Numbering Boundary Nodes

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1 Introduction

The purpose of this paper is to explore how numbering electrical resistor networks effects their response matrix, Λ. Moreover, what can be learned from Λ about the topology of the network? This paper considers specific networks and their Λ, and shows how the boundary nodes were placed in the original network. In other words, no matter how the boundary nodes are numbered, can the information from Λ and about the network still be recovered?

Most of the networks looked at in this paper are edge conductivity networks where current at p, $\phi(p)$, is defined as such:

$$\sum_{q \sim p} \gamma_{p,q}(u_p - u_q)$$

where $\gamma_{p,q}$ is the conductivity of the edge joining node p to q.

The Kirchhoff matrix, K, for a network with n nodes, is an $n \times n$ symmetric matrix where

- $K_{i,j} = -\gamma_{i,j}$ when $i \neq j$
- $K_{i,i} = \sum_{j \neq i} \gamma_{i,j}$

It is useful to write K in block form where the interior nodes fall in block C and the boundary nodes fall in A.

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

Then the response matrix, Λ is formed by taking the Schur complement of K in C. That is

$$\Lambda = A - BC^{-1}B^T$$

In order to determine sign conditions in Λ, select two distinct sets of boundary nodes, $P = (p_1, p_2, \ldots, p_k)$ and $Q = (q_1, q_2, \ldots, q_k)$. Then

$$det\Lambda(P;Q) \cdot detK(I;I) = (-1)^k \sum_{\tau \in S_k} sgn(\tau) \left\{ \sum_{\substack{\alpha \\ \tau_\alpha = \tau}} \prod_{e \in E_\alpha} \gamma(e) \cdot D_\alpha \right\}$$

shows what the sign of det $\Lambda(P;Q)$ is. So if there is only one distinct pairing between the nodes in P and the nodes in Q, then the sign of det $\Lambda(P;Q)$ is known. This background information comes from [1].

This paper looks at annular networks, many of which have conductivities constant on layers. Due to the symmetry of the networks, the ordering of the boundary nodes can be found in these cases, mainly because of the symmetries found in Λ and sign conditions of Λ.

2 Annular Network with Three Rays and Two Circles

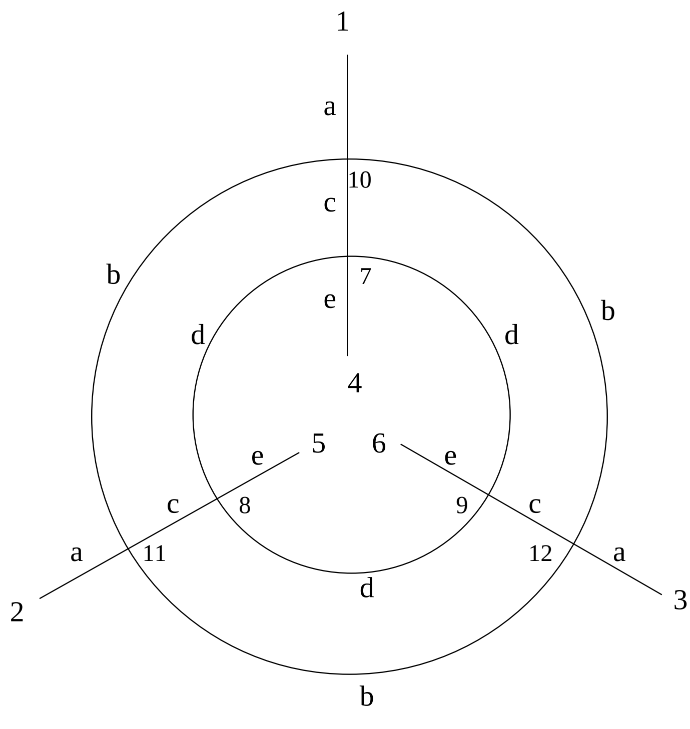

Figure 1: G(3,2)

First consider G(3,2), that is a graph defined as such: G(*# of rays*, *# of circles*), and assume that conductivity is constant on layers. Then the response matrix, Λ, is as follows:

$$\Lambda = \begin{pmatrix} \Sigma & \alpha & \alpha & \beta & \gamma & \gamma \\ \alpha & \Sigma & \alpha & \gamma & \beta & \gamma \\ \alpha & \alpha & \Sigma & \gamma & \gamma & \beta \\ \beta & \gamma & \gamma & \Sigma & \delta & \delta \\ \gamma & \beta & \gamma & \delta & \Sigma & \delta \\ \gamma & \gamma & \beta & \delta & \delta & \Sigma \end{pmatrix}$$

where

$$\alpha = \frac{-\ a^2(bc^2 + 3bcd + c^2d + 2bce + 3bde + be^2)}{(ac + ae + ce)(ac + 3bc + 3ad + 9bd + 3cd + ae + 3be + ce)}$$

$$\beta = \frac{-\ ace(ac + bc + ad + 3bd + cd + ae + be + ce)}{(ac + ae + ce)(ac + 3bc + 3ad + 9bd + 3cd + ae + 3be + ce)}$$

$$\gamma = \frac{-\ ace(bc + ad + 3bd + cd + be)}{(ac + ae + ce)(ac + 3bc + 3ad + 9bd + 3cd + ae + 3be + ce)}$$

$$\delta = \frac{-\ (bc^2 + a^2d + 3abd + 2acd + 3bcd + c^2d)e^2}{(ac + ae + ce)(ac + 3bc + 3ad + 9bd + 3cd + ae + 3be + ce)}$$

and

$$\Sigma = -\ (\textit{Sum of the row entries})$$

Regardless of how the boundary nodes are numbered the response matrix still contains four distinct entries which provide information about the structure of the network. The information may not be as obvious as before because the ordering of the entries in Λ is no longer the same, but the information is still there. For instance, if $\lambda_{x,y}$ is β then x and y are boundary nodes on the same ray. It is fairly easy to identify the β term in the response matrix because it is the only term that shows up six times and is in every row and every column. Now, which nodes appear on the inside of the graph and which are on the outside? Technically, there is no way to tell between the inside and outside of this network because it could easily be inverted. But, it is possible to tell which three nodes are grouped together on the inside or outside. The terms in Λ that show this are δ and α, each of which are the two other terms which appear six times. So for every x and y such that $\lambda_{x,y}$ is a δ, then those x and y's are grouped together. Likewise for every x and y such that $\lambda_{x,y}$ is a α.

There can be many different Λ matrices that have the same structure as the Λ for this network. In order for a Λ to be a response matrix for this network, not only must the above symmetries hold, but these sign conditions must also hold, where (x_1, x_2, x_3) are grouped together on the inside (or outside) and (y_1, y_2, y_3) are the outside, and (x_1, y_1), (x_2, y_2), and (x_3, y_3) are paired on the same ray.

1. det $\Lambda(x_1, x_2, x_3; y_1, y_2, y_3)$ < 0
2. det $\Lambda(x_1, x_2, y_2; x_3, y_1, y_3)$ > 0
3. det $\Lambda(x_1, x_2; y_1, y_2)$ > 0
4. det $\Lambda(x_1, y_1; x_2, y_2)$ > 0

3 Annular Network with Three Rays and Two Circles (With Less Symmetry)

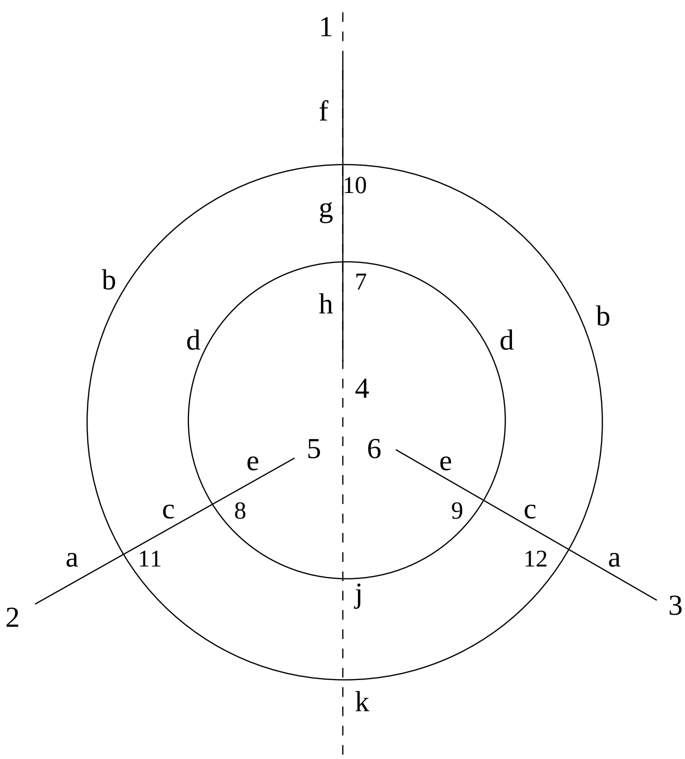

Figure 2: G(3,2) with less symmetry

This is another G(3,2) graph, but symmetric only with respect to the "y-axis". The response matrix, Λ is as follows:

$$\Lambda = \begin{pmatrix} \Sigma & \alpha & \alpha & \beta & \delta & \delta \\ \alpha & \Sigma & \epsilon & \sigma & \rho & \theta \\ \alpha & \epsilon & \Sigma & \sigma & \theta & \rho \\ \beta & \sigma & \sigma & \Sigma & \gamma & \gamma \\ \delta & \rho & \theta & \gamma & \Sigma & \omega \\ \delta & \theta & \rho & \gamma & \omega & \Sigma \end{pmatrix}$$

where

$$\begin{aligned}
\alpha \quad = \quad & -(e(2abd + 2bcd + bcg + 2adg + 2bdg + 2cdg)h)/(4abcd + 4abde + 4bcde + \\
& + 2abcg + 2abdg + 4acdg + 2bcdg + 2abeg + 2bceg + 4adeg + 4bdeg + 4cdeg + \\
& + acg^2 + bcg^2 + adg^2 + bdg^2 + cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + 2abdh + \\
& + 2bcdh + 2abeh + 2bceh + 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh + 2aegh + \\
& + 2begh + 2cegh)
\end{aligned}$$

$$\begin{aligned}
\beta \quad = \quad & -(f(2bcd + acg + bcg + adg + bdg + cdg + aeg + beg + ceg)h)/(4abcd + \\
& + 4abde + 4bcde + 2abcg + 2abdg + 4acdg + 2bcdg + 2abeg + 2bceg + 4adeg + \\
& + 4bdeg + 4cdeg + acg^2 + bcg^2 + adg^2 + bdg^2 + cdg^2 + aeg^2 + beg^2 + ceg^2 + \\
& + 2abch + 2abdh + 2bcdh + 2abeh + 2bceh + 2acgh + 2bcgh + 2adgh + 2bdgh + \\
& + 2cdgh + 2aegh + 2begh + 2cegh)
\end{aligned}$$

$$\begin{aligned}
\delta \quad = \quad & -(a(2bcd + bcg + bdg + 2cdg + beg)h)/(4abcd + 4abde + 4bcde + 2abcg + \\
& + 2abdg + 4acdg + 2bcdg + 2abeg + 2bceg + 4adeg + 4bdeg + 4cdeg + \\
& + acg^2 + bcg^2 + adg^2 + bdg^2 + cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + 2abdh + \\
& + 2bcdh + 2abeh + 2bceh + 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh + \\
& + 2aegh + 2begh + 2cegh)
\end{aligned}$$

$$\begin{aligned}
\epsilon \quad = \quad & -(e^2(2b^2c^2d + 2a^2bd^2 + 2ab^2d^2 + 4abcd^2 + 2b^2cd^2 + 2bc^2d^2 + b^2c^2g + \\
& + 2abcdg + 2b^2cdg + 2bc^2dg + 2a^2d^2g + 4abd^2g + 2b^2d^2g + 4acd^2g + \\
& + 4bcd^2g + 2c^2d^2g + b^2c^2h + 4a^2bdj + 4ab^2dj + 8abcdj + 4b^2cdj + \\
& + 4bc^2dj + 2a^2bgj + 2ab^2gj + 4abcgj + 2b^2cgj + 2bc^2gj + 4a^2dgj + \\
& + 8abdgj + 4b^2dgj + 8acdgj + 8bcdgj + 4c^2dgj + a^2g^2j + 2abg^2j + b^2g^2j + \\
& + 2acg^2j + 2bcg^2j + c^2g^2j + 2a^2bhj + 2ab^2hj + 4abchj + 2b^2chj + 2bc^2hj + \\
& + 2a^2ghj + 4abghj + 2b^2ghj + 4acghj + 4bcghj + 2c^2ghj + 4bc^2dk + \\
& + 4abd^2k + 4bcd^2k + 2bc^2gk + 4bcdgk + 4c^2dgk + 4ad^2gk + 4bd^2gk + \\
& + 4cd^2gk + c^2g^2k + 2bc^2hk + 2c^2ghk + 8abdjk + 8bcdjk + 4abgjk + 4bcgjk +
\end{aligned}$$

$$+\, 8adgjk + 8bdgjk + 8cdgjk + 2ag^2jk + 2bg^2jk + 2cg^2jk + 4abhjk +$$
$$+\, 4bchjk + 4aghjk + 4bghjk + 4cghjk)/(4abcd + 4abde + 4bcde + 2abcg +$$
$$+\, 2abdg + 4acdg + 2bcdg + 2abeg + 2bceg + 4adeg + 4bdeg + 4cdeg +$$
$$+\, acg^2 + bcg^2 + adg^2 + bdg^2 + cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + 2abdh +$$
$$+\, 2bcdh + 2abeh + 2bceh + 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh +$$
$$+\, 2aegh + 2begh + 2cegh)$$

$$\sigma = -\,(ef(2bcd + bcg + adg + bdg + cdg + bch))/(4abcd + 4abde + 4bcde + 2abcg +$$
$$+\, 2abdg + 4acdg + 2bcdg + 2abeg + 2bceg + 4adeg + 4bdeg + 4cdeg +$$
$$+\, acg^2 + bcg^2 + adg^2 + bdg^2 + cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + 2abdh +$$
$$+\, 2bcdh + 2abeh + 2bceh + 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh +$$
$$+\, 2aegh + 2begh + 2cegh)$$

$$\rho = -\,(ae(4abc^2d + 2b^2c^2d + 2abcd^2 + 2b^2cd^2 + 2bc^2d^2 + 4abcde + 2b^2cde +$$
$$+\, 4bc^2de + 2abc^2g + b^2c^2g + 3abcdg + 2b^2cdg + 4ac^2dg + 4bc^2dg + abd^2g +$$
$$+\, b^2d^2g + 2acd^2g + 3bcd^2g + 2c^2d^2g + 2abceg + b^2ceg + 2bc^2eg + abdeg +$$
$$+\, b^2deg + 4acdeg + 5bcdeg + 4c^2deg + ac^2g^2 + bc^2g^2 + acdg^2 + bcdg^2 + c^2dg^2 +$$
$$+\, aceg^2 + bceg^2 + c^2eg^2 + 2abc^2h + b^2c^2h + 2abcdh + b^2cdh + 2bc^2dh + 2abceh +$$
$$+\, b^2ceh + 2bc^2eh + 2ac^2gh + 2bc^2gh + 2acdgh + 2bcdgh + 2c^2dgh + 2acegh +$$
$$+\, 2bcegh + 2c^2egh + 4abcdj + 4b^2cdj + 4bc^2dj + 2abcgj + 2b^2cgj + 2bc^2gj +$$
$$+\, 2abdgj + 2b^2dgj + 4acdgj + 6bcdgj + 4c^2dgj + acg^2j + bcg^2j + c^2g^2j +$$
$$+\, 2abchj + 2b^2chj + 2bc^2hj + 2acghj + 2bcghj + 2c^2ghj + 4bc^2dk + 4bcd^2k +$$
$$+\, 4bcdek + 2bc^2gk + 4bcdgk + 4c^2dgk + 2bd^2gk + 4cd^2gk + 2bcegk + 2bdegk +$$
$$+\, 4cdegk + c^2g^2k + cdg^2k + ceg^2k + 2bc^2hk + 2bcdhk + 2bcehk + 2c^2ghk +$$
$$+\, 2cdghk + 2ceghk + 8bcdjk + 4bcgjk + 4bdgjk + 8cdgjk + 2cg^2jk + 4bchjk +$$
$$+\, 4cghjk))/(4abcd + 4abde + 4bcde + 2abcg + 2abdg + 4acdg + 2bcdg +$$
$$+\, 2abeg + 2bceg + 4adeg + 4bdeg + 4cdeg + acg^2 + bcg^2 + adg^2 + bdg^2 +$$
$$+\, cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + 2abdh + 2bcdh + 2abeh + 2bceh +$$
$$+\, 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh + 2aegh + 2begh + 2cegh)$$

$$\theta = -\,(ae(2b^2c^2d + 2abcd^2 + 2b^2bcd^2 + 2bc^2d^2 + 2b^2cde + b^2c^2g + abcdg + 2b^2cdg +$$
$$+\, 2bc^2dg + abd^2g + b^2d^2g + 2acd^2g + 3bcd^2g + 2c^2d^2g + b^2ceg + abdeg + b^2deg +$$
$$+\, bcdeg + b^2c^2h + b^2cdh + b^2ceh + 4abcdj + 4b^2cdj + 4bc^2dj + 2abcgj + 2b^2cgj +$$
$$+\, 2bc^2gj + 2abdgj + 2b^2dgj + 4acdgj + 6bcdgj + 4c^2dgj + acg^2j + bcg^2j +$$
$$+\, c^2g^2j + 2abchj + 2b^2chj + 2bc^2hj + 2acghj + 2bcghj + 2c^2ghj + 4bc^2dk +$$
$$+\, 4bcd^2k + 4bcdek + 2bc^2gk + 4bcdgk + 4c^2dgk + 2bd^2gk + 4cd^2gk + 2bcegk +$$
$$+\, 2bdegk + 4cdegk + c^2g^2k + cdg^2k + ceg^2k + 2bc^2hk + 2bcdhk + 2bcehk +$$

$$
\begin{aligned}
& + 2c^2ghk + 2cdghk + 2ceghk + 8bcdjk + 4bcgjk + 4bdgjk + 8cdgjk + 2cg^2jk + \\
& + 4bchjk + 4cghjk))/(4abcd + 4abde + 4bcde + 2abcg + 2abdg + 4acdg + 2bcdg + \\
& + 2abeg + 2bceg + 4adeg + 4bdeg + 4cdeg + acg^2 + bcg^2 + adg^2 + bdg^2 + \\
& + cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + 2abdh + 2bcdh + 2abeh + 2bceh + \\
& + 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh + 2aegh + 2begh + 2cegh)
\end{aligned}
$$

$$
\begin{aligned}
\gamma \quad = \quad & - af(2bcd + 2bde + bcg + bdg + cdg + beg + bch + bdh + beh))/(4abcd + 4abde + \\
& + 4bcde + 2abcg + 2abdg + 4acdg + 2bcdg + 2abeg + 2bceg + 4adeg + 4bdeg + \\
& + 4cdeg + acg^2 + bcg^2 + adg^2 + bdg^2 + cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + \\
& + 2abdh + 2bcdh + 2abeh + 2bceh + 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh + \\
& + 2aegh + 2begh + 2cegh)
\end{aligned}
$$

$$
\begin{aligned}
\omega \quad = \quad & - (a^2(2b^2c^2d + 2b^2cd^2 + 2bc^2d^2 + 4b^2cde + 2b^2d^2e + 2b^2de^2 + b^2c^2g + 2b^2cdg + \\
& + 2bc^2dg + b^2d^2g + 2bcd^2g + 2c^2d^2g + 2b^2ceg + 2b^2deg + 2bcdeg + b^2e^2g + \\
& + b^2c^2h + 2b^2cdh + b^2d^2h + 2b^2ceh + 2b^2deh + b^2e^2h + 4b^2cdj + 4bc^2dj + \\
& + 4b^2dej + 2b^2cgj + 2bc^2gj + 2b^2dgj + 4bcdgj + 4c^2dgj + 2b^2egj + c^2g^2j + \\
& + 2b^2chj + 2bc^2hj + 2b^2dhj + 2b^2ehj + 2c^2ghj + 4bc^2dk + 4bcd^2k + 8bcdek + \\
& + 4bd^2ek + 4bde^2k + 2bc^2gk + 4bcdgk + 4c^2dgk + 2bd^2gk + 4cd^2gk + 4bcegk + \\
& + 4bdegk + 8cdegk + 4d^2egk + 2be^2gk + 4de^2gk + c^2g^2k + 2cdg^2k + d^2g^2k + \\
& + 2ceg^2k + 2deg^2k + e^2g^2k + 2bc^2hk + 4bcdhk + 2bd^2hk + 4bcehk + 4bdehk + \\
& + 2be^2hk + 2c^2ghk + 4cdghk + 2d^2ghk + 4ceghk + 4deghk + 2e^2ghk + 8bcdjk + \\
& + 8bdejk + 4bcgjk + 4bdgjk + 8cdgjk + 4begjk + 8degjk + 2cg^2jk + 2dg^2jk + \\
& + 2eg^2jk + 4bchjk + 4bdhjk + 4behjk + 4cghjk + 4dghjk + 4eghjk)/(4abcd + \\
& + 4abde + 4bcde + 2abcg + 2abdg + 4acdg + 2bcdg + 2abeg + 2bceg + 4adeg + 4bdeg + \\
& + 4cdeg + acg^2 + bcg^2 + adg^2 + bdg^2 + cdg^2 + aeg^2 + beg^2 + ceg^2 + 2abch + \\
& + 2abdh + 2bcdh + 2abeh + 2bceh + 2acgh + 2bcgh + 2adgh + 2bdgh + 2cdgh + \\
& + 2aegh + 2begh + 2cegh)
\end{aligned}
$$

and

$$\Sigma = -\ (Sum\ of\ the\ row\ entries)$$

Regardless of how the boundary nodes are numbered the response matrix still contains nine distinct entries which provide information about the structure of the network. The information may not be as obvious as before because the ordering of the entries in Λ is no longer the same, but the information is still there. The first thing to notice in Λ is that there are three entries that only appear twice: β, ϵ, and ω. Upon further inspection, β can be distinguished from ϵ and ω because every row and column that contains a β also contains a Σ and two pairs. Note this is only true for β. So if $\lambda_{x,y}$ is β, then boundary nodes x and y are contained on the same ray. Moreover x and y are on the non-symmetric ray. Consider the other two entries that appear only twice: ϵ and ω. If $\lambda_{x,y}$ is ϵ or ω then x and y are grouped together on the inside or the outside.

Now one of the inside nodes will be paired with one of the outside nodes, but which one? In order to determine this, first select one of the boundary nodes not on the symmetric ray, call it x. Next, select the group of nodes (y, z) that does not contain x. That is if x is on the inside, then select the group known to be on the outside. There are two choices, x and y can be on the same ray or x and z. If $\lambda_{x,y} > \lambda_{x,z}$, then x and z are on the same ray. Otherwise x and y are together. The remaining two nodes are on the last ray. This is true because $\theta > \rho$ due to $\det \Lambda(1,2;4,5) > 0$ and $\det \Lambda(2,1,4;3,5,6) > 0$.

Now, which nodes are grouped together on the inside of the graph and which are on the outside? Technically, there is no way to tell between the inside and outside of this network because it could easily be inverted. But, it is possible to tell which three nodes are together on the inside or outside, two are already known due to ϵ and ω. To help simplify things call the nodes on the non-symmetric ray (x_1, x_2) and call the two sets of nodes that are grouped together on the inside or outside (y_1, y_2) and (z_1, z_2). Now there are two options: (x_1, y_1, y_2) and (x_2, z_1, z_2) can be grouped together or (x_1, z_1, z_2) and (x_2, y_1, y_2). If $\lambda_{x_1,y_1}\lambda_{x_2,z_1} < \lambda_{x_1,z_1}\lambda_{x_2,y_1}$, then (x_1, z_1, z_2) and (x_2, y_1, y_2) are the correct grouping, otherwise (x_1, y_1, y_2) and (x_2, z_1, z_2) are the correct grouping. This is true because $\gamma\alpha > \sigma\delta$ due to $\det \Lambda(1,4;2,5) > 0$. Note, in the inequality y_2 and z_2 could replace y_1 and z_1 respectively because they are the same.

There can be many different Λ matrices that have the same structure as the Λ for this network. In order for a Λ to be a response matrix for this network, not only must the above symmetries hold, but these sign conditions must also hold, where (x_1, x_2, x_3) are grouped together on the inside (or outside) and (y_1, y_2, y_3) are on the outside, and (x_1, y_1), (x_2, y_2), and (x_3, y_3) are paired on the same ray, note (x_1, y_1) is the non-symmetric ray.

1. $\det \Lambda(x_1, x_2, x_3; y_1, y_2, y_3) < 0$
2. $\det \Lambda(x_1, x_2, y_2; x_3, y_1, y_3) > 0$
3. $\det \Lambda(x_2, x_1, y_1; x_3, y_2, y_3) > 0$
4. $\det \Lambda(x_1, x_2; y_1, y_2) > 0$
5. $\det \Lambda(x_2, x_3; y_2, y_3) > 0$
6. $\det \Lambda(x_1, y_1; x_2, y_2) > 0$
7. $\det \Lambda(x_2, y_2; x_3, y_3) > 0$

4 Graph with Three Holes

4.1 Completely Symmetric Case

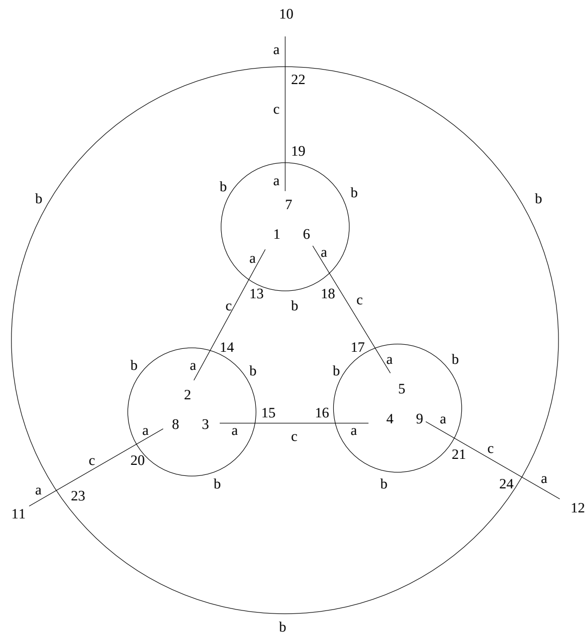

Figure 3: Graph with Three Holes

Consider this graph with conductivities on layers completely constant.

$$\Lambda = \begin{pmatrix}
\Sigma & \alpha & \beta & \delta & \beta & \gamma & \gamma & \beta & \kappa & \beta & \delta & \kappa \\
\alpha & \Sigma & \gamma & \beta & \delta & \beta & \beta & \gamma & \kappa & \delta & \beta & \kappa \\
\beta & \gamma & \Sigma & \alpha & \beta & \delta & \kappa & \gamma & \beta & \kappa & \beta & \delta \\
\delta & \beta & \alpha & \Sigma & \gamma & \beta & \kappa & \beta & \gamma & \kappa & \delta & \beta \\
\beta & \delta & \beta & \gamma & \Sigma & \alpha & \beta & \kappa & \gamma & \delta & \kappa & \beta \\
\gamma & \beta & \delta & \beta & \alpha & \Sigma & \gamma & \kappa & \beta & \beta & \kappa & \delta \\
\gamma & \beta & \kappa & \kappa & \beta & \gamma & \Sigma & \delta & \delta & \alpha & \beta & \beta \\
\beta & \gamma & \gamma & \beta & \kappa & \kappa & \delta & \Sigma & \delta & \beta & \alpha & \beta \\
\kappa & \kappa & \beta & \gamma & \gamma & \beta & \delta & \delta & \Sigma & \beta & \beta & \alpha \\
\beta & \delta & \kappa & \kappa & \delta & \beta & \alpha & \beta & \beta & \Sigma & \gamma & \gamma \\
\delta & \beta & \beta & \delta & \kappa & \kappa & \beta & \alpha & \beta & \gamma & \Sigma & \gamma \\
\kappa & \kappa & \delta & \beta & \beta & \delta & \beta & \beta & \alpha & \gamma & \gamma & \Sigma
\end{pmatrix}$$

where

$$\alpha = -\frac{ac(a^3+5a^2b+7ab^2+3b^3+2a^2c+6abc+2b^2c)}{(a+3b)(a+3b+2c)(a^2+3ab+2ac+4bc)}$$

$$\beta = -\frac{abc(a^2+4ab+3b^2+ac+2bc)}{(a+3b)(a+3b+2c)(a^2+3ab+2ac+4bc)}$$

$$\delta = -\frac{abc(ab+3b^2+ac+2bc)}{(a+3b)(a+3b+2c)(a^2+3ab+2ac+4bc)}$$

$$\gamma = -\frac{ab(a^3+6a^2b+9ab^2+2a^2c+7abc+3b^2c+ac^2+2bc^2)}{(a+3b)(a+3b+2c)(a^2+3ab+2ac+4bc)}$$

$$\kappa = -\frac{ab^2c}{(a+3b)(a+3b+2c)(a^2+3ab+2ac+4bc)}$$

$$\Sigma = -(\textit{Sum of the row entries})$$

Again, regardless of how the boundary nodes are numbered the response matrix still contains five distinct entries which provide information about the structure of the network. The information may not be as obvious as before because the ordering of the entries in Λ is no longer the same, but the information is still there. The first thing to notice in Λ is that α is the only entry that appears just 12 times in the matrix. If $\lambda_{x,y} = \alpha$, then x and y are paired on the same ray. Next there are three terms, δ, γ, and κ that appear 24 times each. If $\lambda_{x,y} = \alpha$, then κ will be in the same column of the x^{th} and y^{th} rows. No other element will have that characteristic.

β is easy to find because it appears 48 times in the matrix. Now the question is how can γ distinguished from δ? Using two sign conditions of the network, det $\Lambda(1,2;5,4) > 0$ and det $\Lambda(1,2;6,3) > 0$, it is known that $\gamma^2 > \delta^2$. If $\lambda_{x,y} = \gamma$, then x and y are grouped in the same circle. Hence all boundary nodes in each circle are known.

There can be many different Λ matrices that have the same structure as the Λ for this network. In order for a Λ to be a response matrix for this network, not only must the above symmetries hold, but certain sign conditions must also hold. A list of some of these sign conditions can be found in the appendix. It is not a complete list due to the great quantity of determinants and complexity of the network. The extra symmetry in this network may mean that some of the determinants in the appendix are equivalent.

4.2 Less Symmetric Case

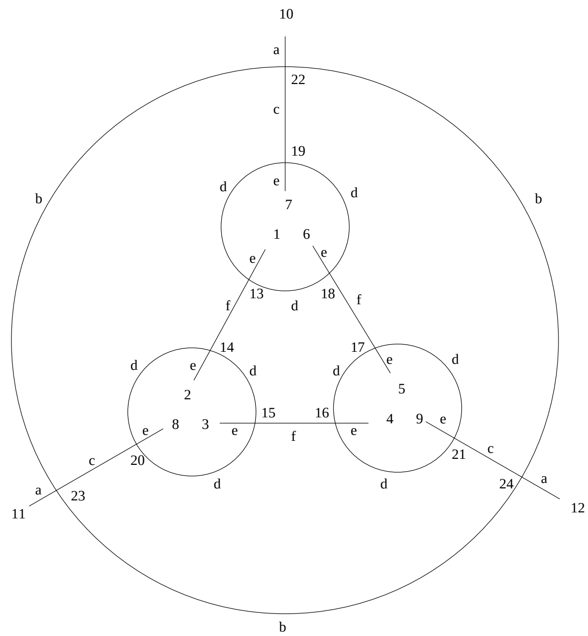

Figure 4: Graph with Three Holes(with less symmetry)

Here is another graph to consider. Assume that conductivity is constant on layers. Then the response matrix, Λ, is as follows:

$$\Lambda = \begin{pmatrix}
\Sigma & \alpha & \beta & \gamma & \beta & \delta & \theta & \kappa & \pi & \psi & \eta & \mu \\
\alpha & \Sigma & \delta & \beta & \gamma & \beta & \kappa & \theta & \pi & \eta & \psi & \mu \\
\beta & \delta & \Sigma & \alpha & \beta & \gamma & \pi & \theta & \kappa & \mu & \psi & \eta \\
\gamma & \beta & \alpha & \Sigma & \delta & \beta & \pi & \kappa & \theta & \mu & \eta & \psi \\
\beta & \gamma & \beta & \delta & \Sigma & \alpha & \kappa & \pi & \theta & \eta & \mu & \psi \\
\delta & \beta & \gamma & \beta & \alpha & \Sigma & \theta & \pi & \kappa & \psi & \mu & \eta \\
\theta & \kappa & \pi & \pi & \kappa & \theta & \Sigma & \sigma & \sigma & \epsilon & \tau & \tau \\
\kappa & \theta & \theta & \kappa & \pi & \pi & \sigma & \Sigma & \sigma & \tau & \epsilon & \tau \\
\pi & \pi & \kappa & \theta & \theta & \kappa & \sigma & \sigma & \Sigma & \tau & \tau & \epsilon \\
\psi & \eta & \mu & \mu & \eta & \psi & \epsilon & \tau & \tau & \Sigma & \chi & \chi \\
\eta & \psi & \psi & \eta & \mu & \mu & \tau & \epsilon & \tau & \chi & \Sigma & \chi \\
\mu & \mu & \eta & \psi & \psi & \eta & \tau & \tau & \epsilon & \chi & \chi & \Sigma
\end{pmatrix}$$

Regardless of how the boundary nodes are numbered the response matrix still contains fourteen distinct entries which provide information about the structure of the network. The information may not be as obvious as before because the ordering of the entries in Λ is no longer the same, but the information is still there. The first thing to notice in Λ is that in the upper triangular part of the matrix, β is the only term that appears six times in the upper triangle and is only in four columns. Hence β is distinguished from the other entries of the matrix. Now looking at the whole matrix, τ can also be distinguished. In the matrix, β appears in six different rows and columns; τ also appears in six different rows and columns, none of which contain a β. So, τ and β can be identified in Λ.

Next look at the 6×6 submatrix containing only rows and columns which have a β. In this submatrix there are three other entries: α, γ, and δ. γ can be distinguished from the other two by using some sign conditions of determinants. γ is the only one of the three, such that $\gamma^2 < \beta^2$. This is because $\det \Lambda(1,2;6,3) > 0$, $\det \Lambda(1,4;2,3) > 0$, and $\det \Lambda(1,2;5,4) > 0$.

Now that β and γ have been identified in Λ the arrangement of nodes in the inner triangle can be determined. If $\lambda_{x,y} = \beta$, then x is two nodes away from y. If $\lambda_{x,y} = \gamma$, then x is three nodes away from y. Using this information it is now known which nodes are paired on the interior rays and two of the nodes grouped together in each of the three interior circles.

Another element can be identified by taking the six by six submatrix of every row and column containing τ. Within this matrix, ϵ is the only element appearing six times and is in every row and every column. If $\lambda_{x,y} = \epsilon$ then x and y are on the same outer ray. Upon further inspection, it becomes clear that this submatrix has the same structure as G(3,2), thus having four different elements. As in the G(3,2), σ and χ can not be distinguished from each other, but they are different entries. Thus if $\lambda_{x,y} = \sigma$, then x and y are grouped together on the outside circle or with the nodes from the inner triangle. Similarly for $\lambda_{x,y} = \chi$.

Furthermore, it would be nice to know which outer ray is grouped with each inner circle. For each of the inner circles, let (x_1, y_1), (x_2, y_2), and (x_3, y_3) be the pair of boundary nodes that are grouped together in one of the inner circles. Now is a given ray grouped with (x_1, y_1), (x_2, y_2), or (x_3, y_3)? Select one of the boundary nodes on the outer ray, call it z. If $\lambda_{x_1,z} = \lambda_{y_1,z}$, then x_1, y_1, and z are grouped together in the same inner circle. If $\lambda_{x_1,z} \neq \lambda_{y_1,z}$, then try the same thing using (x_2, y_2) and (x_3, y_3); they will only be equal in one of these cases. So each outer ray is grouped with one of the inner circles.

In order to determine which nodes of the outer rays are in the outer circle and which are in the inner circles a determinant must be checked. Let (x_1, x_2, x_3) and (y_1, y_2, y_3) be the grouped together on the outer circle or in the inner circles, this is know from σ and χ. Now for the moment, group x_1, x_2,and x_3 with y_1. Then group y_2 with one of the elements in the inner circle grouped with y_1 call it w, and with the two elements, (z_1, z_2), that share an inner ray that connects the two inner circles which do not grouped with y_1. If $\det \Lambda(x_1, x_2, x_3, y_1; y_2, w, z_1, z_2) = 0$, then (x_1, x_2, x_3) are the nodes on the outer circle and (y_1, y_2, y_3) are the nodes in the inner circles. If it is not equal to zero, then (x_1, x_2, x_3) are the nodes in the inner circles and (y_1, y_2, y_3) are the nodes on the outer circle.

There can be many different Λ matrices that have the same structure as the Λ for this network. In order for a Λ to be a response matrix for this network, not only must the above symmetries hold, but certain sign conditions must also hold. A list of some of these sign conditions can be found in the appendix. It is not a complete list due to the great quantity of determinants and complexity of the network.

5 Structure of Entries in Λ

While working with the response matrix a pattern emerged. No matter how large or small the entries were, the off-diagonal entries were all ratios of the opposite of sums of monomials with positive integer coefficients.

Monomials with Positive Coefficients 5.1 *Let $\lambda_{i,j}$ be an entry in the response matrix. It can be written in the following form.*

- $\lambda_{i,j} = -\frac{m_{i,j}}{d}$, *when* $i \neq j$
- $\lambda_{i,i} = \frac{m_{i,i}}{d}$, *when* $i = j$

Where $m_{i,j}$,$m_{i,i}$, and d are sums of monomials with positive integer coefficients.

Proof: (by induction) In order to go from the Kirchhoff matrix, K, to the response matrix, Λ, Gaussian elimination can be done one row at a time until Λ is the result. In other words take the Schur complement but one element at a time.

In the first case this is obviously true, that is when $\lambda_{i,j}$ is an element of the Kirchhoff matrix. This is because $K_{i,j} = -\gamma_{i,j}$ when $i \neq j$ and $K_{i,i} = -\sum_{j \neq i} K_{i,j}$, and $\gamma_{i,j}$ is a sum of positive monomials with integer coefficients.

Now assume this is true for $\lambda_{i,j}$. So, $\lambda_{i,j} = -\frac{m_{i,j}}{d}$ when $i \neq j$ and $\lambda_{i,i} = \frac{m_{i,i}}{d}$ when $i = j$. Where $m_{i,j}$, $m_{i,i}$, and d are sums of monomials with positive integer coefficients. (This is the inductive hypothesis.) Then by Gaussian elimination

$$\lambda'_{i,j} = -\frac{m_{i,j}}{d} - \frac{d}{m_{n,n}} \frac{m_{i,n}}{d} \frac{m_{n,j}}{d}$$

when $i \neq j$. So

$$\lambda'_{i,j} = \frac{1}{d}\big(- m_{i,j} - \frac{m_{i,n} m_{n,j}}{m_{n,n}}\big) = -\frac{m_{n,n} m_{i,j} + m_{i,n} m_{n,j}}{d m_{n,n}} = -\frac{m'_{i,j}}{d'}$$

by the inductive hypothesis. When $i = j$, then

$$\lambda'_{i,i} = \frac{1}{d'} \sum_{j \neq i} m'_{i,j}$$

Thus the theorem has been proven.

6 G(3,2) with Vertex Conductivity

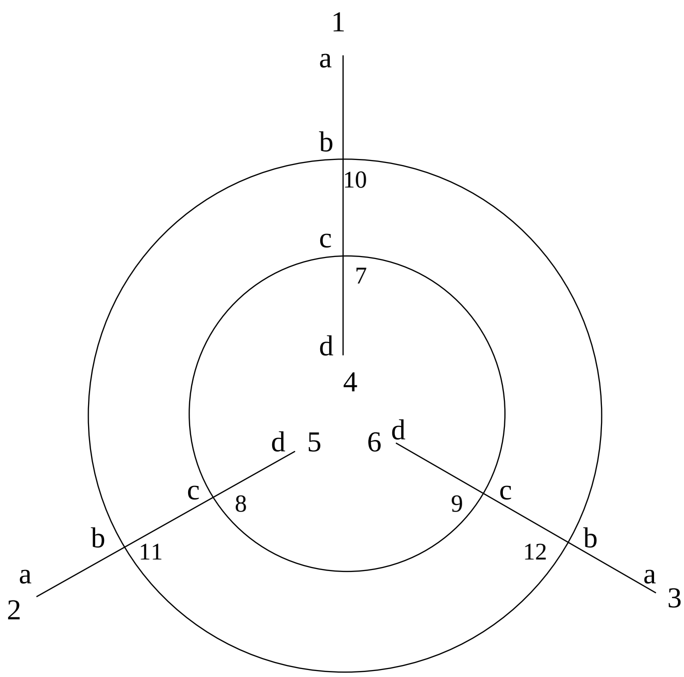

Figure 5: G(3,2) with vertex conductivity

Now consider a network with vertex conductivity instead of edge conductivity. So the current at a node is defined as:

$$\sum_{q \sim p} \gamma_q (u_q - u_p)$$

where γ_q is the conductivity at node q. Using this definition of current, then the response matrix, Λ, is as follows:

$$\Lambda = \begin{pmatrix} \Sigma & \alpha & \alpha & \beta & \gamma & \gamma \\ \alpha & \Sigma & \alpha & \gamma & \beta & \gamma \\ \alpha & \alpha & \Sigma & \gamma & \gamma & \beta \\ \delta & \kappa & \kappa & \Sigma & \epsilon & \epsilon \\ \kappa & \delta & \kappa & \epsilon & \Sigma & \epsilon \\ \kappa & \kappa & \delta & \epsilon & \epsilon & \Sigma \end{pmatrix}$$

where

$$\alpha = -\frac{ab^2(b^2 + 3bc + c^2 + 2bd + 3cd + d^2)}{(ab + ad + cd)(ab + 3b^2 + 3ac + 9bc + 3c^2 + ad + 3bd + cd)}$$

$$\beta = -\frac{bcd(ab + b^2 + ac + 3bc + c^2 + ad + bd + cd)}{(ab + ad + cd)(ab + 3b^2 + 3ac + 9bc + 3c^2 + ad + 3bd + cd)}$$

$$\gamma = -\frac{bcd(b^2+ac+3bc+c^2+bd)}{(ab+ad+cd)(ab+3b^2+3ac+9bc+3c^2+ad+3bd+cd)}$$

$$\delta = -\frac{abc(ab+b^2+ac+3bc+c^2+ad+bd+cd)}{(ab+ad+cd)(ab+3b^2+3ac+9bc+3c^2+ad+3bd+cd)}$$

$$\kappa = -\frac{abc(b^2+ac+3bc+c^2+bd)}{(ab+ad+cd)(ab+3b^2+3ac+9bc+3c^2+ad+3bd+cd)}$$

$$\epsilon = -\frac{c^2(a^2+3ab+b^2+2ac+3bc+c^2)d)}{(ab+ad+cd)(ab+3b^2+3ac+9bc+3c^2+ad+3bd+cd)}$$

$$\Sigma = -(\textit{Sum of the row entries})$$

Regardless of how the boundary nodes are numbered the response matrix still contains six distinct entries which provide information about the structure of the network. The information may not be as obvious as before because the ordering of the entries in Λ is no longer the same, but the information is still there. The first thing to remember is that Λ is no longer symmetric. This is because Λ is for a network with vertex conductivity. β and δ are the only two entries that appear three times. If $\lambda_{x,y} = \beta$ or $\lambda_{x,y} = \delta$, then x and y are paired on the same ray. Now, which nodes appear on the inside of the graph and which are on the outside? Technically, there is no way to tell between the inside and outside of this network because it could easily be inverted. But, it is possible to tell which three nodes are grouped together on the inside or outside. Consider all $\lambda_{x,y} = \beta$, or ϵ. It does not matter which entry is chosen, just that one of the entries that only appears three times. Then all x are grouped on the inside and all y are grouped on the outside.

There can be many different Λ matrices that have the same structure as the Λ for this network. In order for a Λ to be a response matrix for this network, not only must the above symmetries hold, but certain sign conditions must also hold.

References

[1] Curtis, E.B. and J.A. Morrow. *Inverse Problems for Electrical Networks*, World Scientific, New Jersey, 2000.