The Discrete Transport Equation

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June 18, 2003

Abstract

For background I first describe the continuous transport equation and the forward and inverse problems associated with it (as posed in [1]). I then formulate discrete analogs of these problems.

1 The Continuous Problem

Let $X \subset \mathbf{R}^n$ be an open convex set with piecewise smooth boundary ∂X, and let $v \in \mathbf{S}^{n-1}$ be any direction and $x \in X$. The *Transport Equation*, also known as the *Linearized Boltzmann Equation*, is

$$-v \cdot \nabla\phi(x,v) - \sigma(x,v)\phi(x,v) + \int_{\mathbf{S}^{n-1}} \kappa(x,v,w)\phi(x,w)dw = 0. \qquad (1)$$

This equation describes the flow of a stream of particles as they move through a medium and are absorbed and scattered. The interpretation of the symbols in this equation is as follows.

- The *Flux* ϕ is defined on $X \times \mathbf{S}^{n-1}$, where $\phi(x,v)$ is the flux of particles through the point x and in the direction v. Note that in general $\phi(x,v) \neq \phi(x,-v)$—imagine two beams of photons moving past each other in opposite directions. One does however usually require that $\phi \geq 0$.
- The *Absorption Coefficient* σ is defined on $X \times \mathbf{S}^{n-1}$ and must be nonnegative, and represents the rate at which particles are absorbed as they move through the point x in the direction v. In many situations one takes σ to be independent of direction and depend only on location.
- The *Collision Kernel* κ is defined on $X \times \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ and takes on nonnegative values. $\kappa(x,v,w)$ gives the rate at which particles moving in the w direction at the point x are deflected into the v direction.

In posing the forward and inverse problem in this setting, we will need the following notation. For $x \in \partial X$, let $n(x)$ be the unit outward pointing normal. Define Γ_+ and Γ_- by $\Gamma_\pm = \{(x,v) \in \partial X \times \mathbf{S}^{n-1} | \pm n \cdot v > 0\}$.

Given a nonnegative function ϕ_- on Γ_- as boundary data, the forward problem is to find a function ϕ on $X \times \mathbf{S}^{n-1}$ which satisfies the transport equation and has $\phi|_{\Gamma_-} = \phi_-$.

Let $\phi_+ = \phi|_{\Gamma_+}$. The map $\mathcal{A} : \phi_- \longmapsto \phi_+$ is called the *Albedo operator*. The inverse problem is to calculate σ and κ from $\mathcal{A}$.

2 Now we do it Discretely!

Definition 2.1 *Let $G = (V, V_B, E)$ be a finite directed graph with boundary, and with the following additional conditions.*

1. *Each boundary node $p \in V_B$ has degree one.*
2. *At each interior node p there is a bijection $in(p) \longleftrightarrow out(p)$. The image of an incoming edge e is denoted e^+ and the image of an outgoing edge f is denoted f^-. Note that this implies that each interior node has even degree.*

*Then G is said to be a **Transport Graph**.*

Let $in(p)$ denote the set of edges incoming at the node p and let $out(p)$ denote the set of edges outgoing at node p. If P is a set of nodes in G, then $in(P) = \bigcup_{p\in P} in(p)$ and similarly for $out(P)$. Thus $in(V_B)$ is the set of all edges which have no successor, and $out(V_B)$ is the set of all edges which have no predecessor. The sets $out(V_B$ and $in(V_B)$ take the place of Γ_- and Γ_+ respectively.

The relevant functions for the discrete transport equation are represented on the graph as follows:

- The flux $\phi : E \to \mathbf{R}^+$ is an *edge* function. $\phi(e)$ is interpreted as giving the flux along the edge e in the direction that e points.
- The absorption coefficient is a function $\sigma : V \to (0, 1)$, where $\sigma(p)$ is now interpreted as the probability that a particle coming into an interior node p along any edge will be absorbed before it leaves. We will need σ to be nonzero in order to prove the uniqueness of the solution to the forward problem.
- The collision kernel is a set of functions $\{\kappa_p : out(p) \times in(p) \to [0, \delta) | \forall p \in V\}$, (for δ sufficiently small as will be described shortly) defined at each interior node $p \in V$, and satisfying the condition that $\kappa_p(e^+, e) = 0$, $\forall p \in V$, $\forall e \in in(p)$ since direct transmission is mediated by absorption rather than collision. We have the interpretation that $\kappa_p(e, f)$ is the probability that a particle entering node p along edge f will leave along the edge e.

Replacing each term in the continuous equation with its discrete analog, one obtains

$$-\left[\phi(e^+) - \phi(e)\right] - \sigma(p)\phi(e) + \sum_{g\in in(p)} \kappa_p(e^+, g)\phi(g) = 0.$$

However, this equation does not correctly conform to the probabilistic interpretation of σ, so we modify it by replacing $\sigma(p)$ with: (absorption probability) + (total collision probability for particle incoming along edge e). Therefore, the correct equation is

$$-\left[\phi(e^+) - \phi(e)\right] - \left[\sigma(p) + \sum_{f \in out(p)} \kappa_p(f, e)\right] \phi(e) + \sum_{g \in in(p)} \kappa_p(e^+, g)\phi(g) = 0,$$

which may be written in a more compact form as,

$$\phi(e^+) = \sum_{f \in in(p)} \tau_p(e^+, f)\phi(f). \tag{2}$$

Here τ_p is the *transport function at node p*. Its values are given by

$$\tau_p(e^+, f) = \begin{cases} 1 - \sigma(p) - \sum_{g \in out(p)} \kappa_p(g, f) & f = e \\ \kappa(e^+, f) & f \neq e \end{cases} \tag{3}$$

The value of $\tau_p(e^+, f)$ can be interpreted as the probability that a particle entering node p along the edge f will leave p along the edge e^+. For eq. 3 to make sense under this interpretation, it is apparent that the entries in the collision kernel must be sufficiently small. This consideration places an upper bound on δ. We will thus assume that δ is small enough so that the values of τ_p are all in the range $[0, 1)$. In the continuous setting there is a similar constraint on κ for there to be a well posed forward problem.

Definition 2.2 *A* ***transport network*** Γ *is pair* (G, τ)*, where* G *is a transport network, and* τ *is a transport function (which can be decomposed into an absorption coefficient* $\sigma : V \to [0, 1]$ *and a collision kernel* κ *(with* δ *sufficiently small).*

Number the elements of $in(p)$ as $e_1, e_2, \ldots, e_n$. We form a matrix T_p, called the *transport matrix at node p*, with entries $[T_p]_{ij} = \tau_p(e_i^+, e_j)$. Let ϕ be a nonnegative function on the edges of Γ and let Φ_p^{in} denote the column vector $[\phi(e_1), \phi(e_2), \ldots, \phi(e_n)]^T$; define Φ_p^{out} similarly. To say that ϕ satisfies the transport equation is to say that $\forall p \in V$,

$$\Phi_p^{out} = T_p \Phi_p^{in}. \tag{4}$$

In this case we say that ϕ is a *τ-flow*.

2.1 The Discrete Forward Problem

Given transport network Γ and a nonnegative function ψ_- defined on $out(V_B)$, the forward problem is to find a τ-flow ϕ such that $\phi|_{out(V_B)} = \psi_-$. Equation 2 gives exactly one linear equation for each edge with an unknown flux. Putting these together results in a system of n equations with n unknowns, where n is the number of edges with un-imposed fluxes. This system is of the form $Qx = b$.

Lemma 2.1 *The matrix Q has the following properties:*

1. *Each diagonal entry q_{ii} of Q is 1.*
2. *The non-diagonal entries q_{ij} of Q all satisfy $-1 < q_{ij} \leq 0$.*
3. *In each row or column there is at most a single non-diagonal entry that is larger than δ in magnitude.*

Proof: Property (1) follows directly from the form of eq. 2. Property (2) also follows directly from eq. 2 combined with the fact that the values of the transport function are always in the range $[0, 1)$. To prove property (3), first note that $\tau_p(e, f) < \delta$ if f is not the successor of e. From eq. 2, in each row of Q there is at most a single non-diagonal entry that is greater than δ because each edge has only a single predecessor and only the successor-predecessor term can be greater than δ. Similarly, each edge has at most one successor, so by the same reasoning in each column of Q there is at most a single non-diagonal entry that is greater than δ. □

Theorem 2.1 *Let σ_0 be the minimum value of σ. If $\delta < \sigma_0/(n-2)$ then Q is diagonally dominant and hence nonsingular.*

Proof: Computing a row sum of Q in the ith row,

$$\sum_{j=1}^{n} q_{ij} = q_{ii} + \sum_{j=1, j \neq i}^{n} q_{ij} > 1 - (n-2)\delta - (1 - \sigma_0) = \sigma_0 - (n-2)\delta$$

And clearly if $\delta < \sigma/(n-2)$ then this quantity is positive. A nearly identical argument shows that the column sums are all positive. □

Corollary 2.1 *Given a transport network Γ, $\exists \delta$ such that if $\forall p \in V$, $\|\kappa_p\| < \delta$ then there is a solution to the forward problem, and this solution is unique.*

2.2 The Discrete Inverse Problem

Given ψ_- and assuming that the forward problem has a unique solution, we define $\psi_+ = \phi|_{in(V_B)}$. The Albedo operator is then the linear map

$$\mathcal{A} : \psi_- \longmapsto \psi_+.$$

The inverse problem is to recover information about σ and κ from $\mathcal{A}$.

References

[1] Plemen Stefanov, Gunther Uhlmann, "Optical Tomography in two dimensions", 2002. Preprint available at www.math.purdue.edu/∼ stefanov