# The Discrete Scattering Problem

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## 1 Background

### 1.1 The Normal Inverse Problem

Consider a graph $G$ made up of vertices or nodes connected by edges. Let $\Gamma = (V, V_B, E)$ where $V$ is the set of vertices, $V_B \subset V$ is the set of vertices on the boundary, and $E$ is the set of edges connecting vertices in $V$. That is, each $e \in E$ can be defined by some $p, q \in V$ such that $e$ connects node $p$ to $q$, and $e = (p, q)$.
Let $\Gamma = (G, \gamma)$ be a resistor network in which each edge $e = (p, q)$ has a conductivity $\gamma(e)$, allowing a current to flow through it. The current flowing through the edges connected to a node $p$ is defined in terms of the conductivities of those edges and the voltages of $p$ and its neighbors, which are the nodes $q$ connected to $p$ by an edge, denoted by $q \sim p$:

$$\phi(p) = \sum_{q \sim p} \gamma(p, q)[u(p) - u(q)]$$

An interior node $p$ is one that has no net flow of current through its edges. That is, it satisfies the following equation:

$$\phi(p) = \sum_{q \sim p} \gamma(p, q)[u(p) - u(q)] = 0$$

A boundary node need not satisfy the above property.
From this network $\Gamma$ a Kirchhoff matrix $K$ can be constructed by numbering each node, the boundary nodes first, then the interior nodes, and defining each entry in the matrix as such:

$$\begin{aligned} K_{i,j} &= -\gamma(i, j) \qquad i \neq j \\ K_{i,i} &= \textstyle\sum_{j \neq i} \gamma(i, j) \end{aligned}$$

Because the boundary nodes were labeled first, the Kirchhoff matrix can be written in another form:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{1}$$

where $A$ represents the boundary to boundary edges, $B$ the boundary to interior edges, and $C$ the interior to interior edges.

A response matrix $\Lambda$ is found by taking the Schur complement of $C$ in $K$. That is,

$$\Lambda = K/C = A - BC^{-1}B^T$$

The inverse problem deals with taking a given response matrix $\Lambda$ and trying to recover the original Kirchhoff matrix $K$. Curtis and Morrow go into further detail on this problem in their book [1].

### 1.2 A Physical Interpretation

The scattering problem deals with acoustic waves scattered by a given region. Consider a region of mass with density that varies through space but remains constant over time. A velocity potential $u$ applied to the boundary of the region causes waves of certain velocities to be propagated throughout the region. Whereas the usual inverse problem requires that $\nabla \cdot (\gamma \nabla u) = 0$, the scattering problem requires instead that

$$\nabla \cdot (\gamma \nabla u) + \lambda u = 0 \tag{2}$$

where $\gamma = \frac{1}{\rho}$, $\rho$ being the density, and $\lambda$ is the frequency of the sound waves propagated. Discretizing (2) results in

$$-Ku + \lambda u = -(K - \lambda I)u = 0 \tag{3}$$

with $K$ representing the usual Kirchhoff matrix.

Recovering the $\gamma$ values in the Kirchhoff matrix allow us to know the density of the mass at discrete points within the region. The response matrix $\Lambda$ consists of entries based on how waves of a particular frequency $\lambda$ exit the region in scattered form.

One particular applcation of this technique is in the case of oil drilling. The surface of the earth is considered to be the boundary, where sound waves are produced. By recovering the densities of the mass below the surface, the location of oil can be determined.

A more in-depth interpretation of scattering can be found in Erkki Heikkola's thesis [2], while a discretization of the continuous case can be found in a paper by Michelle Covell and Krzystof Fidkowski [3].

### 1.3 The Scattering Problem

The scattering problem has to do with a slightly different Kirchhoff matrix than that in (1), now dependent on a frequency $\lambda$:

$$K - \lambda I = \begin{bmatrix} A - \lambda I & B \\ B^T & C - \lambda I \end{bmatrix}$$

Taking the Schur complement of $(C - \lambda I)$ in $(K - \lambda I)$ gives a response matrix $\Lambda(\lambda)$ in terms of the frequency $\lambda$:

$$\Lambda(\lambda) = (A - \lambda I) - B(C - \lambda I)^{-1}B^T \tag{4}$$

Since $(C - \lambda I)^{-1}$ is not defined when $\lambda$ is an eigenvalue of $C$, we consider only values of $\lambda > \|C\|$, i.e. $\lambda$ is larger than any eigenvalue of $C$. Then $(C - \lambda I)^{-1}$ can be expanded in (4) to write $\Lambda(\lambda)$ as a power series:

$$\Lambda(\lambda) = A - \lambda I + \frac{\Lambda_1}{\lambda} + \frac{\Lambda_2}{\lambda^2} + \ldots + \frac{\Lambda_n}{\lambda^n} + \ldots, \tag{5}$$

where each $\Lambda_n = BC^{n-1}B^T$. Assuming we have $\Lambda(\lambda)$ for any value of $\lambda$, we can now attempt to recover as much of the original Kirchhoff matrix $K$ as possible.

# 2 Matrix $A$

## 2.1 Recovering $A$

A close look at the series expansion for $\Lambda(\lambda)$ in (5) shows $A$ to always be recoverable. Since $\Lambda(\lambda)$ and $\lambda$ are known, we can take a limit:

$$\lim_{\lambda\to\infty} [\Lambda(\lambda) + \lambda I] = \lim_{\lambda\to\infty} A + \frac{\Lambda_1}{\lambda} + \frac{\Lambda_2}{\lambda^2} + \ldots = A$$

We now know the conductivities of all the boundary to boundary edges in our network $\Gamma$.

## 2.2 Equations Derived From $A$

By construction, the diagonal entries of $A$ are the negative sum of all the nondiagonal entries of their row in $K$. That is,

$$A_{ii} = -\sum_{j\neq i, j\in V_B} A_{ij} - \sum_{j\notin V_B} B_{ij} \tag{6}$$

Because all values of $A$ are known, we can define values $k_i$ such that

$$k_i = -\sum_{j\in V_B} A_{ij} \tag{7}$$

Combining equations (6) and (7) we get

$$k_i = \sum_{j\notin V_B} B_{ij} \tag{8}$$

# 3 Matrix $B$

## 3.1 Characterizing $\Lambda_1$

Let $B = (b_{ij})$ be an $m \times n$ matrix. That is, $\Gamma$ consists of $m$ boundary nodes and $n$ interior nodes. As in (5), $\Lambda_1 = BB^T$ and can be written as

$$\Lambda_1 = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & & \\ \vdots & & \ddots & \\ b_{m1} & & & b_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & \dots & b_{m1} \\ b_{12} & b_{22} & & \\ \vdots & & \ddots & \\ b_{1n} & & & b_{mn} \end{bmatrix}$$

Looking at the diagonal entries, we can see that

$$(\Lambda_1)_{ii} = \sum_{j=1}^{n} b_{ij}^2$$

In particular, since $(\Lambda_1)_{ii}$ is a sum of squares, we can make certain observations about $(\Lambda_1)_{ii}$:

**Observation 3.1** *A boundary node $i$ is not connected to any interior nodes* $\iff (\Lambda_1)_{ii} = 0$.

**Observation 3.2** *A boundary node $i$ is connected to exactly one interior node* $\iff (\Lambda_1)_{ii} = k_i^2$.

In considering the entries of $\Lambda_1$ that are not along the diagonal, note that

$$(\Lambda_1)_{ij} = \sum_{l=1}^{n} b_{il} b_{jl} \quad i \neq j$$

In particular, we can make the following observation:

**Observation 3.3** *Boundary nodes $i, j$ are connected to the same interior node(s)* $\iff (\Lambda_1)_{ij} \neq 0$.

## 3.2 One Interior Node Networks

In a one interior node network with $m$ boundary nodes, $B$ has dimension $m \times 1$. That is,

$$B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}$$

where $b_{ij}$ is the $i,j$th entry of $B$. Then, by (8),

$$k_i = b_{i1} \qquad \forall i = 1, 2, \dots, m$$

Hence $B$ is fully recoverable by looking solely at $A$. $C$ consists of only one term which, by definition, is the negative sum of all of the entries in $B$. More concretely,

$$c = -\sum_{i=1}^{m} b_{i1}$$

and so $C$ is also recoverable.

## 3.3 Two Interior Node Networks

In a two interior node network, $B$ is $m \times 2$ and can be written as

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ \vdots & \vdots \\ b_{m1} & b_{m2} \end{bmatrix}$$

Then, by (8),

$$k_i = b_{i1} + b_{i2} \qquad \forall i = 1, 2, \ldots, m \tag{9}$$

and one column of $B$ can be put in terms of the other:

$$B = \begin{bmatrix} b_{11} & k_1 - b_{11} \\ b_{21} & k_2 - b_{21} \\ \vdots & \vdots \\ b_{m1} & k_m - b_{m1} \end{bmatrix}$$

The terms in the first column of $\Lambda_1$ form $m$ equations with $m$ unknowns, namely $b_{11}, b_{21}, \ldots b_{m1}$:

$$\begin{gathered} (\Lambda_1)_{11} = b_{11}^2 + (k_1 - b_{11})^2 \\ (\Lambda_1)_{21} = b_{11}b_{21} + (k_1 - b_{11})(k_2 - b_{21}) \\ \vdots \\ (\Lambda_1)_{m1} = b_{11}b_{m1} + (k_1 - b_{11})(k_m - b_{m1}) \end{gathered}$$

Looking at the first equation, letting $\lambda_{11} = (\Lambda_1)_{11}$, and rearranging the terms, we get

$$2b_{11}^2 - 2k_1 b_{11} + (k_1^2 - \lambda_{11}) = 0$$

Solving for $b_{11}$ gives two solutions:

$$b_{11} = \frac{1}{2}k_1 \pm \frac{1}{2}\sqrt{2\lambda_{11} - k_1^2} \tag{10}$$

Going back to (9) and solving for $b_{12}$, we get

$$b_{12} = \frac{1}{2}k_1 \mp \frac{1}{2}\sqrt{2\lambda_{11} - k_1^2},$$

the same two values as $b_{11}$ but in reverse. Setting aside for the moment the ambiguity of the ordering and letting $\lambda_{i1} = (\Lambda_1)_{i1}$,

$$\begin{aligned} \lambda_{i1} &= b_{11}b_{i1} + (k_1 - b_{11})(k_i - b_{i1}) \\ &= b_{11}b_{i1} + k_1^2 - b_{11}k_i - b_{i1}k_1 + b_{11}b_{i1} \end{aligned}$$

This results in a linear equation for $b_{i1}$ which allows us to solve for the remaining values of $B$ in terms of $b_{11}$:

$$b_{i1} = \frac{\lambda_{i1} + b_{11}k_i - k_1^2}{2b_{11} - k_1}$$

This equation has the added requirement that $k_1 \neq 2b_{11}$, which occurs only when $b_{11} = b_{12}$. For now we assume that $b_{11} \neq b_{12}$. Using the value of $b_{11}$ corresponding to the positive squareroot in (10) to solve for $b_{i1}$, we get

$$b_{i1} = \frac{1}{2}k_i + \frac{\lambda_{i1} + k_1k_i - k_1^2}{\sqrt{2\lambda_{11} - k_1^2}}$$

Solving for $b_{i2}$ using (9) gives the solution

$$b_{i2} = \frac{1}{2}k_i - \frac{\lambda_{i1} + k_1k_i - k_1^2}{\sqrt{2\lambda_{11} - k_1^2}}$$

Had $b_{11}$ been chosen to correspond to the value with the negative squareroot in (10), i.e. switching the values of $b_{11}$ and $b_{12}$, the two values for $b_{i1}$ and $b_{i2}$ would have also been reversed.

In the case that $b_{11} = b_{12}$, the first equation exactly determines the two values. However, the remaining equations cannot be used to solve for the other unknown entries in $B$. Suppose for some $i$, $1 \leq i \leq m$, $b_{i1} \neq b_{i2}$, which can be verified by examining $(\Lambda_1)_{ii}$. Then the entries along the $i$th column can be used to recover each value of the $B$ matrix. If no such $i$ exists, then the diagonal entries are sufficient to recover $B$.

Thus, $B$ is recoverable, save for an ambiguity in the ordering of the two columns, in any network with exactly two interior nodes.

### 3.4 Larger Networks

Consider a network $\Gamma$ with $n$ interior nodes, $n > 2$, such that each boundary node is connected to at most two interior nodes. $B$ in this network is somewhat sparse. The diagonal entries of $\Lambda_1$ can be used to determine the number of interior nodes a particular boundary node is connected to.

If $(\Lambda_1)_{ii} = 0 \Longrightarrow$ node $i$ is not connected to any interior nodes. (by 3.1)

If $(\Lambda_1)_{ii} = k_i^2 \Longrightarrow$ node $i$ is connected to exactly one interior node by an edge of conductivity $-k_i$. (by 3.2)

Otherwise, because of the restriction on our network $\Gamma$, node $i$ is connected to exactly two interior nodes by edges whose conductivities can be found using $b_{i1}^2 + b_{i2}^2 = (\Lambda_1)_{ii}$ and $b_{i1} + b_{i2} = k_i$.

Once the values of each row are recovered, the next step is to determine which column in $B$ they each fall into. To do this, we can look at the nondiagonal entries of $\Lambda_1$.

By 3.1, we know that the $i,j$th entry is nonzero when boundary nodes $i$ and $j$ are adjacent to the same interior node(s). Suppose nodes $i$, $j$ are each connected to two interior nodes $i_1, i_2, j_1, j_2$ by edges with conductivities $b_{i1}, b_{i2}, b_{j1}, b_{j2}$, respectively. Then there are six ways for the two nodes to be connected by the interior nodes, each corresponding to a particular value in $\Lambda_1$:

| *matching nodes* | $(\Lambda_1)_{ij}$ |
|---|---|
| $i_1 = j_1$ | $b_{i1}b_{j1}$ |
| $i_1 = j_2$ | $b_{i1}b_{j2}$ |
| $i_2 = j_1$ | $b_{i2}b_{j1}$ |
| $i_2 = j_2$ | $b_{i2}b_{j2}$ |
| $i_1 = j_1$ *and* $i_2 = j_2$ | $b_{i1}b_{j1} + b_{i2}b_{j2}$ |
| $i_1 = j_2$ *and* $i_2 = j_1$ | $b_{i1}b_{j2} + b_{i2}b_{j1}$ |

Comparing these six potential values to the actual value in $\Lambda_1$ reveals which of the connections is the accurate one (assuming only one of the six is equivalent to the value in $\Lambda_1$). In this way, we can determine which boundary nodes are connected to a particular interior node, and reconstruct the columns of $B$. If either node $i$ or $j$ (or both) is connected to only one node, the same method applies, but with fewer possible connections between the two nodes.

A problem arises when $\Gamma$ contains an interior node $i$ which is not connected to any boundary nodes. In this case, the $i$th column of $B$ is made up entirely of zeros. This information is lost when looking at $\Lambda_1$, and the existance of such an interior node cannot be recovered by the method described above.

Thus, $B$ can be determined up to an ambiguity in the order of its columns and without any zero columns in this case.

## 3.5 When $B$ has rank 1

In the case that $B$ has rank 1, that is, all columns of $B$ can be written as a scalar multiple of any one column of $B$, and $B$ is of dimension $m \times n$, $B$ can be written in the form

$$B = \begin{bmatrix} b_{11} & k_2 b_{11} & \dots & k_n b_{11} \\ b_{21} & k_2 b_{21} & \dots & k_n b_{21} \\ \vdots & \vdots & & \vdots \\ b_{m1} & k_2 b_{m1} & \dots & k_n b_{m1} \end{bmatrix}$$

Then, $\Lambda_1$ can be written as

$$\Lambda_1 = (1 + k_2^2 + k_3^2 + \ldots + k_n^2) \begin{bmatrix} b_{11}^2 & b_{11}b_{21} & \dots & b_{11}b_{m1} \\ b_{11}b_{21} & b_{21}^2 & & \\ \vdots & & \ddots & \\ b_{11}b_{m1} & & & b_{m1}^2 \end{bmatrix}$$

B has $(m + n - 1)$ unknowns in need of being recovered, namely $b_{11}, b_{21}, \ldots, b_{m1}, k_2, k_3, \ldots, k_n$. From $\Lambda_1$, we can only recover the ratios of the conductances, that is

$$\frac{\lambda_{21}}{\lambda_{11}} = \frac{b_{21}}{b_{11}}, \quad \frac{\lambda_{31}}{\lambda_{11}} = \frac{b_{31}}{b_{11}}, \quad \ldots, \quad \frac{\lambda_{m1}}{\lambda_{11}} = \frac{b_{m1}}{b_{11}} \tag{11}$$

The other entries of $\Lambda_1$ only provide similar ratios which are easily derived from (11). Thus, if any one of the $b_{i1}$ values is known, the remaining values can be recovered as well. Otherwise, we can only determine the ratios between any two conductances.

If the $b_{i1}$ values can be recovered, only then will two equations for the $k_i$ values make themselves known, derived from $A$ and $\Lambda_1$. These two matrices give us values $\alpha$ and $\beta$ such that

$$1 + k_2^2 + k_3^2 + \ldots + k_n^2 = \alpha \qquad \text{from } \Lambda_1$$

$$1 + k_2 + k_3 + \ldots + k_n = \beta \qquad \text{from } A$$

The $k_i$ values, since there are exactly two equations, can only be recovered if $n \leq 3$, that is, the network $\Gamma$ has no more than 3 interior nodes.

# 4 Matrix $C$

## 4.1 When $B$ Has Full Rank

Let $B$ be a matrix of dimension $m \times n$ whose columns are linearly independent. This implies that $n \leq m$ and $B$ has rank $n$, as does $B^T$. Then the $n \times n$ matrix $B^T B$ is also of rank $n$, and has an inverse $(B^T B)^{-1}$. Using the matrix $\Lambda_2 = BCB^T$, consider

$$(B^T B)^{-1} B^T \Lambda_2 B (B^T B)^{-1} = (B^T B)^{-1} B^T BCB^T B (B^T B)^{-1} = C$$

Thus $C$ can be recovered when $B$ has linearly independent columns.

## 4.2 Characterizing the $\Lambda_n$ Sequence

The $\Lambda(\lambda)$ series contains infinitely many terms, but only finitely many of them have the potential to be useful. The following theorem places an upper bound on the number of terms which are not redundant.

**Theorem 4.1** *If $C$ is $n \times n$ the terms $\Lambda_{n+1}, \Lambda_{n+2}, \ldots$ can be written in terms of $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$.*

*Proof.* Let $p(\lambda) = det(\lambda I - C)$ be the characteristic equation of $C$. Then, because $C$ has dimension $n \times n$, $p(\lambda)$ must have degree $n$ with leading coefficient 1. As the characteristic equation of $C$, $p(\lambda)$ also satisfies the property that $p(C) = 0$.

Let $p(\lambda)$ be rewritten as

$$p(\lambda) = \lambda^n - \alpha_n \lambda^{n-1} - \alpha_{n-1} \lambda^{n-2} - \ldots - \alpha_2 \lambda - \alpha_1 \tag{12}$$

Then, substituting $C$ into (12), we get

$$p(C) = C^n - \alpha_n C^{n-1} - \alpha_{n-1} C^{n-2} - \ldots - \alpha_2 C - \alpha_1 I = 0 \tag{13}$$

Rearranging the terms of (13), $C^n$ can be written in terms of the lower powers of $C$:

$$C^n = \alpha_n C^{n-1} + \alpha_{n-1} C^{n-2} + \ldots + \alpha_2 C + \alpha_1 I$$

Now consider $\Lambda_{n+1}$:

$$\begin{aligned}
\Lambda_{n+1} &= BC^n B^T \\
&= B[\alpha_n C^{n-1} + \alpha_{n-1} C^{n-2} + \ldots + \alpha_2 C + \alpha_1 I]B^T \\
&= \alpha_n (BC^{n-1}B^T) + \alpha_{n-1}(BC^{n-2}B^T) + \ldots + \alpha_2(BCB^T) + \alpha_1(BB^T) \\
&= \alpha_n \Lambda_n + \alpha_{n-1}\Lambda_{n-1} + \ldots + \alpha_2 \Lambda_2 + \alpha_1 \Lambda_1
\end{aligned}$$

Thus $\Lambda_{n+1}$ can be written as a linear combination of $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$. We can now consider $\Lambda_{n+2}$:

$$\begin{aligned}
\Lambda_{n+2} &= BC^{n+1} B^T \\
&= B[C(\alpha_n C^{n-1} + \alpha_{n-1} C^{n-2} + \ldots + \alpha_2 C + \alpha_1 I)]B^T \\
&= B[\alpha_n C^n + \alpha_{n-1} C^{n-1} + \ldots + \alpha_2 C^2 + \alpha_1 C]B^T \\
&= \alpha_n (BC^n B^T) + \alpha_{n-1}(BC^{n-1}B^T) + \ldots + \alpha_2(BC^2B^T) + \alpha_1(BCB^T) \\
&= \alpha_n \Lambda_{n+1} + \alpha_{n-1}\Lambda_n + \ldots + \alpha_2 \Lambda_3 + \alpha_1 \Lambda_2
\end{aligned}$$

The $\Lambda_{n+1}$ term can be replaced by the linear combination of $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ above, allowing $\Lambda_{n+2}$ to be written in terms of only $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$.

Continuing in this manner, all terms $\Lambda_{n+1}, \Lambda_{n+2}, \ldots$ can be written in terms of $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$.

### 4.3 When $C$ is $2 \times 2$, $B$ has rank 1

Let $B$ be a matrix of dimension $m \times n$. If $m < n$, that is, if the network has more interior nodes than boundary nodes, then $B$ is guaranteed to have linearly dependent columns. It is also possible that the columns of $B$ may be linearly dependent (i.e. $B$ has rank $< n$) even when $m \geq n$.

In the case that B is $m \times 2$ with rank 1, there is a real constant $k$ such that $B$ can be written in the form

$$B = \begin{bmatrix} b_{11} & kb_{11} \\ b_{21} & kb_{21} \\ \vdots & \vdots \\ b_{m1} & kb_{m1} \end{bmatrix}$$

Let $c$ be the conductivity between the two interior nodes, $\Sigma_1 = -c - \sum b_{i1}$, and $\Sigma_2 = -c - k\sum b_{i1}$. Then

$$C = \begin{bmatrix} \Sigma_1 & c \\ c & \Sigma_2 \end{bmatrix}$$

Looking at $\Lambda_2 = BCB^T$ we get

$$\Lambda_2 = \begin{bmatrix} b_{11} & kb_{11} \\ b_{21} & kb_{21} \\ \vdots & \vdots \\ b_{m1} & kb_{m1} \end{bmatrix} \begin{bmatrix} \Sigma_1 & c \\ c & \Sigma_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & \dots & b_{m1} \\ kb_{11} & kb_{21} & \dots & kb_{m1} \end{bmatrix}$$

Let $\lambda_{11} = (\Lambda_2)_{11}$. Then

$$\begin{aligned} \lambda_{11} &= (b_{11}\Sigma_1 + kb_{11}c)b_{11} + (b_{11}c + kb_{11}\Sigma_2)kb_{11} \\ \tfrac{\lambda_{11}}{b_{11}^2} &= -(\textstyle\sum b_{i1}) - c + kc + kc - k^3(\textstyle\sum b_{i1}) - k^2c \\ &= -c(k^2 - 2k + 1) - (k^3 + 1)(\textstyle\sum b_{i1}) \\ &= -\big(c(k-1)^2 + (k^3+1)(\textstyle\sum b_{i1})\big) \end{aligned}$$

Solving for $c$ results in

$$c = \frac{-\lambda_{11} - b_{11}^2(k^3+1)(\sum b_{i1})}{b_{11}^2(k-1)^2}$$

Thus, when $k \neq 1$, $C$ is recoverable in a network of $m$ boundary nodes and 2 interior nodes. However, it is possible for $C$ to be irrecoverable when $k = 1$, that is, when the columns of $B$ are identical.

# 5 Unfinished Business

This section concerns paths begun but not yet finished.

## 5.1 Recovering the Eigenvalues of $C$

As stated in theorem (4.1) and its proof, the $\Lambda_n$ sequence forms a recurrence relation based on the characteristic polynomial of $C$. The theorem deals only with the existence of the relation. What is as yet unknown is whether the relation is recoverable by examining the $\Lambda_n$ sequence, though it appears that it should be. If we can actually determine the relation, that is, the $\alpha_i$ values in section 4.2, then we can recover the characteristic polynomial of $C$. The characteristic polynomial (12) has the eigenvalues of $C$ as its roots. Thus, there may be a way to determine the eigenvalues of $C$, which would possibly help in recovering $C$ as well.

## 5.2 Nonrecoverable Edges of $C$

**Conjecture 5.1** *Suppose the ith and jth columns of $B$ are identical. Then the i,jth element of $C$, $c_{ij}$ is not recoverable.*

Let $B$ be a $3 \times 3$ matrix of rank 1. Then $B$ can be written as

$$B = \begin{bmatrix} b_{11} & k_1 b_{11} & k_2 b_{11} \\ b_{21} & k_1 b_{21} & k_2 b_{21} \\ b_{31} & k_1 b_{31} & k_2 b_{31} \end{bmatrix}$$

for some real values $k_1, k_2$, and C is then a $3 \times 3$ matrix in the form

$$C = \begin{bmatrix} \Sigma_1 & c_{12} & c_{13} \\ c_{12} & \Sigma_2 & c_{23} \\ c_{13} & c_{23} & \Sigma_3 \end{bmatrix}$$

where each $\Sigma_i$ is the negative sum of row entries of the Kirchhoff matrix. That is,

$$\begin{aligned} \Sigma_1 &= -(b_{11} + b_{21} + b_{31} + c_{12} + c_{13}) \\ \Sigma_2 &= -(k_1 b_{11} + k_1 b_{21} + k_1 b_{31} + c_{12} + c_{23}) \\ \Sigma_3 &= -(k_2 b_{11} + k_2 b_{21} + k_2 b_{31} + c_{13} + c_{23}) \end{aligned} \tag{14}$$

$\Lambda_2 = BCB^T$, and if we examine $(\Lambda_2)_{11}$ we get

$$\begin{aligned} (\Lambda_2)_{11} &= b_{11}\left[b_{11}(\Sigma_1 + k_1 c_{12} + k_2 c_{13})\right] + k_1 b_{11}\left[b_{11}(c_{12} + k_1 \Sigma_2 + k_2 c_{23})\right] \\ &\quad + k_2 b_{11}\left[b_{11}(c_{13} + k_1 c_{23} + k_2 \Sigma_3)\right] \end{aligned}$$

Dividing through by the $b_{11}$ terms, and replacing the $\Sigma_i$ terms, the equation becomes

$$\begin{aligned} \tfrac{(\Lambda_2)_{11}}{b_{11}^2} &= (-\Sigma b_{i1} - c_{12} - c_{13}) + k_1 c_{12} + k_2 c_{13} + k_1 c_{12} + k_1^2(-k_1 \Sigma b_{i1} - c_{12} - c_{23}) + k_1 k_2 c_{23} \\ &\quad + k_2 c_{13} + k_1 k_2 c_{23} + k_2^2(-k_2 \Sigma b_{i1} - c_{13} - c_{23}) \\ &= -\Sigma b_{i1}(1 + k_1^3 + k_2^3) - c_{12}(k_1^2 - k_1 - k_1 + 1) - c_{13}(k_2^2 - k_2 - k_2 + 1) - c_{23}(k_1^2 - k_1 k_2 - k_1 k_2 + k_2^2) \\ &= -\Sigma b_{i1}(1 + k_1^3 + k_2^3) - c_{12}(k_1 - 1)^2 - c_{13}(k_2 - 1)^2 - c_{23}(k_1 - k_2)^2 \end{aligned}$$

Note that columns 1 and 2 of $B$ being identical corresponds to $k_1 = 1$. In this case, the term containing $c_{12}$ drops out of the above equation, and cannot be used to recover that particular edge. The same follows if any other set of columns of $B$ are identical. This is true for the equations of all entries of $\Lambda_2$. Looking further to $\Lambda_3 = BC^2B^T$, we can again look at the entry in position 1,1:

$$\begin{aligned} (\Lambda_3)_{11} &= b_{11}^2\left[(\Sigma_1^2 + c_{12}^2 + c_{13}^2) + k_1(c_{12}\Sigma_1 + c_{12}\Sigma_2 + c_{13}c_{23}) + k_2(c_{13}\Sigma_1 + c_{12}c_{23} + c_{13}\Sigma_3)\right] \\ &\quad + k_1 b_{11}^2\left[(c_{12}\Sigma_1 + c_{12}\Sigma_2 + c_{13}c_{23}) + k_1(c_{12}^2 + \Sigma_2^2 + c_{23}^2) + k_2(c_{12}c_{13} + c_{23}\Sigma_2 + c_{23}\Sigma_3)\right] \\ &\quad + k_2 b_{11}^2\left[(c_{13}\Sigma_1 + c_{12}c_{23} + c_{13}\Sigma_3) + k_1(c_{12}c_{13} + c_{23}\Sigma_2 + c_{23}\Sigma_3) + k_2(c_{13}^2 + c_{23}^2 + \Sigma_3^2)\right] \end{aligned} \tag{15}$$

Through algebraic manipulations, this equation eventually becomes

$$\begin{aligned}
\frac{(\Lambda_3)_{11}}{b_{11}^2} &= \left(\sum b_{i1}\right)\left[2(k_1+1)(k_1-1)^2c_{12} + 2(k_2+1)(k_2-1)^2c_{13} + 2(k_1+k_2)(k_1-k_2)^2c_{23}\right] \\
&\quad -\left(\sum b_{i1}\right)^2(1+k_1^4+k_2^4) + 2(k_1-1)^2c_{12}^2 + 2(k_2-1)^2c_{13}^2 + 2(k_1-k_2)^2c_{23}^2 \\
&\quad +2(k_1-1)(k_2-1)c_{12}c_{13} + 2(k_1-k_2)(k_1-1)c_{12}c_{23} + 2(k_1-k_2)(k_2-1)c_{13}c_{23}
\end{aligned}$$

As in the previous equation (15), the terms containing $c_{12}$ in this equation drop out when $k_1 = 1$, causing $c_{12}$ to not be recoverable from this equation. That is, when the first two columns of $B$ are identical, $\Lambda_3$ cannot be used to recover $c_{12}$, similar to $\Lambda_2$. The same is true when a different pair of columns of $B$ are identical for the remaining entries of $C$.
While this has not yet been proven, it seems that this should be true of the remaining matrices in the $\Lambda_n$ sequence.

## 5.3 Patterns in the $\Lambda_n$ Sequence

Suppose $B$ has dimension $3 \times 3$ and rank 1. We can write $B$ and $C$ in the form

$$B = \begin{bmatrix} b_{11} & k_1 b_{11} & k_2 b_{11} \\ b_{21} & k_1 b_{21} & k_2 b_{21} \\ b_{31} & k_1 b_{31} & k_2 b_{31} \end{bmatrix}$$

$$\mathrm{C} = \begin{bmatrix} \Sigma_1 & c_{12} & c_{13} \\ c_{12} & \Sigma_2 & c_{23} \\ c_{13} & c_{23} & \Sigma_3 \end{bmatrix}$$

where the $\Sigma_i$ values are as in (14). Then $\Lambda_n$ turns out to be

$$\Lambda_n = y \begin{bmatrix} b_{11}^2 & b_{11}b_{21} & b_{11}b_{31} \\ b_{11}b_{21} & b_{21}^2 & b_{21}b_{31} \\ b_{11}b_{31} & b_{21}b_{31} & b_{31}^2 \end{bmatrix} \tag{16}$$

with $y$ varying with $n$. For $\Lambda_1$,

$$y = 1 + k_1^2 + k_2^2$$

For $\Lambda_2$,

$$y = \Sigma_1 + k_1^2\Sigma_2 + k_2^2\Sigma_3 + 2k_1c_{12} + 2k_2c_{13} + 2k_1c_{23}k_2$$

The $y$ values for each $\Lambda_n$ become increasingly complicated as $n$ increases, but a definite pattern shows up. This pattern can be generalized to a $B$ matrix with higher dimensions.

When $B$ has rank 2 with dimension $3 \times 3$, $B$ can be written as

$$B = \begin{bmatrix} b_{11} & b_{12} & \alpha b_{11} + \beta b_{12} \\ b_{21} & b_{22} & \alpha b_{21} + \beta b_{22} \\ b_{31} & b_{32} & \alpha b_{31} + \beta b_{32} \end{bmatrix}$$

with $C$ as before. Then each $\Lambda_n$ matrix has the form

$$\Lambda_n = \begin{bmatrix} b_{11}^2 x + 2b_{11}b_{12}y + b_{12}^2 z & \begin{matrix} b_{11}b_{21}x + (b_{11}b_{22} + b_{12}b_{21})y \\ +b_{12}b_{22}z \end{matrix} & \begin{matrix} b_{11}b_{31}x + (b_{11}b_{32} + b_{12}b_{31})y \\ +b_{12}b_{32}z \end{matrix} \\ \begin{matrix} b_{11}b_{21}x + (b_{11}b_{22} + b_{12}b_{21})y \\ +b_{12}b_{22}z \end{matrix} & b_{21}^2 x + 2b_{21}b_{22}y + b_{22}^2 z & \begin{matrix} b_{21}b_{31}x + (b_{21}b_{32} + b_{22}b_{31})y \\ +b_{22}b_{32}z \end{matrix} \\ \begin{matrix} b_{11}b_{31}x + (b_{11}b_{32} + b_{12}b_{31})y \\ +b_{12}b_{32}z \end{matrix} & \begin{matrix} b_{21}b_{31}x + (b_{21}b_{32} + b_{22}b_{31})y + b_{22}b_{32}z \\ +b_{22}b_{32}z \end{matrix} & b_{31}^2 x + 2b_{31}b_{32}y + b_{32}^2 z \end{bmatrix} \tag{17}$$

where $x, y, z$ vary with $n$, as in the previous case. For $\Lambda_1$,

$$x = 1 + \alpha^2, \quad y = \alpha\beta, \quad z = 1 + \beta^2$$

For $\Lambda_2$,

$$x = \Sigma_1 + 2\alpha c_{13} + \alpha^2\Sigma_3, \quad y = \alpha\beta\Sigma_3 + \beta c_{13} + \alpha c_{23} + c_{12}, \quad z = \Sigma_2 + 2\beta c_{23} + \beta^2\Sigma_3$$

And so the pattern continues for all of the matrices $\Lambda_n$, with messier values for $x, y$, and $z$.

Clearly, patterns will ensue for the $\Lambda_n$ sequence based on the rank of $B$. Besides the pattern among the $\Lambda_n$ matrices, a pattern also shows up in the general form of the $\Lambda_n$ for a matrix $B$ of rank k. That is, a pattern shows up between the forms (16) and (17), and could easily be extended to $B$ matrices of higher rank.

The question remains as to how useful these patterns are. One possible outcome may be to put a lower bound on the recurrence relation than the one already in place in theorem (4.1). That is, if $C$ is an $n \times n$ matrix, there may be a recurrence relation between the $\Lambda_n$ matrices where there is some $k < n$ such that

$$\Lambda_k = \alpha_k\Lambda_k + \alpha_{k-1}\Lambda_{k-1} + \ldots + \alpha_2\Lambda_2 + \alpha_1\Lambda_1$$

## 5.4 Further Questions

Here are some other questions that have come up, but that have been explored in less detail...

- What happens in the case of larger networks, when there are more than two interior nodes, in terms of recovering $B$? in terms of recovering $C$ when $B$ does not have full rank?
- How does the rank of $B$ affect the recoverability of itself and $C$?
- Is it possible to recover $C$ without first recovering $B$?
- What restrictions must be placed on $B$ and/or $C$ to make them recoverable? In what cases are they not recoverable?

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