1}^{k} (1 + a_{2n+2p+1}s_{n+p}) \tag{81}$$

are different from zero, and the infinite product

$$\lim_{k\to\infty} \pi_k = \prod_{p=1}^{k} (1 + a_{2n+2p+1}s_{n+p}) \tag{82}$$

converges (to a non-zero value), because the sum $\sum a_{2p+1}s_p$ converges.

Lemma 5.13 *Let*

$$\begin{aligned}
U_{2k} &= \frac{\beta_{2n+2k+1}}{\pi_k},\ V_{2k} = \frac{\beta_{2n+2k}}{\pi_k}, && (83)\\
U_{2k+1} &= (\beta_{2n+2k+2} - s_{n+k+1}\beta_{2n+2k+1})\,\pi_k, && (84)\\
V_{2k+1} &= (\beta_{2n+2k+1} - s_{n+k+1}\beta_{2n+2k})\,\pi_k,\ k > 0; && (85)\\
c_{2k} &= \frac{a_{2n+2k+1}}{\pi_{k-1}\pi_k},\ c_{2k+1} = -a_{2n+2k+1}s_{n+k}^2\pi_{k-1}\pi_k,\ k \geq 1. && (86)
\end{aligned}$$

Then

$$\begin{aligned}
U_k &= c_kU_{k-1} + U_{k-2},\ k \geq 2, && (87)\\
V_k &= c_kV_{k-1} + V_{k-2},\ k \geq 2. && (88)
\end{aligned}$$

Assume for the moment that the lemma is true. Since the series $\sum a_{2p+1}$ and $\sum a_{2p+1}s_p^2$ converge, and from the convergence of

$$\pi_k = \prod_{p=1}^{k} \left(1 + a_{2n+2p+1}s_{n+p}\right),$$

we can conclude that the series $\sum c_p$ converges. As in the proof of the previous theorem, it follows that the sequences $\{U_{2k}\}, \{V_{2k}\}, \{U_{2k+1}\}, \{V_{2k+1}\}$ converge to finite, non-zero limits. Therefore the limits

$$\lim_{p\to\infty} \beta_{2p+1} = F_1 \text{ and } \lim_{p\to\infty} \beta_{2p} = F_0 \tag{89}$$

are finite and non-zero.

From the equations for U_{2k+1} and V_{2k+1}, we can substitute $k = p - 1$ to get

$$\frac{\beta_{2n+2p}}{\beta_{2n+2p-1}} = \frac{s_{n+p}\beta_{2n+2p-1} + \dfrac{U_{2p-1}}{\pi_{p-1}}}{s_{n+p}\beta_{2n+2p-2} + \dfrac{V_{2p-1}}{\pi_{p-1}}} = \frac{\beta_{2n+2p-1} + \dfrac{U_{2p-1}}{s_{n+p}\pi_{p-1}}}{\beta_{2n+2p-2} + \dfrac{V_{2p-1}}{s_{n+p}\pi_{p-1}}}. \tag{90}$$

Then as $p \to \infty$, we have shown that

$$\frac{\beta_{2p}}{\beta_{2p-1}} = \frac{F_0}{F_1} = \frac{\beta_{2p-1}}{\beta_{2p-2}} = \frac{F_1}{F_0}. \tag{91}$$

Hence, under these conditions, the RCF must converge to a value of 1. $\square$

We now establish the proof of the lemma so as to complete the proof of the above theorem.

Proof. [of Lemma] The proof of the lemma deviates very little from Wall's [W] corresponding proof. From the recurrence relation (65), we have

$$\begin{aligned}
\beta_{2n+2p+1} &= a_{2n+2p+1}\beta_{2n+2p} + \beta_{2n+2p-1} \\
&= a_{2n+2p+1}\left(\beta_{2n+2p} - s_{n+p}\beta_{2n+2p-1}\right) + \left(1 + a_{2n+2p+1}s_{n+p}\right)\beta_{2n+2p-1}.
\end{aligned} \tag{92}$$

We can rewrite this using equations (83) through (86) to get

$$\pi_p U_{2p} = a_{2n+2p+1}\frac{U_{2p-1}}{\pi_{p-1}} + \pi_p U_{2p-2}, \tag{93}$$

or

$$U_{2p} = c_{2p}U_{2p-1} + U_{2p-2}. \tag{94}$$

This establishes equation (87) with $k = 2p$. Equation (88) results, with $k = 2p$, if we replace $\beta_{2n+2p+1}$ with β_{2n+2p} above.

To establish equation (87) for odd values of k, we proceed as follows:

$$\begin{aligned}
& (1 + a_{2n+2p+1}s_{n+p})\left(\beta_{2n+2p+2} - s_{n+p+1}\beta_{2n+2p+1}\right) \\
= \;& (1 + a_{2n+2p+1}s_{n+p})\left(\beta_{2n+2p+2} - a_{2n+2p+2}\beta_{2n+2p+1} - s_{n+p}\beta_{2n+2p+1}\right) \\
= \;& (1 + a_{2n+2p+1}s_{n+p})\left(\beta_{2n+2p} - s_{n+p}\beta_{2n+2p+1}\right) \\
= \;& \beta_{2n+2p} - s_{n+p}\left(\beta_{2n+2p+1} - a_{2n+2p+1}\beta_{2n+2p}\right) - a_{2n+2p+1}s_{n+p}^2\beta_{2n+2p+1} \\
= \;& -a_{2n+2p+1}s_{n+p}^2\beta_{2n+2p+1} + \left(\beta_{2n+2p} - s_{n+p}\beta_{2n+2p-1}\right). \qquad (95)
\end{aligned}$$

Thus,

$$\frac{U_{2p+1}}{\pi_{p-1}} = -a_{2n+2p+1}s_{n+p}^2\pi_p U_{2p} + \frac{U_{2p-1}}{\pi_{p-1}}, \qquad (96)$$

or

$$U_{2p+1} = c_{2p+1}U_{2p} + U_{2p-1}. \qquad (97)$$

This establishes equation (87) with $k = 2p + 1$. Equation (88) results, with $k = 2p + 1$, if we replace $\beta_{2n+2p+1}$ with β_{2n+2p} above. □

Remark 5.14 *We believe that under appropriate modifications, the results of this section can be modified so that the RCF converges to a value other than* 1*. It seems plausible that the RCF should converge to the limiting value of the sequence* $\left\{\frac{1}{b_{n+2}} + \frac{1}{c_{n+2}}\right\}$*, which would be the same as the limiting value of* $\{d_{n+2}\}$*. Perhaps it is necessary to redefine the transformations* $F_p(w)$ *and the convergents so that the odd and even convergents are not reciprocals of each other. Perhaps other theorems on continued fractions can be better adapted to the RCF. It may even be the case that we want the RCF to diverge in order to have appropriate meaning for the infinite ladder network. We regret not having more time to explore these issues.*

5.4.3 Another Result for the Conductance σ_{34}

The problem of the convergence of σ_{34} can be made more tractable by imposing the following symmetry conditions on the conductances of the infinite ladder network. Suppose that

$$d_i = d \text{ and } \frac{1}{b_i} + \frac{1}{c_i} = d \text{ for all } i > 1, \qquad (98)$$

where $d \in \mathbb{R}^+$. Under these conditions, the effective conductance σ_{34} is given by

$$\sigma_{34} = \ddots + \cfrac{1}{d + \ddots \cfrac{1}{d + \cfrac{1}{d + \cfrac{1}{d_1}}}}, \tag{99}$$

which can be generated by iterating the transformation

$$F(v) = d + \frac{1}{v}, \tag{100}$$

starting at some initial point $x_0 = d_1 > 0$ (note that x_0 cannot be arbitrary; it will have to be contained in some neighborhood of the limit ξ, which will be specified later).

Let $F(x_0) = x_1, F(F(x_0)) = F(x_1) = x_2, \ldots, F(x_n) = x_{n+1}$. Then if the RCF approaches some limit ξ, the sequence $\{x_n\}$ must approach ξ, or equivalently, $F(\xi) = \xi$. Hence we are interested in the fixed point ξ of F. Assume for the moment that the fixed point exists. Then it satisfies

$$d + \frac{1}{\xi} = \xi, \tag{101}$$

which implies

$$\xi^2 - d\xi - 1 = 0. \tag{102}$$

The roots of this equation are

$$\frac{d \pm \sqrt{d^2 + 4}}{2}, \tag{103}$$

one being positive and the other being negative. Since all of the partial denominators are greater than zero, the limiting value of the continued fraction cannot possibly be negative, and so we discard the negative root above. Thus ξ is given by

$$\xi = \frac{d + \sqrt{d^2 + 4}}{2}, \tag{104}$$

which is greater than 1 (indeed, $\xi = 1$ if and only if $d = 0$, which is a contradiction).

We now show that ξ is an attractive fixed point. Observe that

$$|F'(\xi)| = \frac{1}{\xi^2} < 1, \tag{105}$$

since $\xi > 1$. The following lemma establishes that our fixed point ξ is indeed attractive.

Lemma 5.15 *The fixed point ξ is attractive if $|F'(\xi)| < 1$.*

Proof. Choose $k \in (|F'(\xi)|, 1)$. From the definition of the derivative

$$\left|\frac{F(x) - F(\xi)}{x - \xi}\right| \to |F'(\xi)| \text{ as } x \to \xi.$$

Hence we can choose a symmetric interval I around ξ such that for $x \in I$, we have

$$0 < \left|\frac{F(x) - F(\xi)}{x - \xi}\right| < k,$$

or, equivalently,

$$|F(x) - F(\xi)| < k|x - \xi|.$$

We begin the iteration with some $x_0 = d_1 \in I$. Then

$$\begin{aligned}
|x_1 - \xi| &= |F(x_0) - F(\xi)| < k|x_0 - \xi|, \\
|x_2 - \xi| &= \left|F^2(x_0) - F^2(\xi)\right| < k|F(x_0) - F(\xi)| < k^2|x_0 - \xi| \\
&\vdots \\
|x_n - \xi| &< k^n|x_0 - \xi|.
\end{aligned}$$

Since $0 < k < 1$, we conclude that $|x_n - \xi| \to 0$ as $n \to \infty$. Thus ξ is an attractive fixed point. □

Thus by imposing certain symmetry conditions on the infinite ladder network, we have caused the RCF representing the conductance σ_{34} to converge to some arbitrary $\xi > 1$.

Example 5.16 *For the infinite ladder network of figure 15, $d = 1$, and thus*

$$\sigma_{34} = \frac{d + \sqrt{d^2 + 4}}{2} = \frac{1 + \sqrt{5}}{2} = \sigma_{12}, \tag{106}$$

as expected.

5.4.4 The Other Four Effective Conductances

At this point, little is known about how to obtain expressions for the effective conductances $\sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}$. Finding these effective conductances is straightforward if the ladder has up to 4 vertical conductances, since paths in series and in parallel can be

12cm3.5cm//Borg/MSCC/home1/saksena/Pic19.bmp

Figure 17: $Y - \Delta$ Transform?

clearly identified. As more rungs are added however, paths between the boundary nodes have significant overlaps, and it is difficult to begin computing one of the above four effective conductances. One possibility is to $Y - \Delta$ transform all ladder networks with 5 or more vertical conductances into a ladder with 4 vertical conductances (figure 19). Such a sequence of $Y - \Delta$ transformations can be shown to exist, but it requires many steps and the conductances become difficult to obtain very quickly.

16cm17cm//Borg/MSCC/home1/saksena/Pic12.bmp

Figure 18: Diagram Illustrating Steps of the Algorithm.

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