The Map L and Partial Recovery in Circular Planar Non-Critical Networks

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Abstract

It has been proven [1] that a circular planar electrical network is recoverable (i.e. the conductance function γ is uniquely determined by the response matrix Λ) if and only if the underlying graph is critical. In this paper I expand on these results by showing that for critical circular planar graphs, the map L which sends γ to Λ_γ is a diffeomorphism. I then construct a fibration of L for an arbitrary non-critical graph and use this structure to determine the extent to which the network can be partially recovered. More specifically, given a circular planar graph $G = (V, V_B, E)$ and a response matrix Λ compatible with this graph, I describe a method for constructing an explicit parametrization of the set $L^{-1}(\Lambda)$ of conductance functions γ such that the electrical network $\Gamma = (G, \gamma)$ has response matrix Λ.

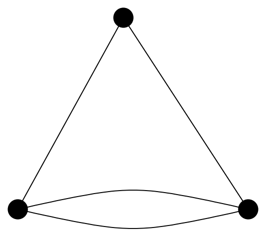
1 Introduction

Consider an electrical network Γ which consists of a pair (G, γ) where $G = (V, V_B, E)$ is a graph with boundary and γ is a conductivity function defined on the edges of G. We are interested in the problem of computing the conductance γ from the response matrix Λ. For circular planar networks this problem is well understood—the conductivity is uniquely determined by Λ if and only if the underlying graph is critical. In §3 I show that this correspondence is actually a diffeomorphism (a result which is stated but not proven in [1]).

Even though unique recovery is impossible on a non-critical graph, we may still be able to obtain partial information about the conductance. For example, consider a Δ network, but with one edge replaced by two edges in parallel (see figure 1). The conductance on the two parallel edges cannot be computed from Λ; only the sum of the two can be recovered. Nevertheless, the conductances of the two single edges can be exactly calculated from Λ.

If the graph is not critical then one cannot uniquely determine a γ from Λ; there will be many conductivity functions which result in electrically equivalent networks. These sets are the fibers of the map L which sends γ to Λ_γ.

Figure 1: Example of a partially recoverable network.



2 Preliminaries

The following notation will make the discussion more clear. Suppose G is a circular planar graph with n boundary nodes and m edges.

Definition 2.1 *The symbol $\pi(G)$ will denote the set of connections $P \leftrightarrow Q$ through G where the $(P;Q)$ are circular pairs.*

Definition 2.2 *Let $\Omega(\pi)$ be the set of $n \times n$ matrices A which satisfy the following properties:*

- *A is symmetric.*
- *The sum of the entries is each row is 0.*
- *$(-1)^k \det A(P;Q) > 0$ for each pair $(P;Q) \in \pi$, and all other sub-determinants are 0.*

Thus $\Omega(\pi(G))$ is the set of response matrices which are possible for a network with underlying graph G. We say that Λ and G are **compatible** if $\Lambda \in \Omega(\pi(G))$. If G has m edges, then a conductivity function γ on G may be thought of as a point in $(\mathbb{R}^+)^m$. The map

$$L_G : (\mathbb{R}^+)^m \longrightarrow \Omega(\pi(G))$$

sends a conductivity γ on G to the response matrix of the network (G, γ). Later on I will work with the L maps or multiple graphs simultaneously, so to avoid confusion I will often label L with a subscript specifying the graph for which it gives response matrices.

3 L is a diffeomorphism when G is critical

In §4.6 of [1] there is a proof that L is a diffeomorphism for a specific class of critical graphs; the proof proceeds by way of a computation which works for any critical graph, but at one step it hinges on a lemma which is only proven for the case of the well-connected circular graphs. I give a general proof of lemma 3.3, and from this the general case of the theorem follows.

In proving lemma 3.3, I will need to construct a set of special functions which are γ-harmonic and satisfy some addition relationships. In §4.4 of [1] the set of special functions is constructed on the well-connected circular graphs G_n. I generalize the construction of special functions to an arbitrary critical circular planar graph.

Lemma 3.1 CONSTRUCTION OF SPECIAL FUNCTIONS *Let* $\Gamma = (G, \gamma)$ *be a critical circular planar network, and let* e *be either a boundary spike or a boundary edge. Then there exist* ***special functions*** f *and* g *which are* γ*-harmonic, and satisfy the addition condition that for each edge with endpoints* (p, q)*:*

$$\begin{array}{ll} (f(p) - f(q))(g(p) - g(q)) = 0 & \textit{if } pq \neq e \\ (f(p) - f(q))(g(p) - g(q)) \neq 0 & \textit{if } pq = e \end{array}$$

Proof: Let $\mathcal{M}$ be the medial graph of G, embedded in the disc, and let A and B be the two geodesics whose point of intersection v_e corresponds to the edge e. Recall that G is critical so $\mathcal{M}$ is lensless, and therefore these two geodesics intersect only once at the point v_e Furthermore, e is a boundary edge or boundary spike, so these two geodesics intersect each other at v_e before they intersect any other geodesics. Let the endpoints of A and B be (a_1, a_2) and (b_1, b_2) respectively, with a_1 and b_1 being the endpoint of each geodesic that is closer to the point of intersection. Further, suppose that a_1 is immediately clockwise from b_1. (If this is not so, then simply swap the names of A and B.)

If p and q are two points on the boundary circle then we will denote the clockwise open arc from p to q by $\widehat{pq}$ and the counterclockwise open arc by $\widehat{qp}$.

The cells of the cell-complex constructed from $\mathcal{M}$ can be 2-colored, so color them with black and white so that the black segments of the boundary circle correspond to the boundary nodes.

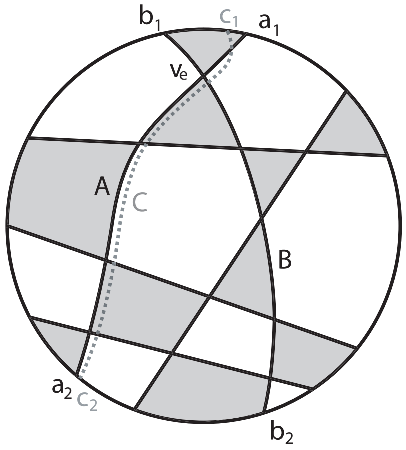
I will first describe the construction for the case that e is a boundary spike. The boundary edge case is a simple modification which I will describe later.

Insert two auxiliary points on the boundary circle: a point c_1 immediately counterclockwise from a_1 and a point c_2 immediately counterclockwise from a_2. Now add an auxiliary chord C with endpoints c_1 and c_2 such that C crosses A between v_e and a_1, and between this crossing and c_2 it runs parallel to A and close enough so that C crosses only the geodesics that A crosses and in the same order. This is shown in figure 2.

The chord C divides the disc into two regions R_1 and R_2, with boundary given by $C \cup \widehat{c_1c_2}$ and $C \cup \widehat{c_2c_1}$ respectively. Each of these regions is homeomorphic to a disc, so $\mathcal{M}_1 = \mathcal{M} \cap R_1$ and $\mathcal{M}_2 = \mathcal{M} \cap R_2$ may each be thought of as medial graphs themselves. We color the cells of these two subgraphs as they were colored in $\mathcal{M}$.

Let G_1 and G_2 be the graphs associated to $\mathcal{M}_1$ and $\mathcal{M}_2$ by letting black cells correspond to nodes. Each of these is a subset of G (with boundary nodes now designated so that they correspond to the black boundary intervals on the

Figure 2: Placement of the auxiliary chord C for a boundary spike.



boundary of R_1 or R_2). This gives a natural division of Γ into two subnetworks Γ_1 and Γ_2; define Γ_1 as G_1 together with the restriction of the conductivity γ to G_1, and use the analogous definition for Γ_2. From Γ_1 and Γ_2 we can reconstruct the original network Γ by identifying and interiorizing the set of boundary nodes which correspond to the black intervals that are at least partially contained in C. For illustration see figure 3.

Let w the boundary node corresponding to the black interval $\widehat{b_1 a_1}$. This is the boundary node at the end of the boundary spike. Let v be the node at the other end of the spike. Notice that by construction w is not connected to anything in Γ_1 since v is not present.

I will construct the desired special function f on each part separately and then piece it together to form f on all of Γ. Following the same steps with the geodesic B and with clockwise and counterclockwise reversed will lead to the construction of the special function g.

Consider the medial graph $\mathcal{M}_1$. Now we will make use of the Cut-Point Lemma of [1] (p. 152) by inserting two cut points X and Y at c_2 and c_1 respectively. Since $\mathcal{M}$ is lensless, no geodesic intersects A twice, and hence no geodesic intersects C twice, so there are no re-entrant geodesics boundary arc of R_1 formed by C. By the Cut-Point Lemma,

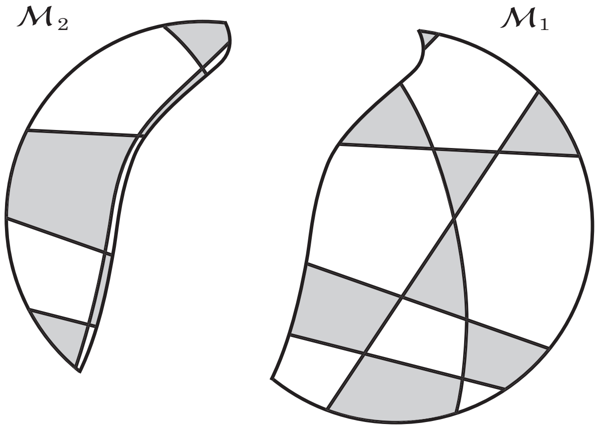
$$m(X,Y) + r(X,Y) - n(X,Y) = 0 \tag{1}$$

where

- $m(X,Y) =$ the maximum integer k such that there is a k-connection which respects the cut-points X and Y.

- $r(X,Y) =$ the number of re-entrant geodesics in the clockwise open boundary arc from X to Y.

Figure 3: The chord C separates the medial graph into two subgraphs $\mathcal{M}_1$ and $\mathcal{M}_2$ for the case of a boundary spike.



- $n(X,Y)$ = the number of black intervals which are entirely within the clockwise open boundary arc from X to Y.

In our situation, $r(X,Y) = 0$, so eq. 1 reduces to saying that the size largest connection which respects the cut-points is given exactly by the number of black intervals along C. Denote this largest connection by the circular pair (of black boundary intervals) $(\tilde{P}, \tilde{Q})$, where the intervals of $\tilde{P}$ are *all* the black intervals that are entirely contained in C, and the intervals of $\tilde{Q}$ are some subset of the black intervals that lie in $\widehat{YX}$. See figure 4.

Let (P,Q) denote the circular pair of boundary nodes in Γ_1 corresponding to the circular pair of boundary intervals $(\tilde{P}, \tilde{Q})$ in $\mathcal{M}_1$. Since there is a connection $\tilde{P} \leftrightarrow \tilde{Q}$ through $\mathcal{M}_1$, there is a naturally corresponding connection $P \leftrightarrow Q$ through G_1.

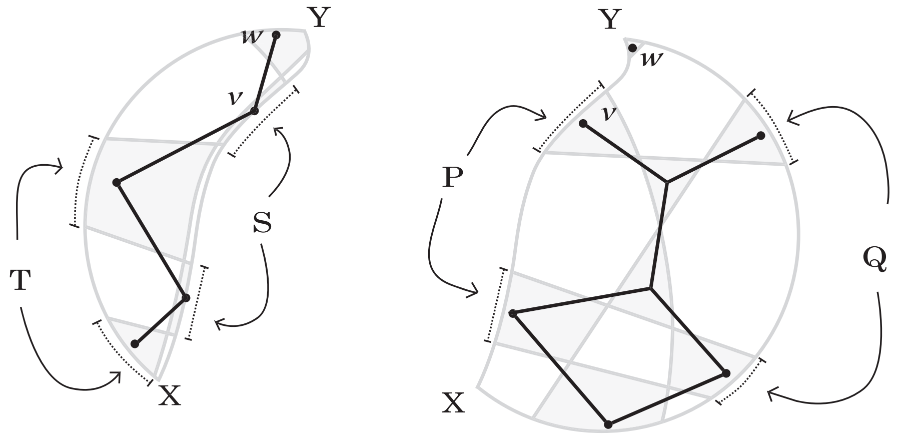
Consider imposing the following mixed boundary conditions on the network Γ_1:

- A potential of 1 at the boundary node w (corresponding to the black interval $\widehat{b_1 a_1}$).
- A potential of 0 at each other boundary node that is in the complement of P.
- Zero current flow into the network at each node in Q.

By theorem 4.2 of [1] (p. 70) and in light of the existence of the connection $P \leftrightarrow Q$, there is a unique γ-harmonic function f_1 on Γ_1 that satisfies these boundary data. The function

$$f_1(p) = \begin{cases} 1 & \text{if } p = w \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Figure 4: The relevant connections $P \leftrightarrow Q$ and $S \leftrightarrow T$ in Γ_1 and Γ_2.



is a solution, so it is the unique solution.

Now we shift our attention to Γ_2 and $\mathcal{M}_2$. Inserting cut-points X at c_2 and Y at c_1 on the boundary of R_2, the Cut-Point Lemma again implies that the size of the largest k-connection is exactly given by the number of black intervals that are entirely contained in the interval C. Let $(\tilde{S}, \tilde{T})$ denote this connection, where the intervals of $\tilde{S}$ are entirely contained in C and the intervals of $\tilde{T}$ are entirely contained in $\widehat{XY}$. The set S corresponds exactly to the set P.

Now impose the following mixed boundary conditions on Γ_2:

- A potential of 1 at the boundary node w (the node that corresponds to the black interval $\widehat{b_1 a_1}$). Note that this node is in the complement of T because the connection $S \leftrightarrow T$ must respect the cut-points, and one of these was placed in this interval.

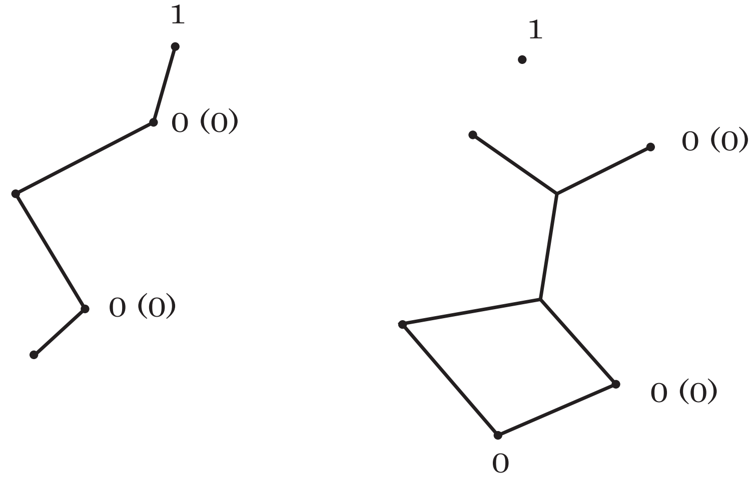
- A potential of 0 at all other boundary nodes that are in the complement of T.

- Zero current flow into the network at each node in S.

Once again, by theorem 4.2 of [1] (p. 70) and in light of the existence of a connection $S \leftrightarrow T$, there is a unique γ-harmonic function f_2 on Γ_2 that satisfies these boundary data. The imposed boundary conditions described above are shown in figure 5.

Now we combine f_1 on Γ_1 and f_2 on Γ_2 to produce a γ-harmonic function f on all of Γ. By construction we have ensured that f_1 and f_2 agree where they overlap and the currents flowing between the two subgraphs agree because they are all zero. This f is one of the desired special functions. It has the property that

Figure 5: Boundary conditions imposed on Γ_1 and Γ_2.



- f is γ-harmonic.
- $f(p) = 0$ for all nodes $p \neq w$ corresponding to black cells that are on the R_1 side of the geodesic A.
- $(f(w) - f(v)) \neq 0$. *I.e.* there is a non-zero current flowing through the boundary spike.

It is clear that constructing the special function g similarly so that

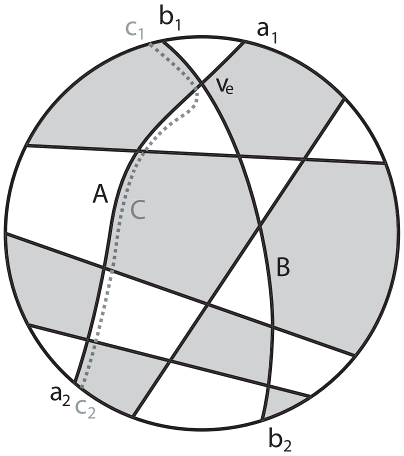
- g is γ-harmonic.
- $g(p) = 0$ for all nodes $p \neq w$ corresponding to black cells that are on the R_2 side of the geodesic B.
- $(g(w) - g(v)) \neq 0$.

leads to a pair of special functions which satisfy the desired properties.

I now describe how to modify the construction for the case of a boundary edge. Insert the point c_1 immediately counterclockwise from b_2, and insert c_2 immediately counterclockwise from b_2. Let the auxiliary chord C have endpoints c_1 and c_2 and let it cross the geodesic A between v_e and the second node along A (counting with v_e as the first). Let it run parallel and close to A from the crossing to c_2. This is shown in figure 6.

Let w be the boundary node corresponding to the black interval immediately counterclockwise from b_1 and let v be the boundary node corresponding to the black interval immediately clockwise from a_1. These nodes w and v are the

Figure 6: The chord C separates the medial graph into two subgraphs for the case of a boundary edge.



endpoints of the boundary edge e. Notice that in the subnetwork Γ_1, e is both a boundary edge and a boundary spike simultaneously.

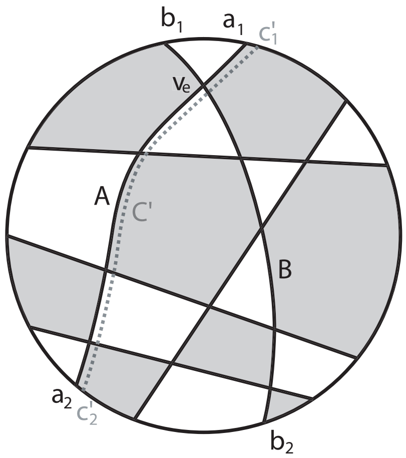
Consider first Γ_1. There is a connection $\tilde{P} \leftrightarrow \tilde{Q}$ where $\tilde{P}$ is the set of black intervals that are entirely contained in C and all the intervals of $\tilde{Q}$ are disjoint from C. Let P and Q be the sets of boundary nodes corresponding to $\tilde{P}$ and $\tilde{Q}$. Here again, theorem 4.2 of [1] (p. 70) guarantees that there is a unique solution f_1 to the Dirichlet problem on Γ_1 with the boundary conditions

- A potential of 1 at w.
- A potential of 0 at all nodes $p \neq w$ in the complement of P.
- Zero current into the network at each node in Q.

Note that we must choose the connection so that the node v is not in Q. Such a connection exists for the following reason. Suppose we had inserted c_1 just clockwise from a_1 rather than counterclockwise from b_1 and drawn a chord C' parallel to A along its entire length (rather than bending it along B near the c_1 end). See figure 7. The black intervals that are entirely contained in C' exactly correspond to the black intervals that were entirely contained in the old chord C. This new chord yields a subnetwork Γ_1' that is precisely Γ_1 with the edge e and the node w both deleted. Applying the Cut-point Lemma to Γ_1' with the cut-points X and Y placed at the ends of this chord C' show the existence of a connection from the same set of black intervals $\tilde{P}$ to a set of intervals that are not contained in C' (and hence not contained in C) and also do not include the interval corresponding to the node v. Thus there is a connection $P \leftrightarrow Q$ where

P is the set of boundary nodes of Γ_1 that correspond to black intervals entirely contained in C and Q does not contain v.

Figure 7: Placement of the second auxiliary chord C'.



Now consider Γ_2. Using the Cut-Point Lemma, there is a connection $\tilde{S} \leftrightarrow \tilde{T}$ through the medial graph $\mathcal{M}_2$, where $\tilde{S} = \tilde{P}$ and $\tilde{T}$ is disjoint from $\tilde{S}$ and does not contain w. Let $T \leftrightarrow S$ be the corresponding connection through the graph G_2. We set f_2 to be the γ-harmonic function which satisfies the boundary conditions

- A potential of 1 at w.
- A potential of 0 at each node in T.
- Potential of 0 and 0 current flow into the network at each boundary node that corresponds to a black interval entirely contained in C.

It is clear that these boundary conditions are allowed.

By patching together f_1 and f_2 one forms f on all of Γ, and this is the desired special function. The conjugate special function g is once again obtained by repeating the construction with clockwise and counterclockwise exchanged, and the roles of the geodesics A and B swapped. □

Lemma 3.2 *Let Γ and Γ' be a circular planar networks where Γ' is obtained from Γ by either contracting a boundary spike or deleting a boundary edge. Suppose u is a γ-harmonic function on Γ'. Then there is a unique extension of u to a γ-harmonic function on Γ.*

Proof: The case where a boundary edge is deleted is trivial. No edges are added at interior nodes when the deleted edge is replaced, so the original u

is still γ-harmonic at each interior node. Only the currents flowing into the network change.

For the case of a boundary spike, let p be the interior endpoint of the spike and let q be the boundary node endpoint of the spike. Let ϕ_p be the current flowing into Γ' at node p. Then in Γ, $u(q)$ is uniquely determined by requiring that the current flowing through the spike into node p be exactly ϕ_p. Writing this relation out,

$$\gamma(pq)(u(q) - u(p)) = \phi_p$$

and so u(q) is given by

$$u(q) = \phi_p/\gamma(pq) + u(p)$$

This is the unique γ-harmonic extension of u to all of Γ. □

The following is a generalization of lemma 4.5 on p. 79 of [1]. This result supplies the key justification in the proof of theorem 3.1.

Lemma 3.3 *Let Γ be a critical circular planar network, and let κ be any real-valued function on the edges of Γ. Suppose that for all γ-harmonic functions u and w,*

$$\sum_{pq \in E} \kappa(pq)(u(p) - u(q))(w(p) - w(q)) = 0$$

Then κ is identically 0.

Proof: Let m be the number of edges in Γ. The network is critical, so it has a boundary spike or a boundary edge—call it e_m, and removing this edge (by contraction if it is a boundary spike and deletion if it is a boundary edge) yields a new critical network Γ_{m-1}. This network again has either a boundary spike or a boundary edge—call it e_{m-1}. By repeating this step we arrive at a sequence of critical circular planar networks

$$\Gamma_1, \Gamma_2, \ldots, \Gamma_m = \Gamma$$

where each Γ_i is obtained from Γ_{i+1} by removing a boundary spike or boundary edge e_{i+1}. Terminate the process at Γ_1, when there are no interior nodes and only a single edge remaining. Thus all the edges in the graph are numbered $e_1, e_2, \ldots, e_m$.

For each $k = 1, \ldots, m$, by lemma 3.1 there exist special functions f_k and g_k on Γ_k so that $(f_k(p) - f_k(q))(g_k(p) - g_k(q)) \neq 0$ only for $pq = e_k$. Clearly

$$\sum \kappa(pq)(f_m(p) - f_m(q))(g_m(p) - g_m(q)) = 0$$

implies that $\kappa(e_m) = 0$. We now proceed inductively; assume that the hypotheses imply that $\kappa(e_j) = 0$ for $j = k, (k+1), \ldots, m$. I will show that the hypotheses also imply that $\kappa(e_{k-1}) = 0$.

By repeated application of lemma 3.2 the special functions f_{k-1} and g_{k-1} extend to γ-harmonic functions $\tilde{f}_{k-1}$ and $\tilde{g}_{k-1}$ on Γ which have the property that

$$(\tilde{f}_{k-1}(p) - \tilde{f}_{k-1}(q))(\tilde{g}_{k-1}(p) - \tilde{g}_{k-1}(q)) = 0$$

for $pq = e_1, e_2, \ldots, e_{k-2}$, and by the induction hypothesis $\kappa(e_j) = 0$ for $j = (k-1), \ldots, m$. Therefore $\sum \kappa(pq)(\tilde{f}_{k-1}(p) - \tilde{f}_{k-1}(q))(\tilde{g}_{k-1}(p) - \tilde{g}_{k-1}(q)) = 0$ implies that $\kappa(e_{k-1}) = 0$. We are done by induction. □

We are now ready to prove the main theorem of this section, which is a generalization of theorem 4.6 on p. 80 of [1].

Theorem 3.1 *If G is a critical circular planar graph then L is a diffeomorphism.*

Proof: Since G is critical, the recovery algorithm gives that L is a bijection, so it remains to be shown that the differential is injective everywhere. Rather than show this directly, I will compute the differential of the map $\tilde{L}$ which sends the conductivity γ to the bilinear form

$$B_\gamma(x, y) = < y, \Lambda_\gamma x >$$

associated to Λ_γ. From $d\tilde{L}$ being injective it follows that dL is injective.

We proceed as in [1] (p. 77). The following notation will simplify the computation greatly. If σ is a function on the edges of G and f is a function defined on the nodes of G then

- $\sigma_{i,j}$ will stand for $\sigma(e)$ if there is an edge e joining v_i to v_j, and $\sigma_{i,j} = 0$ if there is no such edge. In particular, $\gamma_{i,j} = \gamma(e)$ if there is an edge e joining v_i to v_j and $\gamma_{i,j} = 0$ otherwise. Note that since G is critical, each edge can be uniquely specified by its endpoints.
- $f_i = f(v_i)$ is the value of f at node v_i
- $\nabla_{i,j} f = f_i - f_j$
- $\phi_f(p)$ is the current into the network at node p.

Let κ be a real-valued function defined on the m edges of G and let t be a real parameter sufficiently small so $\gamma + t\kappa$ is positive on all the edges of G. We will differentiate $B_{\gamma+t\kappa}(x, y)$ with respect to t and then set $\kappa = 0$ to obtain the result. For each pair of functions x and y defined on the boundary nodes of G, let u_t and w_t be the $(\gamma + t\kappa)$-harmonic functions with boundary values x and y respectively. Then $u_t = u_0 + \delta u_t$ and similarly $w_t = w_0 + \delta w_t$, where δu_t and δw_t are functions defined on the nodes of G which are 0 on the boundary nodes. The Kirchhoff matrix can be used to show that u_t and w_t depend differentiably on t. Suppose the Kirchhoff matrix K_t for $(G, \gamma + t\kappa)$ is:

$$K_t = \begin{bmatrix} A_t & B_t \\ B_t^T & D_t \end{bmatrix}$$

where $D = K(I; I)$ is the block corresponding to the interior nodes of G. The values of u_t at interior nodes are given by:

$$u_t(p) = [-D_t^{-1} B_t^T x](p)$$

and since B_t and D_t are affine functions of t, $u_t(p)$ is a rational function of t and hence smooth. This allows us to write $\delta u_t = t\tilde{u}_t$ where $\lim_{t\to 0}\tilde{u}_t$ exists and is finite. Now,

$$B_\gamma(x,y) =< y, \Lambda x >= \sum \gamma_{i,j}\nabla_{i,j}u_0\nabla_{i,j}w_0$$

and more generally we have

$$\begin{aligned}
B_{\gamma+t\kappa}(x,y) &= \sum(\gamma_{i,j}+t\kappa_{i,j})(\nabla_{i,j}u_0+\nabla_{i,j}\delta u_t)(\nabla_{i,j}w_0+\nabla_{i,j}\delta w_t)\\
&= \sum\gamma_{i,j}\nabla_{i,j}u_0\nabla_{i,j}w_0 + t\sum\kappa_{i,j}\nabla_{i,j}u_0\nabla_{i,j}w_0\\
&+ \sum\gamma_{i,j}(\nabla_{i,j}u_0\nabla_{i,j}\delta w_t+\nabla_{i,j}\delta u_t\nabla_{i,j}w_0+\nabla_{i,j}\delta u_t\nabla_{i,j}\delta w_t)\\
&+ t\sum\kappa_{i,j}(\nabla_{i,j}u_0\nabla_{i,j}\delta w_t+\nabla_{i,j}\delta u_t\nabla_{i,j}w_0+\nabla_{i,j}\delta u_t\nabla_{i,j}\delta w_t)
\end{aligned}$$

Now we can simplify this equation by noting that

$$\sum\gamma_{i,j}\nabla_{i,j}u_0\nabla_{i,j}\delta w_t = \delta w_t(p)\phi_u(p) = 0$$
$$\sum\gamma_{i,j}\nabla_{i,j}\delta u_t\nabla_{i,j}w_0 = \delta u_t(p)\phi_w(p) = 0$$

since $\phi_u(p) = \phi_w(p) = 0$ when $p \in \text{int } G$, and when $p \in \partial G$ then $\delta w_t(p) = \delta u_t(p) = 0$. Thus eq. 3 becomes

$$B_{\gamma+t\kappa}(x,y) = \sum\gamma_{i,j}(\nabla_{i,j}u_0\nabla_{i,j}w_0+\nabla_{i,j}\delta u_t\nabla_{i,j}\delta w_t)$$

$$+t\sum\kappa_{i,j}(\nabla_{i,j}u_0\nabla_{i,j}w_0+\nabla_{i,j}u_0\nabla_{i,j}\delta w_t+\nabla_{i,j}\delta u_t\nabla_{i,j}w_0+\nabla_{i,j}\delta u_t\nabla_{i,j}\delta w_t)$$

Putting this in terms of $\tilde{u}_t$ and $\tilde{w}_t$ gives

$$B_{\gamma+t\kappa}(x,y) = \sum\gamma_{i,j}\nabla_{i,j}u_0\nabla_{i,j}w_0 + t\sum\kappa_{i,j}\nabla_{i,j}u_0\nabla_{i,j}w_0$$

$$+t^2\sum\gamma_{i,j}\nabla_{i,j}\tilde{u}_t\nabla_{i,j}\tilde{w}_t$$

$$+t^2\sum\kappa_{i,j}(\nabla_{i,j}u_0\nabla_{i,j}\tilde{w}_t+\nabla_{i,j}\tilde{u}_t\nabla_{i,j}w_0+\nabla_{i,j}\tilde{u}_t\nabla_{i,j}\tilde{w}_t)$$

Hence

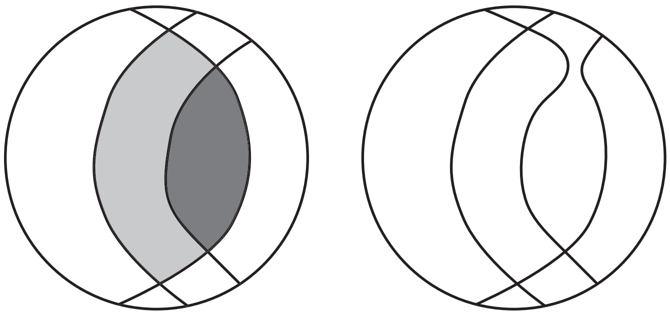
$$\left.\frac{d}{dt}B_{\gamma+t\kappa}(x,y)\right|_{t=0} = \sum\kappa_{i,j}\nabla_{i,j}u_0\nabla_{i,j}w_0 \tag{3}$$

and from lemma 3.3 it follows that this quantity is 0 if and only if κ is identically 0. Thus $d\tilde{L}$ is injective and hence dL is injective. □

4 The Fibers of L

The following theorem from [1] (p. 137) will be important in the construction which I will describe.

Figure 8: The larger lens cannot be emptied by switching arcs, but uncrossing the smaller lens removes both.



Theorem 4.1 *Suppose that $\mathcal{A}$ is a family of arcs that has one or more lenses. Then by a finite sequence of switches and uncrossings of arcs that form empty lenses, $\mathcal{A}$ can be reduced to a family that is lensless.*

A crucial aspect of this theorem is that *only* empty lenses are uncrossed, and at each stage in the process the lens to be emptied next is one for which the number of regions contained within it is minimal. It is not true that an arbitrary lens can be emptied. Consider for example figure 8 in which the larger lens cannot be emptied by switching arcs. However, uncrossing the smaller lens yields a lensless family.

Theorem 4.2 *Given a circular planar graph G, there exists a diffeomorphism Φ and a critical circular planar graph G' such that*

$$\pi = \pi(G) = \pi(G')$$

and the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{R}^+)^{m-k} \times I^k \arrow[r, "proj"] \arrow[d, dotted, "\Phi"', "\cong"] & (\mathbb{R}^+)^{m-k} \arrow[d, "\cong"', "L_{G'}"] \\
(\mathbb{R}^+)^m \arrow[r, "L_G"] & \Omega(\pi)
\end{tikzcd} \tag{4}$$

Proof: The strategy is to construct the critical graph G' and then relate conductances on this graph to conductances on G in such a way that the two networks formed have the same response matrix and are thus electrically equivalent.

Step 1: Construct the medial graph $\mathcal{M}$ associated to G.

Step 2: Apply theorem 4.1 to obtain a sequence of medial graphs

$$\{\mathcal{M} = \mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n = \mathcal{M}'\}$$

where $\mathcal{M}'$ is lensless and each $\mathcal{M}_i$ is obtained from $\mathcal{M}_{i-1}$ by either switching an arc or uncrossing an empty lens at a pole. Associated to the above sequence of medial graphs is a sequence of graphs

$$\{G = G_0, G_1, G_2, \ldots, G_n = G'\}$$

where G' is the desired critical graph (G' is guaranteed to have the same set of connections as G by theorem 4.1) and each G_i has $\mathcal{M}_i$ as its associated medial graph and is obtained from G_{i-1} by either a $Y - \Delta$ transformation, contracting two edges in series to a single edge, or deleting an edge from a pair in parallel (see figure 9). Let m_i be the number of edges in G_i.

Step 3: Repeat this step for $i = n, (n-1), \ldots, 1$. There are three cases, depending on how G_i is obtained from G_{i-1}.

Case 3a— Suppose G_i is obtained from G_{i-1} by transforming a Δ into a Y (an analogous argument will handle the case of when G_i is obtained from G_{i-1} by transforming a Y into a Δ). We would like to relate conductances on G_i to electrically equivalent conductances on G_{i-1}—that is, we would like to construct a diffeomorphism φ_i to complete the diagram

$$\begin{tikzcd} (\mathbb{R}^+)^{m_i} \arrow[rr, dotted, "\varphi_i", "\cong"'] \arrow[dr, "L_{G_i}"'] & & (\mathbb{R}^+)^{m_{i-1}} \arrow[dl, "L_{G_{i-1}}"] \\ & \Omega(\pi) & \end{tikzcd}$$

Note that $m_i = m_{i-1}$ since $Y - \Delta$ transformation preserves the edge count.

The $Y - \Delta$ transformation formula gives a bijection

$$\varphi_i : (\mathbb{R}^+)^{m_i} \longrightarrow (\mathbb{R}^+)^{m_{i-1}}$$

Let the edges of G_{i-1} and G_i be ordered so that the three edges involved in the transformation are the final three, and each of these edges in the Y of G_i corresponds to the opposing edge of the Δ in G_{i-1}. Let (a, b, c) be the conductivities on the Y of G_i. Then [1]

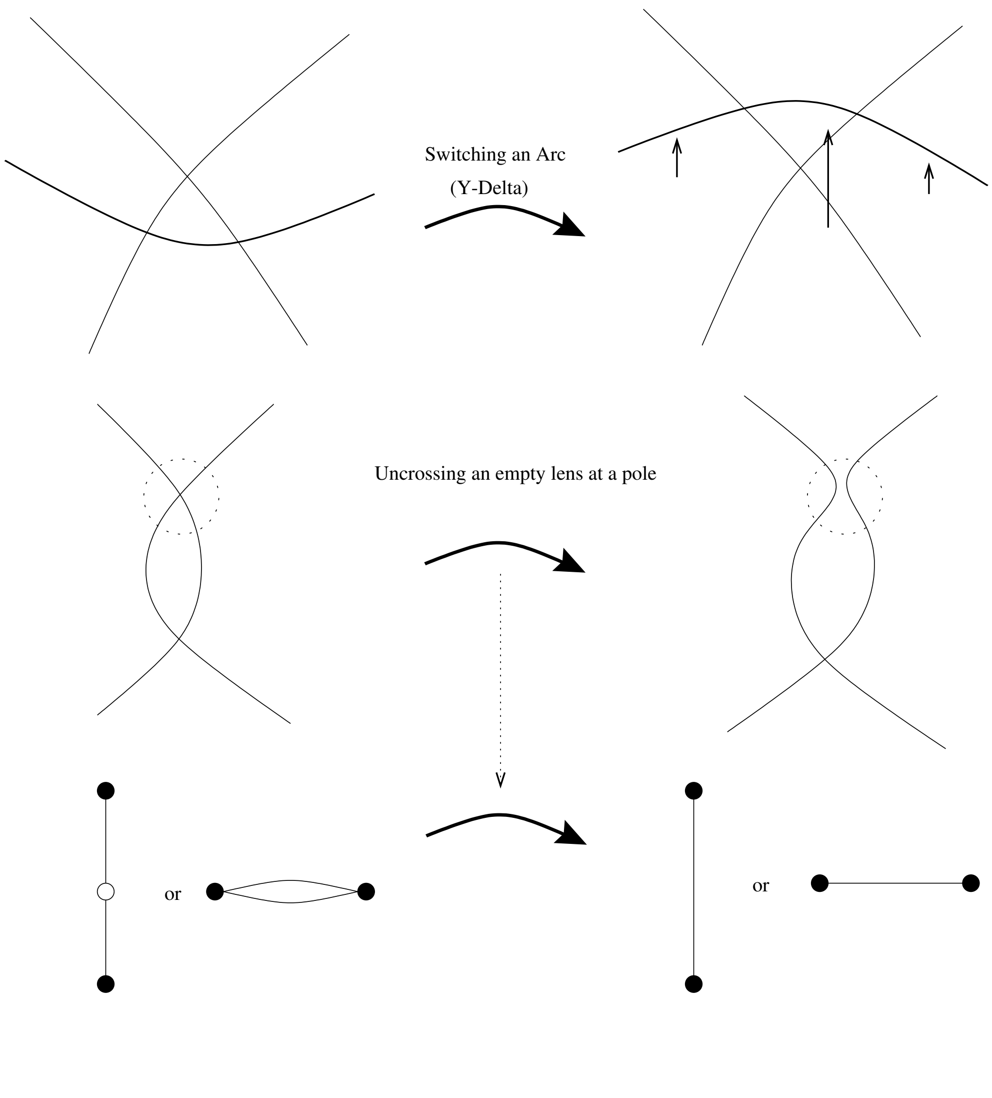
$$\varphi_i = \mathrm{Id}_{m_i - 3} \times \varphi_{Y\Delta}$$

[1]Here Id_{m_i-3} is the identity on the first $(m_i - 3)$ edges, and the $\times$ operation is defined by the commutative diagram:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times B \arrow[l, "\pi"'] \arrow[r, "\pi"] \arrow[d, "f \times g"] & B \arrow[d, "g"] \\ C & C \times D \arrow[l, "\pi"'] \arrow[r, "\pi"] & D \end{tikzcd}$$

i.e. $f \times g(x \times y) = f(x) \times g(y)$.

Figure 9: Allowed operations on medial graphs



where

$$\varphi_{Y\Delta} : (a, b, c) \longmapsto (\frac{bc}{a+b+c}, \frac{ac}{a+b+c}, \frac{ab}{a+b+c})$$

is the function which sends conductivities on a Y network to their equivalent conductivities on a Δ. Computing $\det d\varphi_i$ will show that this is actually a diffeomorphism: $d\varphi_i = \mathrm{Id}_{m_i-3} \times d\varphi_{Y\Delta}$, and taking the partial derivatives gives

$$d\varphi_{Y\Delta} = \begin{bmatrix} \frac{-bc}{(a+b+c)^2} & \frac{c(a+c)}{(a+b+c)^2} & \frac{b(a+b)}{(a+b+c)^2} \\ \frac{c(b+c)}{(a+b+c)^2} & \frac{-ac}{(a+b+c)^2} & \frac{a(a+b)}{(a+b+c)^2} \\ \frac{b(b+c)}{(a+b+c)^2} & \frac{a(a+c)}{(a+b+c)^2} & \frac{-ab}{(a+b+c)^2} \end{bmatrix}$$

Some algebraic manipulation yields

$$\det d\varphi_{Y\Delta} = \frac{abc}{(a+b+c)^3}$$

so

$$\det d\varphi_i = \det \mathrm{Id}_{m_i-3} \cdot \det d\varphi_{Y\Delta} = \frac{abc}{(a+b+c)^3}$$

The conductivities a, b, and c are strictly positive, so this determinant is always non-zero. Thus $d\varphi_i$ is injective, so φ_i is a diffeomorphism.

Case 3b— Suppose G_i is obtained from G_{i-1} by replacing two edges e, f in parallel with a single edge g. So

$$m_{i-1} = m_i + 1$$

Then the conductances $\gamma_{i-1}(e)$ and $\gamma_{i-1}(f)$ cannot be uniquely written in terms of $\gamma_i(g)$, so we must introduce a parameter $t_j \in (0, 1)$ (where $j-1$ is the number of parameters previously introduced in this manner) and write

$$\gamma_{i-1}(e) = \gamma_i(g)(t_j)$$

$$\gamma_{i-1}(f) = \gamma_i(g)(1 - t_j)$$

This gives a bijection

$$\varphi_i : (\mathbb{R}^+)^{m_i} \times (0, 1) \longrightarrow (\mathbb{R}^+)^{m_{i-1}}$$

Let the edges of G_i and G_{i-1} be ordered so that the edges which are being modified come last, and let $a = \gamma_i(g)$. Then

$$\varphi_i = \mathrm{Id}_{m_i-1} \times \varphi_{par}$$

where

$$\varphi_{par} : (a, t) \longmapsto (at, a(t-1))$$

We again compute the differential to show that this map is a diffeomorphism:

$$d\varphi_i = \mathrm{Id}_{m_i-1} \times d\varphi_{par}$$

where

$$d\varphi_{par} = \begin{bmatrix} t & a \\ (1-t) & -a \end{bmatrix}$$

from which one computes that

$$\det d\varphi_i = (\det \mathrm{Id}_{m_i-1}) \cdot (\det d\varphi_{par}) = (1) \cdot (-a) = -a \neq 0$$

where the last inequality is again because conductances are strictly positive. Thus φ_i is a diffeomorphism.

Case 3c— Proceeding along the lines of (3b), suppose G_i is obtained from G_{i-1} by contracting two edges e, f in series to a single edge g. So as in (3b)

$$m_{i-1} = m_i + 1$$

We must again introduce a parameter $t_j \in (0,1)$ and write

$$\gamma_{i-1}(e) = \frac{\gamma_i(g)}{t_j}$$

$$\gamma_{i-1}(f) = \frac{\gamma_i(g)}{(1-t_j)}$$

Again this gives a bijection

$$\varphi_i : (\mathbb{R}^+)^{m_i} \times (0,1) \longrightarrow (\mathbb{R}^+)^{m_{i-1}}$$

Let the edges of G_i and G_{i-1} be ordered so that the edges which are being modified come last, and let $a = \gamma_i(g)$. Then

$$\varphi_i = \mathrm{Id}_{m_i-1} \times \varphi_{ser}$$

where

$$\varphi_{ser} : (a,t) \longmapsto \left(\frac{a}{t}, \frac{a}{1-t}\right)$$

We again compute the differential to show that this map is a diffeomorphism:

$$d\varphi_i = \mathrm{Id}_{m_i-1} \times d\varphi_{ser}$$

where

$$d\varphi_{ser} = \begin{bmatrix} 1/t & -a/t^2 \\ 1/(1-t) & a/(1-t)^2 \end{bmatrix}$$

from which one computes that

$$\det d\varphi_i = (\det \mathrm{Id}_{m_i-1}) \cdot (\det d\varphi_{par}) = (1) \cdot \left(\frac{a}{t^2(1-t)^2}\right) \neq 0$$

where the last inequality is again because conductances are strictly positive. Thus φ_i is a diffeomorphism.

Step 4: Putting the maps constructed in step 3 together yields the desired diffeomorphism

$$\Phi : (\mathbb{R}^+)^{m-k} \times \mathbb{I}^k \longrightarrow (\mathbb{R}^+)^m$$

given by

$$\Phi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1$$

where each φ_i is extended as identity to the proper domain in the obvious way. By construction, diagram 4 commutes. □

The number k has a special significance. In reducing G to G', k is the total number of times that an edge is removed. Recall, $Y - \Delta$ transformations preserve the number of edges, so the edge count is a constant over each $Y - \Delta$ equivalence class. Alternatively, k can be thought of as the difference between the number of edges in the non-critical graph G that we began with and any critical graph which has the same set of connections as G (and is thus in the $Y - \Delta$ equivalence class of G').

It is currently an open question as to whether or not there is a direct method for extracting k from a graph or medial graph without first going through the process of emptying and uncrossing lenses. In some cases a medial graph may appear to have several lenses, but only a single uncrossing is required to reduce it to a lensless medial graph.

Corollary 4.1 *Given a circular planar graph G and a compatible response matrix Λ, the set $L_G^{-1}(\Lambda)$ is diffeomorphic to $\mathbb{I}^k = \underbrace{(0,1) \times \ldots \times (0,1)}_{k \text{ times}}$ for some integer k.*

Proof: Construct the critical graph G' and the diffeomorphism Φ as in theorem 4.2. By restricting the domain of Φ to the fiber of $(L_{G'} \circ \mathrm{proj})$ over Λ, we obtain a diffeomorphism

$$L_{G'}^{-1}(\Lambda) \times \mathbb{I}^k \longrightarrow L_G^{-1}(\Lambda)$$

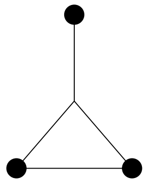
Since G' is a critical graph, there is a unique conductance function γ' on the edges of G' such that the network (G', γ') has the desired Λ as its response matrix. This γ' can be computed from Λ. Thus $L_{G'}^{-1}(\Lambda) = \{\gamma'\}$—i.e. it consists of only a single point in $(\mathbb{R}^+)^m$. Therefore this component can be trivially projected off to yield a diffeomorphism Ψ. This is visualized with the following commutative diagram:

$$\begin{tikzcd}
L_{G'}^{-1}(\Lambda) \times \mathbb{I}^k \arrow[d, "\mathrm{proj}"'] \arrow[dr, "\cong"'] & \\
\mathbb{I}^k \arrow[r, "\Psi", "\cong"'] & L_G^{-1}(\Lambda)
\end{tikzcd}$$

This $\Psi : \mathbb{I}^k \longrightarrow L_G^{-1}(\Lambda)$ is the desired diffeomorphism. $\square$

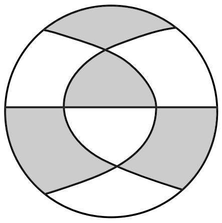
This corollary completely answers the question of partial recovery for circular planar networks. The proof provides a method for constructing an explicit formula for the diffeomorphism φ. With this formula one can directly see which edges have conductivities which are constant over $\mathbb{I}^k$ (and are thus recoverable), and which edges do not (and thus cannot be recovered). Furthermore, for the set of edges that fall into the latter class, the formula reveals all relationships between these irrecoverable conductances which can be recovered from the response matrix.

Example Consider the *Hershey's Kiss* graph H shown below.

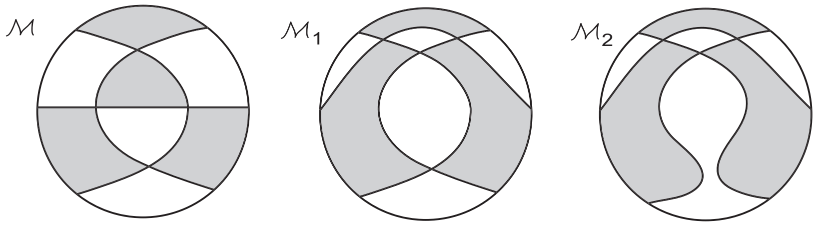


This graph is non-critical, so only partial information about the conductance can be recovered.

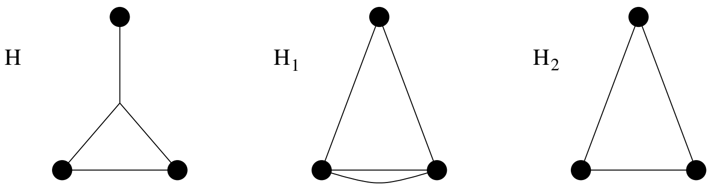
The medial graph $\mathcal{M}$ of H is



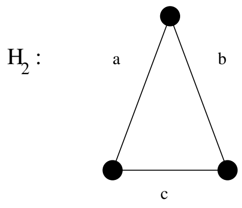
The lens in this medial graph can be emptied by a single arc switch (either up or down) to yield $\mathcal{M}_1$ which is lensless. Next, the lens in $\mathcal{M}_1$ can be uncrossed to yield the lensless medial graph $\mathcal{M}_2$. This is shown below.



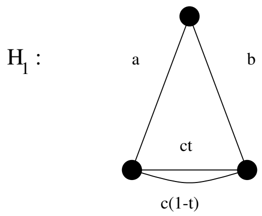
These medial graphs correspond to the sequence of graphs H, H_1, H_2, where H_2 is critical.



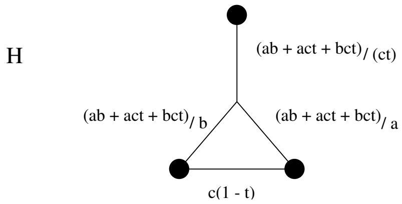
Given a response matrix Λ compatible with H, we now recover conductances on H_2. In the present example this is unusually simple because Λ happens to also be a Kirchhoff matrix for H_2, so the conductances can be read off directly as entries in Λ. Let a, b, and c be the conductances recovered on H_2.



Then the conductances on H_1 are a and b on the single edges, and they are ct and $c(1-t)$ on the two edges in parallel.



Now applying the $Y-\Delta$ transformation from H_1 to H gives the recovered conductances on H as



With these expressions we have completely parametrized all conductivity functions on H which give rise to the response matrix Λ. We can see that all four conductivities have some dependence of t, so no edge in this graph is entirely recoverable.

5 The Rank of the differential of L

We wish to compute the rank of dL_G for a circular planar non-critical graph G. This is a trivial computation in light of theorem 4.2.

Theorem 5.1 *Let G be a circular planar graph. The differential of L_G has rank equal to the number of edges m' in any circular planar graph G' such that $\pi(G) = \pi(G') = \pi$.*

Proof: First note that m' is well defined since the set of critical circular planar graphs that have the same set of connections as G (and the same number of boundary nodes) is a $Y - \Delta$ equivalence class, and the number of edges is invariant over these equivalence classes. Apply theorem 4.2 to obtain diagram 4. Hitting this diagram with the differential functor gives the diagram of tangent spaces

$$\begin{tikzcd} \mathbb{R}^{m-k} \times \mathbb{R}^k \arrow[r, "\text{proj}"] \arrow[d, "d\Phi"', "\cong"] & \mathbb{R}^{m-k} \arrow[d, "\cong"', "dL_{G'}"] \\ \mathbb{R}^m \arrow[r, "dL_G"] & T_\Lambda \Omega(\pi) \end{tikzcd}$$

where $T_\Lambda \Omega(\pi)$ is the tangent space to $\Omega(\pi)$ at the appropriate point Λ, m is the number of edges in G, and $(m-k) = m'$ is the number of edges in G'. From here it is easy to compute the rank of dL_G. Clearly the map proj has rank m', and since $d\Phi$ and $dL_{G'}$ are vector space isomorphisms, dL_G also has rank m'. $\square$

Example Consider the Hershey's Kiss graph H discussed in the example of §4. The graph H has four edges, and the connection-equivalent critical graph that it was reduced to in the example of §4 was a Δ graph with three edges. So, $m = 4$ and $k = 1$, and thus

$$\text{rank } dL_H = (m - k) = 3$$

without any computation at all.

The beauty of this result is that it is entirely topological. The rank is constant over the entire domain of conductivity functions.

We can take this idea a little further. Given a circular planar non-critical graph G, one could think of the set of connections $\pi(G)$ together with the rank of dL_G as uniquely specifying the $Y - \Delta$ equivalence class of G.

6 A More Direct Recoverability Result

The methods described in the §4 give a complete description of all conductance information that can be calculated from Λ, but those methods are computationally difficult. Only after many steps of manipulating the graps and calculating conductances does one arrive at the final answer. Therefore it may sometimes be useful to have a more direct method for deciding whether or not the conductance on a given edge can actually be recovered. This is the motivation for the theorem I will prove at the end of this section. The idea we will exploit is that we can cut a region out of a medial graph and replace it with something that is connection-equivalent to the part we removed and this operation does not affect the edges that are outside of the region where we are performing this surgery.

We can cut out the non-critical parts of the graph where there are lenses and replace them with critical subgraphs to construct an electrically equivalent critical graph and then recover all edges that were ouside of the surgery region.

Before we can can proceed we must extend our concept of a medial graph and be precise about what it means to extract a subgraph of a medial graph.

Definition 6.1 *A generalized medial graph is a pair* $(R, \mathcal{A})$*, where* R *is a compact region in the plane* [2] *with piecewise smooth boundary and* $\mathcal{A} = \{\alpha_i\}_{i=1}^n$ *is a finite collection of piecewise smooth curves* α_i *(called geodesics) in* R *such that the following conditions hold:*

- *Each* α_i *is either closed and contained entirely in int* R*, or begins and ends on* ∂R *and intersects* ∂R *nowhere else.*
- *If* α_i *and* α_j *intersect at a point* p *then no other* α_k *passes through that point (i.e. every vertex has degree exactly four).*
- *The intersection of each connected component of* ∂R *with the the image of* $\{\alpha_i\}$ *contains an even number of points.*

Note that these conditions imply that the cell-complex constructed from $\mathcal{A}$ is two-colorable, so color it black and white.

From an abstract medial graph we can construct a graph (or its dual) in the same way that we do for a medial graph in the disc by indentifying black cells with nodes, black boundary intervals with boundary nodes, and nodes in the medial graph with edges in the graph. If the region R is not a disc then the graph constructed will not be circular planar.

We can also go in other direction and define an abstract medial graph associated to a (not necessarily circular planar) graph. One must be careful in doing this because there may be multiple ways to draw the medial graph of a given graph if the region is not simply connected and we are not careful in how we specify the construction procedure. If e and f are two edges which meet at a node p then it is sufficient to require that one draw the geodesic connecting the midpoints m_e and m_f of e and f so that it is path homotopic to $m_e p m_f$. This ensures that the geodesics of the medial graph go the right way around the holes in R.

Let $\mathcal{M}$ be a medial graph embedded in the unit disc and let α be a simple closed curve such that α intersects each edge of $\mathcal{M}$ at most once, does not intersect any node, and intersects the boundary of the disc either never or twice. Let R_1 be the intersection of the disc with the region bounded by α, and let R_2 be the disc minus R_1. By the restriction imposed on α each of these two regions is homeomorphic to a disc or an. As in the proof of lemma 3.1, we split $\mathcal{M}$ into subgraphs $\mathcal{M}_1 = \mathcal{M} \cap R_1$ and $\mathcal{M}_2 = \mathcal{M} \cap R_2$ (this is a slight abuse of notation–what I mean is to restrict the curves in $\mathcal{A}$ to R_1 or R_2).

[2] We could easily extend this definition by replacing R with a manifold with boundary, and this may provide an interesting area for further investigation, but for our purposes we can be content to stay in the plane.

If G is associated graph of $\mathcal{M}$ then the associated graphs of $\mathcal{M}_1$ and $\mathcal{M}_2$ may be thought of as subgraphs of G which, when amalgamated through the proper set of boundary nodes[3] reconstructs the original graph G.

We are now ready to state and prove the main theorem of this section.

Theorem 6.1 *Let e be an edge in a circular planar graph G, and let v_e be the vertex corresponding to e in the medial graph $\mathcal{M}$ associated to G. If v_e is not contained in any lens then $\gamma(e)$ is recoverable.*

Proof: For each lens (or set of overlapping lenses) draw a simple closed curve β_i such that β satisfies the criteria for being a geodesic (i.e. it must not intersect ∂R or any vertices of $\mathcal{M}$) and v_e is outside the region bounded by β_i. This is possible by the hypothesis that v_e is outside of any lens.

For each i let $\mathcal{M}_i$ be the medial subgraph in the region bounded by β_i and let G_i be the graph associated with $\mathcal{M}_i$. The region bounded by β_i is homeomorphic to a disc, so G_i is the medial graph of a circular planar graph and we can appeal to the machinery of lemma 4.2 to computing the set of edges in G_i which can be uniquely recovered. Doing so produces a set of diffeomorphisms Φ_i from (conductivities on G_i) to (conductivities on a critical graph G'_i) $times$(a parameter space) which leave the response matrix invariant.

Let $\mathcal{M}_{shell}$ be the medial subgraph of the disc minus the regions bounded by the β_i, and let G_{shell} be the associated graph. This medial graph $\mathcal{M}_{shell}$ is not circular planar (there are lots of holes) but it is lensless[4]. Pasting in the $\mathcal{M}_i$ yields a lensless medial graph $\mathcal{M}'$ in the disc, so we can safely say that the graph G' associated to $\mathcal{M}'$ is critical and circular planar and hence recoverable.

By construction the graphs G' and G are connection-equivalent and so given a conductivity on one there is a (non unique) conductivity on the other such that the two are electrically equivalent. What we now need to prove is that recovering the conductance on edge e in the graph G' gives the only possible conductance on e in the original graph G.

We do this by extending the maps Φ_i to a diffeomorphism Φ from (conductivitities on G) to (conductivities on the critical graph G')$\times$(cartesian product over i of parameters spaces) such that Φ is the identity when restricted to the edges of G_{shell}. This Φ leaves the response matrix invariant and is the identity when restricted to the edge e. Since G' is critical, the conductivity on e is uniquely recoverable, and hence it is uniquely recoverable in G. $\Box$

7 Current Questions

Theorem 3.1 says that L is a diffeomorphism when the underlying graph is circular planar and critical. However, if we leave the arena of circular planar graphs then little is known. There are many examples of non-planar graphs that are

[3]See [4] for a description of the amalgamation process

[4]Note that in noncircular planar graphs a lensless medial graph does not imply that the graph is recoverable.

still recoverable—for instance, consider a critical circular planar graph with the boundary nodes permuted so that it cannot be embedded in the disc. The graph remains critical but it appears to no longer be circular planar. Nevertheless, this permutation of boundary nodes does not affect the properties of L, so in this case we can say difinitively that the network is still recoverable and L is still a diffeomorphism even though the proofs do not work for non-circular-planar graphs.

For a more interesting example, consider the annular graph with two circles and four rays (see [2]). Any network with this as its underlying graph is recoverable, but we do not yet know if L has any critical points for this graph.

Now consider the annular graph of two circles and three rays. Esser showed in [2] that on this graph L has the unusual and surprising property of being a two-to-one map almost everywhere in its range. However, there are response matrices compatible with this graph which are the image of only a single conductance function. For these response matrices the conductivity can be calculated unambiguously, but L is certainly *not* a local diffeomorphism at these points.

Following the thoughts above, I would like to pose the following:

Conjecture 7.1 *Let G be a graph (not necessarily planar) such that L_G is one-to-one, then L_G is a diffeomorphism.*

I believe that there are no graphs for which a single conductivity function is uniquely specified by each response matrix while L has critical points. Perhaps future REU students will see this as an interesting and exciting question and give an answer someday.

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