Infinite Networks in Two Dimensions

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Abstract

In this paper, 2-Dimensional resistor networks of various configurations are considered. It shall be shown that a unique Green's function defining the potential at each node exists in each case, putting conditions on conductances and potential differences. Additionally, power dissipation in infinite networks is examined.

1 INTRODUCTION

Electrical networks are comprised of graphs containing nodes connected by edges. A conductance (1/resistance) γ_{pq} is associated with each pair of nodes p and q. If p and q are not connected, then $\gamma_{pq} = 0$. The nodes are divided up into two categories: interior and boundary. Boundary nodes need not correspond to the geometric boundary of the graph; rather, they are locations where known currents and voltages can be applied. Interior nodes must satisfy Kirchhoff's Law, which can be summarized as saying the total current flowing into a node must equal the total current flowing out. The degree of a node refers to the number of edges connected to it. Two primary problems are associated with electrical networks. The forward problem takes a known graph and set of conductances γ_{pq}, then uses boundary potentials to find a function $u(p)$ that gives the potential at every point. A linear map Λ called the response map yields boundary currents upon input of boundary potentials. The inverse problem is to use Λ to find all conductances and possibly the structure of the graph as well.

One way to approach the forward problem is to construct a specialized function called the Green's function. The Green's function sets every boundary potential to 0. It yields the potential at every interior node when a unit current is applied at some other interior node. This can be conveniently expressed using an $n \times n$ matrix, where n is the number of interior nodes. An entry G_{pq} gives the potential at node q when a unit current is applied at node p. This matrix can be found using the set of conductances. To find it, one uses a Kirchhoff matrix ,which consists of a square matrix whose size matches the total number of nodes in the graph. Each entry K_{ij} is defined as follows:

$$\text{for } i \neq j, K_{ij} = -\gamma_{ij}$$

$$\text{for } i = j, K_{ij} = \sum_{j \neq i} \gamma_{ij}$$

For more background information, see [3]. I am building on the work of Phillip Lynch [2]. However, in the two-dimensional case Lynch chose to require the potential difference between any 2 nodes to approach zero as distance goes to infinity, while I require the potentials themselves to go to zero near inifinity. I feel that my choice makes more sense in terms of defining what is meant by an infinite network. I am not considering networks that are truly infinite in extent, but rather networks of finite extent but infinite nodes and edges used to represent a continuous shape. The distance function I discuss is thus measuring infitesimal units. In such a case it would be inconceivable for the potential to go to $-\infty$ (as it tends to do unless required to be 0) at the boundary even if the potential differences go to 0.

2 Rectangular Grid networks

fig. 1

5cm5cmgrid1.bmp

Consider a network of squares as in figure 1. There is a single boundary node in the center; all other nodes are interior nodes. Every node has degree 4. Setting the center as my origin and using cartesian coordinates with each conductor having unit length, I define a distance function $d = |x| + |y|$. At a given distance d, there are $4d$ nodes and $8d - 4$ edges connecting these nodes to nodes 1 unit closer. The goal is to prove the existence (uniqueness is already proven for the general case) of a Green's function $g(x, y)$ defining the potential for all interior nodes given a unit current source at the central node. g must be $\gamma - harmonic$ and satisfy the condition that

$$\lim_{d \to \infty} g(x, y) = 0. \quad (1)$$

Additionally, the potential g must be finite at every point. Now, consider any node $p = (x, y)$ at distance d. The difference in potential between p and the center node is given by the sum of the potential differences along a shortest path (or any path, but choose the shortest for convenience) between p and the center. That is,

$$g(0, 0) - g(p) = \sum_{k=1}^{d} \Delta g_k \quad (2)$$

Where Δg_k refers to the difference in potential between a node at distance $k - 1$ and one at distance k. Taking the limit of this equation, $g(p) \to 0$ as required by (1). Thus

$$g(0, 0) = \sum_{k=1}^{\infty} \Delta g_k. \quad (3)$$

Since $g(0,0)$ must be finite, this sum will converge. Let the value of the sum be α. Because current is flowing outward in the system, Δg_k is always non-negative. Thus $g(x,y)$ will be a non-increasing function bounded from above and below:

$$0 \leq g(x,y) \leq \alpha \tag{4}$$

Assuming that, in the limit, g depends generally on d, we have a strong case for existence. A reasonable choice would be for $g(x,y)$ to act like $\frac{1}{d^\epsilon}$, causing $\Delta g(x,y)$ to act like $\frac{1}{d^{1+\epsilon}}$ for large d. That is,

$$0 < \lim_{d\to\infty} \Delta g(x,y)(d^{1+\epsilon}) < \infty \text{ for some } \epsilon > 0 \tag{5}$$

For g to depend generally on d, γ must also depend generally on d and can be written (again in a limit approximation) as γ_d. The unit current flowing into the central node propogates outward so that the total current flowing both into and out of each distance level is unity. This means that

$$\forall d, I_d \approx (8d-4)(\gamma_d)(\Delta g_d) = 1. \tag{6}$$

Because the network is two-dimensional, the number of edges at a certain distance growns linearly. For 1D cases it would be constant; it would grow like d^2 in 3 dimensions. Since we are only concerned with this expression in the limit as $d \to \infty$, only the highest order piece of each term need be kept. Thus the following is obtained:

$$d\frac{1}{d^{1+\epsilon}}\gamma_d \approx 1 \text{ or } \gamma_d \propto d^\epsilon. \tag{7}$$

It makes sense that the conductances must go to ∞ as voltage differences go to 0 in order to maintain current flow, but notice that the required growth rate is fairly low.

2.1 Power Dissipation

Given general behavior of g and γ for large d, power dissipated by the entire network can be examined.

Theorem 1 *The total power dissipated in an infinite rectangular grid network on which a Green's function with potentials going to 0 exists is finite.*

proof. For any network of vertices connected by conductors, power dissipation is generally given by

$$P = \sum_{i<j} \gamma_{ij}(g_i - g_j)^2. \tag{8}$$

For the rectangular network I am considering, this can be approximated by:

$$P \approx \sum_{d=1}^{\infty} (8d-4)\gamma_d(\Delta g_d)^2. \tag{9}$$

That is, the power dissipated at a given distance d is found by multiplying the number of edges at that distance times the approximate power dissipated in any given edge at that distance. The total power is then found by summing over all distances. Again we are concerned with the order of the terms rather than their coefficients. Substituting for γ_d and Δg_d, we obtain

$$P \propto \sum_{d=1}^{\infty}(d)(d^{\epsilon})(\frac{1}{d^{1+\epsilon}})^2 = \sum_{d=1}^{\infty}\frac{1}{d^{1+\epsilon}}. \tag{10}$$

Recalling that $\epsilon > 0$, we can easily see that this sum converges. The power dissipated in the network is therefore finite. QED.

2.2 More Specific Results

Given the existence of a Green's function defining potentials at every point, it seemed feasible to find a pattern for the behavior of potentials at increasing distance. Matlab was used to generate a Kirchhoff matrix for the rectangular grid based on a specified pattern of conductances. The conductances entered were the same at a given distance: $\gamma_d = d$ or $\gamma_d = d^2$. It was found that the Kirchoff matrix was not at all dense (most are not) and so could be handled much more efficiently when declared as sparse. Once the Kichoff matrix was, the rows and columns corresponding to nodes on the edge–boundary nodes were deleted. What was left is called the C matrix, which can be inverted to give the Green's function for the network. Call the inverse of that matrix G. G_{ij} gives the potential at node j when a unit current source is placed at node i. The matrix turns out to be symmetric so that order does not matter. The column corresponding to the current source placed at the center node was selected out, thus obtaining the desired potentials. To analyze the potentials, those from nodes along a single ray (the top of the y axis if the graph is laid on the xy plane) were examined. It seemed logical that any pattern would be easier to discern along these nodes. Despite managing to obtain values up to a distance of 350 units, I was unable to find a pattern among the potentials or their differences. Examined both for polynomial and exponential behavior, they did not achieve a decent fit. It seems unlikely that lower order terms are still dominating, but that is possible. It is also possible that the potentials shrink like 1 over a fractional power of d, which the analysis would not have discovered. Alternatively, the potentials might not act like either a polynomial or exponential, but rather some other decreasing function not considered. As a test of whether the strategy was even feasible, I entered 1 for all the conductances in an attempt to achieve the behavior that Lynch predicted. However, the fact that my Green's function was defined slightly differently made this a futile effort–the behavior was not at all the same. The program was checked for possible errors, but none were found. Small cases worked by hand were found to matched the results generated by the computer. I concluded that using computers to produce a Green's function for larger and larger networks is not the best way to understand infinite networks.

fig. 2

-1cm16cm5cmtri.bmp

In the future, perhaps this will not be the case. For a copy of the matlab code or plots of Δg versus distance, see the appendix.

3 Generalized quadrilateral networks

Rectangular grid networks are actually a specialized case of a more general generating algorithm. To generate an alternate version, begin with a single point that will be the origin. Draw an arbitrary number (> 1) of rays emanating outward from that point. Choose a point on each of the rays. Next, repeat *ad infinitum* the following:
1. Choose a point between each pair of adjacent rays, farther from the origin than either point on the surrounding rays.
2. Connect each of the farthest chosen points on the surrounding rays to the adjacent points chosen in 1.
3. Choose a new point on each ray, farther than any previously chosen point.
Every vertex except the center will have degree 4; the degree of the center is given by the number of beginning rays. Such graphs are in fact far easier to draw than describe. The first 3 iterations of a graph generated using 3 rays are displayed in fig 2 (next page). It is actually quite easy to expand the applicability of the results for rectangular grid networks. (3),(4), and (5) are derivable in the exact same manner. Starting with r original rays, (7) turns into

$$\forall d, I_d \approx (r(2d-1)(\gamma_d)(\Delta g_d) = 1. \tag{11}$$

We still find that $\gamma_d \propto d^\epsilon$, but with different coefficients. The total power dissipated remains finite. Moreover, in the case with perfect symmetry (γ and g the same for a given distance), the power dissipated remains constant regardless of the number of principal rays used. Let $\Delta g_d = \frac{1}{d^{1+\epsilon}}$. Thus $\gamma_d = \frac{d^{1+\epsilon}}{r(2d-1)}$. We find that

$$P = \sum_{d=1}^{\infty}(r(2d-1)\frac{d^{1+\epsilon}}{r(2d-1)}\frac{1}{d^{1+\epsilon}})^2) = \frac{1}{d^{1+\epsilon}}. \tag{12}$$

The sum is independent of r, so it will converge to the same value for any r. One major difference between general infinite quadrilateral networks and rectangular grid networks is in the choice of the central boundary node. In an infinite rectangular grid, all points look the same and so the unit current can be applied anywhere with identical results. However, for other cases there is one and only one node with degree $\neq 1$. The unit current must be applied here for the aforementioned results to hold. However, if the current is applied far from that point, the network reverts to the grid case–it looks rectangular and the Green's function will be very close to that for the rectangular grid.

fig. 3: part of a binary tree

-1cm5cm5cmtree.bmp

4 Tree Networks

I will next consider an alternative type of infinite network I will call tree networks. To construct a tree network, begin with 2 nodes connected by a singe edge. For any branching factor b, connect one node to b new points lying beyond the edge. Now continue connecting each new node to b nodes created farther away from the initial edge. This causes the number of nodes to grow at a fixed exponential rate as farther distances are looked at.

Every node but one will have degree b+1. The single node with degree 1 is designated as the only boundary node. In formulating a Green's function, we will consider a unit current entering that node. There is only one path between any interior node and the boundary node, so a distance function $d(i)$ is simply defined as the number of edges from the boundary node to a node i. There are b^{d-1} nodes at a distance d. It will again be required that

$$\lim_{d\to\infty} g(i) = 0.$$

Morever, I will require symmetry among branches so that g and γ can be expressed as proper functions of d, abbreviated g_d and γ_d. Again the potential at the boundary node can be found by summing potential differences out from infinity:

$$g_1 = \sum_{d=1}^{\infty} \Delta g_d \tag{13}$$

where $\Delta g_d = (g_{d-1} - g_d)$. At a distance d, there will be b^{d-1} nodes and a unit current travelling outward. So,

$$I_d = (b^{d-1})(\Delta g_d)(\gamma_d) = 1. \tag{14}$$

Given a pattern of conductances, the forward and reverse problems can be solved. For instance, let $\gamma_d = c$ so that there is constant conductance everywhere. Then $\Delta g_d = \frac{c}{b^{d-1}}$. g_1 can be found by computing the geometric series

$$\sum_{d=1}^{\infty} \frac{1}{(c)b^{d-1}} = \frac{1}{c(1-\frac{1}{b})}.$$

The value of g at any node could be found by subtracting a finite sum from this quantity. Thus, g_d could be found from the conductance c, or c from g_d. A similar approach can be done for any pattern of γ_d (e.g. $\gamma_d = cn^3$) such that

$$\sum_{d=1}^{\infty} \frac{1}{\gamma_d(b^{d-1})}$$

converges to a computable value. Practically, this means that γ_d must be a relatively simple function with any growth rate or a decay slower than $\frac{1}{b^d}$.

fig. 4: part of a diamond network

4cm5cm4cmdiamond.bmp

fig. 5: reduction of a diamond network

0cm10cm5cmdiam.bmp

5 Diamond Networks

Another infinite network, called a diamond network, I considered consists of rectangles lined up and connected at the corners, as in fig. 4. In the infinite case, the rectangles extend off to infinity in one or both directions.

The nodes where corners of rectangles intersect are interior nodes; all other nodes are boundary nodes. Thus each interior node has degree 4, while each boundary node has degree 2. Any finite diamond network is a critical circular planar graph and therefore clearly recoverable. The infinite case should therefore be theoretically recoverable, albeit in infinite time. Current flow in diamond networks can be analyzed locally. This is primarily because there are no interior to interior connections. In fact, the matrix corresponding to a Green's function for the network is diagonal; the unit current applied to some interior node flows straight out the 4 surrounding boundary nodes–rather uninteresting behavior. If we instead require that no current flow out of the local boundary nodes (i.e. turn them into interior nodes) and apply a current source at some interior node, then the current will propogate in both directions out to infinity. In this case, the diamond network can be reduced because there are now series and parallel connections. Each rectangle can be reduced to a single edge with conductivities as seen in fig. 5, according to the formula

$$\frac{1}{\frac{1}{j}+\frac{1}{m}}+\frac{1}{\frac{1}{k}+\frac{1}{l}}$$

After reduction, the network is simply a straight line, as studied in [1]. For it to have a valid Green's function, the conductances must grow fast enough that $\sum_{d=1}^{\infty}\frac{1}{\gamma_d}$ converges. The conductances will be slightly altered by the reduction, but not enough to affect convergence.

6 Further Results

Theorem 1 *The total power dissipated in any infinite network on which a Green's function with potentials going to 0 exists is finite.*

proof Let $d =$ the number of edges traversed in going from the origin O along the shortest path to some particular node in an infinite network. Let the S_d denote the set of conductors at distance d (that is, conductors connecting nodes at distance $d-1$ to nodes at distance d). Regardless of the configuration of the network, a unit current applied at some node O will propogate outward through each set of conductors at a certain distance (circular currents would violate Kirchoff's law). That is, for any d,

$$\sum_{e \in S_d} \Delta g(e)\gamma(e) = 1$$

Using the definitions above, power dissipation is given by

$$P = \sum_{d=1}^{\infty} (\sum_{e \in S_d} (\Delta g(e))^2 \gamma(e))$$

Substituting Δg_d, the maximum potential difference at distance d, for one factor $\Delta g(e)$, we get

$$P \leq \sum_{d=1}^{\infty} (\Delta g_d \sum_{e \in S_d} (\Delta g(e)\gamma(e)) = \sum_{d=1}^{\infty} \Delta g_d.$$

The argument I presented in prior sections will hold, requiring $\sum_{d=1}^{\infty} \Delta g_d$ to converge so that the potential at the origin will be finite regardless of path used to find it. Since $P \leq$ a convergent expression, it must converge to a finite value itself. QED.

A Matlab Code for Green's Function On Rectangular Grid Network

```
function G = Green(z)
a = (2*z^2+10*z+9); %Total # of nodes
C = sparse(a,a);    %Create Kirchoff Matrix
for n = 1:z
    c = (2*n^2+2*n-4);   %Current level starting node
    d = (2*n^2+6*n);     %Next level starting node
%Individual connections
    C(1+c, 2+c) = n;
    C(2+c,1+c) = n;
    C(1+c, d) = n;
    C(d, 1+c) = n;
    C(1+c, d+1) = n;
    C(d+1, 1+c) = n;
    e = d + n + 3;
    for i = 2+c+n:n+1:d %Generates intralevel connections
        C(i,i-1) = n;
        C(i-1,i) = n;
        C(i,i+1) = n;
        C(i+1,i) = n;
        C(i,e) = n+1;   %interlevel connection
```

```
        C(e,i) = n+1;   %interlevel connection
        e = e + n + 2;
    end
    j = 2 + c;
    k = 2 + d;
    while j <= d
        if mod(j-1-c,n+1) == 0
            j = j + 1;
            k = k + 2;
        end
     %generates remaining interlevel connections
        C(j, k) = (n+1);
        C(k, j) = (n+1);
        C(j, k+1) = (n+1);
        C(k+1,j) = (n+1);
        j = j + 1;
        k = k +1;
    end
end
% enter edges connnecting to center
    C(1,a) = 1;
    C(a,1) = 1;
    C(3,a) = 1;
    C(a,3) = 1;
    C(5,a) = 1;
    C(a,5) = 1;
    C(7,a) = 1;
    C(a,7) = 1;
    D = sum(C);
    for i = 1:a     %Make row and column sums 0
        C(i,i) = -D(i);
    end
    C(:,(a-4*z-8):(a-1)) = [];  %reduce to only interior (C)
    C((a-4*z-8):(a-1),:) = [];  %reduce to only interior (C)
    F = size(C);
   %G = -C\eye(F(1));   old method
    %G = G(:,F(1));     old method
    Q = zeros(F(1),1);
    Q(F(1),1) = 1;
    G = -C\Q;   %alternative way of finding C inverse
```

B Plots of Δg versus distance for rectangular grid networks with $\gamma(d) = d$

5cm4.5cmgraph1.bmp 5cm4.5cmgraph2.bmp

References

[1] Aravkin, Sasha. *Transfinite Networks.* Submitted.

[2] Lynch, Phillip. *Infinite Networks.* Submitted.

[3] Morrow, James K. Curtis, Edward B. *Inverse Problems for Electrical Networks.* Series on Applied Mathematics, Vol. 13. Singapore: World Scientific, 2000.