Transfinite Networks

Sasha Aravkin

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Abstract

This paper considers the Dirichlet problem on infinite networks with finite boundary nodes. Using a discrete analogue to the Green's function and the Poisson Integral Formula, we obtain explicit expressions for the solutions to the Dirichlet problem and the Green's function on several special types of networks.

1 Background on Green's Function

In complex analysis, the Green's function on a relatively compact region Ω is that function which satisfies these properties:

1. $G_p(z) = 0$ for $z \in \partial\Omega$
2. $G_p(z)$ is harmonic for $z \in \Omega - p$
3. $G_p(z)$ blows up like $-log|z-p|$ at p

When we consider the whole plane, our boundary is the point at ∞. Under those conditions, the Green's function as defined above cannot exist. We can prove that one cannot have a harmonic function that is bounded as $z \to \infty$, yet blows up at a point p.

Suppose we have a harmonic function $U(z)$ s.t. $|U(z)| \leq M$ as $|z| \to \infty$. We introduce a conformal mapping, $z = \frac{1}{\zeta}$. We can now work with a harmonic function, $V(\zeta) = U(\frac{1}{\zeta})$, on the punctured disk. We want to show that $V(\zeta)$ has a removable singularity at 0. We begin by looking for an analytic function $F(\zeta) = V(\zeta) + iW(\zeta)$, where $W(\zeta)$ is the harmonic conjugate of V. Since this harmonic conjugate must satisfy the Cauchy-Riemann equations, we know that its gradient must be $(-V_y, V_x)$. We can integrate the gradient along a curve to get the value of W at a point, that is, $\int_p^q \nabla W \cdot \vec{T} ds = W(q)$. There is a problem, however; since we are not working on a simply connected set, the integral over a closed curve containing the origin may not necessarily be zero. However, we can show that if it is not zero, it is a constant. Look at the differences between integrals over two different circles centered at the origin, and apply the Green's Theorem:

$$\int_C -V_y dx + V_x dy - \int_{C'} -V_y dx + V_x dy = \int -(V_y)_y - (V_x)_x = 0 \tag{1}$$

by the definition of harmonic functions. Thus the integral over any circle containing the origin must be a constant; call it P. If $P = 0$, the function is well defined, and we are done. Thus we assume that $P \neq 0$. We now have a multi-valued function

$$\tilde{F} = V + (i(W + nP))\frac{2\pi}{P} = V + i(\frac{2\pi}{P}W + 2n\pi). \tag{2}$$

Exponentiating this function we get

$$e^{\tilde{F}} = e^{V + i(\frac{2\pi}{P}W + 2n\pi)} = \tilde{G} \tag{3}$$

Now, $\tilde{G}(\zeta)$ is a single valued function, and $|\tilde{G}| = e^V$, which is bounded at the origin. We can simplify our notation by using our original definitions:

$$\tilde{G}(\zeta) = \tilde{G}(\frac{1}{z}) = G(z) = e^{F(z)} \tag{4}$$

Furthermore, $G(z)$ is analytic in $\mathbb{C} - p$ and

$$Re(G(z)) = e^{V(\frac{1}{z})} = e^{U(z)} \tag{5}$$

As $|G(z)|$ is bounded at infinity, $U(z)$ must be bounded, and has a removable singularity at infinity. The first step of the argument is now complete: we have shown that we can produce a function analytic on $\mathbb{C} - p$ that has $U(z)$ as its real part, even though we are not working on a simply connected set.

Let us turn our attention to the point p. At the point p, our function is supposed to blow up like $-log|z - p|$, so that the function $U + log|z - p|$ should be harmonic.

$$e^{U+log|z-p|} = e^U|z - p| = (Re(G(z))(|z - p|) = |G(z)||z - p| = |e^{F(z)}(z - p)| = |H(z)| \tag{6}$$

The function $H(z)$ as a removable singularity at p, and is thus entire. Furthermore, $\frac{H(z)}{z-p}$ is bounded at infinity. By a generalization of Liouiville's theorem, we can say that H is a polynomial of degree ≤ 1. In order for this to be true, $e^F = Const \Rightarrow U(z) = Const$. Thus, a function $U(z)$ cannot satisfy all the requirements of the Green's function in the entire complex plane.

The discrete case is interesting because, by setting restrictions on the conductances, we can indeed find a discrete analogue to the Green's function over an infinite network. By specifying our convergence criterion, we force the resistance to go to zero at the ends of the wires. We can thus think of the network as that of finite size, being imbedded in a disk, with more and more branches as you go to the boundary. We are now ready to define the discrete analog.

Set the potential to be zero at the boundary nodes, and apply a unit current to node k. The discrete Green's function, call $u(p)$, shall satisfy the following:

1. $u(p) = 0$ for $p \in \partial\Omega$
2. $Ku = 0$ on $\Omega - p$
3. $Ku = 1$ at the point k

2 Using the Green's Function to Solve the Dirichlet Problem

This paper will describe the discrete analogue to the Green's function and its normal derivative, and will explain its application in solving the discrete Dirichlet problem on special kinds of inifinte networks.

2.1 Continuous Case

Given a Domain $D \subset \mathbb{C}$ whose boundary, Γ, consists of a finite number of disjoint piecewise smooth simple closed curves, we can show that

$$u(p) = -\frac{1}{2\pi} \int_{\Gamma} u(z) \frac{\partial G_p(z)}{\partial n(z)} ds(z) \tag{7}$$

for any function u that is continuous on $D \cup \Gamma$ and harmonic on D. This is the Poisson Integral Formula. Thus if we know the normal derivative of Green's function on the boundary, called the Poisson Kernel, and the boundary values of u, we can solve the Dirichlet problem. It is interesting to note that we don't need the actual Green's function, nor do we need it everywhere; this will produce an interesting result in the discrete analogue.

2.2 Dirichlet Problem on Discrete Network

Curtis and Morrow ([2], p.43) provide a good formulation of the Dirichlet problem. When we multiply the familiar Kirchhoff's matrix by the vector of boundary and interior potentials, you get the boundary and interior currents. The latter are 0 from Kirchhoff's law.

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \cdot \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix} \tag{8}$$

Solving these equations, we obtain an expression for the interior potentials, g, which are the solution to the Dirichlet problem:

$$g = -C^{-1}B^T f \tag{9}$$

Once we find the quantity $-C^{-1}B^T$, we will have our solution. Hence, this expression is the discrete analogue to the Poisson Kernel. If the network is finite, we can find the solution directly by inverting C and multiplying it by B^T. However, since we are considering infinite networks, it is worthwhile to understand what these matrix products actually are, and this will bring us to the solution.

2.3 Discrete Green's Function

Suppose we take a network and apply a unit current to an interior node. We also set the potentials to be zero on the boundary. These are easily recognizable as conditions of the Green's function. Our equation then becomes

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \cdot \begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} \phi \\ \psi \end{bmatrix} \tag{10}$$

Notice that g is a vector of interior potentials which result from a unit current being applied to an interior node. Now, we obtain two linear equations from the matrix equation; lets work with the second:

$$Cg = \psi \tag{11}$$

$$\text{C is invertible, so} \qquad g = C^{-1}\psi \tag{12}$$

Our ψ vector is a vector of interior currents, so it has zeros everywhere except the node to which the current is applied, where it has a 1. Thus, as we vary the node at which we apply the unit current, our interior potentials are appropriate rows of C^{-1}. So if we do this for every interior node, we see that C^{-1} is an $int \times int$ matrix whose columns are indexed by the node k to which the unit current is applied. The rows give the values of the potentials at every interior node p because of the application of the unit current at node k. In other words, C^{-1} is our discrete analogue to the Green's Function.

2.4 Discrete Analogue to the Poisson Kernel

As we saw in the continuous case, one can integrate $\frac{\partial G}{\partial n}u(z)$ around the boundary to obtain the interior potentials. It makes good sense to take $C^{-1}B^T$ to be this analogue, since we multiply this quantity by boundary potentials to find the solution to the Dirichlet problem. Originally, it seemed like we would need to find the Green's function first, before tackling this quantity, but as it turned out, it is possible to understand it as a whole. Lets rewrite our matrix equation yet again, using the Green's function potentials:

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \cdot \begin{bmatrix} 0 \\ C^{-1} \end{bmatrix} = \begin{bmatrix} \phi \\ I \end{bmatrix} \tag{13}$$

Using the first equation of this system, we obtain:

$$\phi = BC^{-1} \cdot I = BC^{-1} \tag{14}$$

Lets take the transpose of the whole system.

$$(\phi)^T = (BC^{-1})^T = C^{-1}B^T \tag{15}$$

Hence, for $-C^{-1}B^T$, we just take $(-\phi)^T$. We are looking to construct is a $int \times \partial$ matrix whose rows are indexed by the node k to which the unit current is applied, and whose columns give the negated currents at the boundary due to that application. Clearly, this is $-C^{-1}B^T$, just what we were looking for, so we can produce the solution to the Dirichlet problem by multiplying this $int \times \partial$ matrix by the $\partial \times 1$ vector of boundary potentials. A natural restriction of the problem is to have a finite number of boundary nodes. Then we will have an infinite collection of finite row vectors of length ∂, and can multiply an arbitrary row i by ϕ to get g_i.

3 Infinite Line Network

Consider an infinite collection of nodes with potentials v_i connected by wires with conductances γ_i. A finite line has two boundary nodes, the first and the last. Analagously, we consider the two boundary nodes of the infinite line to be located at zero and infinity, using limit notation to make sense of the potential.

In [1], Lynch considers all the conductances on the line to be constant. As a result, the power generated on the network diverges. In order for the problem to make sense, we found the following restriction should be placed on the conductances:

$$\sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty \tag{16}$$

This restriction immediately implies that finite power is dissipated.

$$P = \sum_{i=1}^{\infty} I^2 \gamma_i \tag{17}$$

If a constant current or a finite number of constant currents are applied to the network, the power is finite. Now, we can consider the Green's function on this line.

3.1 Finding the Green's Function on the Line

When we apply a unit current at node k, it will split in two directions, which we will call I_+ and I_-, with right flow chosen positive. Since the potential at the boundary nodes is set to 0, we can come up with two expressions for the potential at node k. It is important to note the approach we use here, as it will be useful in more complicated cases. We think of bringing the ends of the wire together, into a single node, since both have voltage of 0. Then, we have two wires connected in parallel, and we can use equivalent resistance formulas familar from elementary physics. We can use this method whenever we have several wires with the same voltage on the ends.

$$g_k(k) = I_- \sum_{i=1}^{k} \frac{1}{\gamma_i} \tag{18}$$

$$g_k(k) = I_+ \sum_{i=k+1}^{\infty} \frac{1}{\gamma_i} \tag{19}$$

Since we also know that $I_+ + I_- = 1$, we can get an expression for $g_k(k)$, in terms of sums over segments of the wire, and ultimately an expression for $u_k(p)$ for any node p:

$$\text{if} \quad p \leq k \quad g_k(p) = \frac{\sum_{i=1}^{p+1} \frac{1}{\gamma_i} \sum_{i=k+1}^{\infty} \frac{1}{\gamma_i}}{\sum_{i=1}^{\infty} \frac{1}{\gamma_i}} \tag{20}$$

$$\text{if} \quad p \geq k \quad g_k(p) = \frac{\sum_{i=1}^{k+1} \frac{1}{\gamma_i} \sum_{i=p+1}^{\infty} \frac{1}{\gamma_i}}{\sum_{i=1}^{\infty} \frac{1}{\gamma_i}} \tag{21}$$

3.2 Solving the Dirichlet Problem on the Line

As previously shown, we can produce a matrix that is $-C^{-1}B^T$ by looking at the currents produced on the boundary when we apply a unit current to interior node k. Thus we are looking for a $int \times \partial$ matrix. Our columns will be I_+ and I_-, while the rows will be indexed by the node k at which the unit current is applied. Here are the expressions:

$$I_-(k) = \frac{\sum_{i=k+1}^{\infty} \frac{1}{\gamma_i}}{\sum_{i=1}^{\infty} \frac{1}{\gamma_i}} \tag{22}$$

$$I_+(k) = \frac{\sum_{i=1}^{k} \frac{1}{\gamma_i}}{\sum_{i=1}^{\infty} \frac{1}{\gamma_i}} \tag{23}$$

Our matrix now looks like this:

$$\begin{bmatrix} -I_-(1) & -I_+(1) \\ -I_-(2) & -I_+(2) \\ ... & ... \\ -I_-(k) & -I_+(k) \\ ... & ... \end{bmatrix} = -C^{-1}B^T \tag{24}$$

Thus, when we multiply this matrix by our boundary potentials, ϕ, we will have the solution to the Dirichlet problem. Conveniently enough, as the reader noticed, it is easier to find the expression for $-C^{-1}B^T$ than the actual expression for the Green's function, and in my calculation, the solution to the latter requires the solution to the former.

3.3 Power Dissipated on the Line

In his work with the Green's Function, Lynch assumed constant conductances. As a result, the power dissipated by the infinite line network is infinite. It is worth checking that the power dissipated is finite for our network.

We can express power as $P = IV = I^2R$. Thus, on the line,

$$P = \sum_{i=1}^{\infty} I^2 R_i = I^2 \sum_{i=1}^{\infty} \frac{1}{\gamma_i} < \infty \tag{25}$$

Thus, if we impose a current on one boundary node, the power dissipated will be finite. Now, impose a voltage of 1 on the first boundary node and a voltage of 0 on the second:

$$\Delta V = IR = 1 \Rightarrow I = \frac{1}{R} = \frac{1}{\sum_{i=1}^{\infty} \frac{1}{\gamma_i}} \tag{26}$$

The power expression, I^2R, still converges.

4 N-Ray Network

Now that we have the Green's function and solution to the Dirichlet problem for a single infinite ray, it is natural to consider a star of several such rays joined at the origin. We will be working on a particular ray, call R_p, and will number the remaining n rays R_1 through R_n clockwise, though the ordering is only for convenience. The Green's function is slightly more complicated here, since to calculate the potential at a point, we must consider both the segment of the particular ray and the n ray star.

4.1 Caluculating the Green's Function

Using the previously explained parallel circuit model, we obtain the expression for the current I_s in a ray R_s when a current I is applied to the origin.

$$I_s = \frac{I}{\Sigma s * \sum_{i=1}^{n} \frac{1}{\Sigma i}} \tag{27}$$

To avoid being bogged down in detail, we simplify our notation. $\sum_{i=1}^{\infty} \frac{1}{\gamma_s} = \Sigma s$, to be read as 'sum of reciprocals of conductances over R_s'. If indices are present, i.e. $\Sigma_a^b s$, this indicates a partial sum of reciprocals of conductances from a to b over R_s. R_p is the ray on which the node k is located. Using the expression for current and methods of the first problem we can write down the Green's function for any point q on any R_s:

$$\text{if} \quad k \leq q \quad \text{on} \quad R_p \quad g_k(q) = \frac{\Sigma_q^\infty p[\Sigma_0^k p \sum_{i=1}^n \frac{1}{\Sigma i} + 1]}{\Sigma p \sum_{i=1}^n \frac{1}{\Sigma i} + 1} \tag{28}$$

$$\text{if} \quad q \leq k \quad \text{on} \quad R_p \quad g_k(q) = \frac{\Sigma_k^\infty p[\Sigma_0^q p \sum_{i=1}^n \frac{1}{\Sigma i} + 1]}{\Sigma p \sum_{i=1}^n \frac{1}{\Sigma i} + 1} \tag{29}$$

$$\text{if} \quad q \quad \text{on} \quad R_s \quad g_k(q) = \frac{\Sigma_q^\infty s \Sigma_k^\infty p}{\Sigma s[\Sigma p \sum_{i=1}^n \frac{1}{\Sigma i} + 1]} \tag{30}$$

It is easy to check that this function is 'continuous' over the intervals; that is, it yields the same values at the end points: $(0, k)$. This function also satisfies all the properties of the Green's function.

4.2 Solution to Dirichlet Problem

We have already calculated an expression for the current at the boundary if the unit current is applied at the origin. It is not difficult to modify this expression to yield the current on the boundary when the unit current is applied to an arbitrary interior node. We will index our k node k_s, so that it is clear that it depends on the ray in question.

$$\text{on particular ray} \quad I_{k_s} = \frac{\Sigma_0^k p \sum_{i=1}^n \frac{1}{\Sigma i} + 1}{\Sigma p \sum_{i=1}^n \frac{1}{\Sigma i} + 1} \tag{31}$$

$$\text{on any s ray} \quad I_{k_s} = \frac{\frac{\Sigma_k^\infty p \sum_{i=1}^n \frac{1}{\Sigma i} + 1}{\Sigma p \sum_{i=1}^n \frac{1}{\Sigma i} + 1}}{\sum_{i=1}^\infty \frac{1}{\gamma_s} * \sum_{i=1}^n \frac{1}{\sum_{i=1}^\infty \frac{1}{\gamma_i}}} \tag{32}$$

This equation is rather involved, because we have to account for partial sums on the particular ray in calculating the currents. However, now we are in a good position to claim to have the solution to the Dirichlet problem. We arrange the currents in a matrix as follows:

$$\begin{bmatrix} -I_{1_1} & : -I_{1_2} & : \ldots & -I_{1_{N+1}} \\ -I_{2_1} & : -I_{2_2} & : \ldots & -I_{2_{N+1}} \\ \ldots & : \ldots & : \ldots & \ldots \\ -I_{k_1} & : -I_{k_2} & : \ldots & -I_{k_{N+1}} \\ \ldots & : \ldots & : \ldots & \ldots \end{bmatrix} = -C^{-1}B^T \tag{33}$$

In this network, instead of having an $\infty \times 2$ matrix, as in the previous example, we will have an $\infty \times (N+1)$ matrix.

4.3 Power Dissipated

Imposing a current I at R_p, a particular ray in the network, we get the following expression for power dissipated:

$$P = I^2 \Sigma p + \sum_{i=1}^N I_n^2 \Sigma n = I^2 (\Sigma p + \sum_{i=1}^n \frac{1}{\Sigma n (\sum_{j=1}^n \frac{1}{\Sigma j})^2}) \tag{34}$$

This result is obviously finite, as all Σ's are finite from our original assumption. The problem becomes interesting when we take the limit as N goes to ∞.

We will first prove that our expression converges for several general cases. Suppose that the equivalent conductances along the wires are bounded above and and below, an assumption that is quite reasonable. That is, $0 < \alpha \leq \Sigma_i \leq \beta < \infty$ for all i. R_p is not important in the question of convergence, so we just look at the troublesome part of our expression:

$$\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{\Sigma n (\sum_{j=1}^n \frac{1}{\Sigma j})^2} \leq \lim_{n\to\infty} \sum_{i=1}^\infty \frac{1}{\alpha (\sum_{j=1}^n \frac{1}{\beta})^2} = \lim_{n\to\infty} \sum_{i=1}^n \frac{\beta^2}{\alpha n^2} = \lim_{n\to\infty} \frac{\beta^2}{\alpha n} = 0 \tag{35}$$

So, if we take this limit, our power dissipated becomes $I^2\Sigma p$. However, the conductances don't necessarily need to have an upper bound. Suppose we define a sequence of Σi's such that $\sum_{i=1}^{\infty} \frac{1}{\Sigma_i}$ converges to a finite limit, call L. Then:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\Sigma n(\sum_{j=1}^{n} \frac{1}{\Sigma j})^2} \leq \lim_{n\to\infty} \Sigma_1^2 \sum_{i=1}^{n} \frac{1}{\Sigma_i} = \Sigma_1^2 L \tag{36}$$

The only case left to consider is when Σ_i's don't have an upper bound, but don't grow fast enough for the sum of their reciprocals to converge. I conjecture that the power dissipated will converge for that case also.

References

[1] Lynch, Phillip. *Infinite Networks.*

[2] Curtis, Edward,& Morrow, James. *Inverse Problems for Electrical Networks*