The Discrete Inverse Scattering Problem

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Abstract

This paper first deals with a physical interpretation of a particular scattering problem involving acoustic waves. The resulting continuous equation is discretized in two ways: using edge conductivities and using vertex conductivities. Boundary spike and boundary edge formulas are derived for both cases and eigenvalues of the vertex conductivity Kirchhoff matrix are investigated. Finally, examples of recoverable and nonrecoverable networks are presented along with several leads and ends - among them a formulation based on the Schroedinger equation.

1 Introduction

The scattering problem occurs in such areas as acoustics, particle physics, and electromagnetics. In this section we use first principles to formulate the scattering problem for the case of acoustic waves. Much of the motivation for the remainder of this section comes from Erkki Heikkola's thesis [3].

Consider the propagation of sound waves in an isotropic inviscid fluid. Let $\vec{v}$ be the velocity field, p the pressure, and ρ the density of the fluid at an arbitrary point. Assume that variations in the density and pressure do not deviate significantly from the static state in which $p = p_0$ and $\rho = \rho_0$. In particular, $\delta\rho \ll \rho_0$, where $\rho = \rho_0 + \delta\rho$. This assumption then allows us to linearize the governing equations as follows:

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0 \qquad \text{(Linearized continuity equation)} \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho_0}\nabla p = 0 \qquad \text{(Linearized Euler equation)} \tag{2}$$

$$\frac{\partial p}{\partial t} = c^2 \frac{\partial \rho}{\partial t} \qquad \text{(State equation)} \tag{3}$$

In the state equation, c denotes the speed of sound in the fluid at that point. We now introduce a time harmonic velocity potential $U(x,t)$ separable into spatial and temporal components: $U(x,t) = u(x)e^{-i\omega t}$. We assume that the velocity field is obtained from this potential as follows:

$$\vec{v} = \frac{1}{\rho}\nabla U = \frac{1}{\rho}\nabla u e^{-i\omega t} \tag{4}$$

Substituting (4) into the Euler equation (2), we have:

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{\rho}\nabla u e^{-i\omega t}\right) + \frac{1}{\rho_0}\nabla p = 0 \\ \frac{1}{\rho}\nabla u e^{-i\omega t}\left(-\frac{1}{\rho}\frac{\partial \rho}{\partial t} - iw\right) + \frac{1}{\rho_0}\nabla p = 0 \end{aligned} \tag{5}$$

Approximating $|\frac{\partial \rho}{\partial t}|$ as $\delta\rho\,\omega$, the first term in parentheses in (5) becomes negligible relative to the second term, since $\delta\rho \ll \rho_o$. Also, $\rho_0/\rho = \rho_0/(\rho_0 + \delta\rho) \approx 1$. Thus, we have:

$$\nabla p = \frac{\rho_0}{\rho} i\omega \nabla u e^{-i\omega t} \approx i\omega \nabla u e^{-i\omega t} \tag{6}$$

This expression suggests that the pressure takes the form:

$$p = i\omega u e^{-i\omega t} = -\frac{\partial U}{\partial t} \tag{7}$$

Implicit in the above formulas is that we are concerned only with the real part of expressions that represent physical observables. Combining (1) and (3) to eliminate the density term, and using (4) and (7), we have:

$$\begin{aligned} \rho_0 \nabla \cdot (\frac{1}{\rho}\nabla u) e^{-i\omega t} + \frac{\omega^2}{c^2} u e^{-i\omega t} = 0 \\ \nabla \cdot (\frac{1}{\rho}\nabla u) + \frac{k^2}{\rho_0} u = 0 \\ \nabla \cdot (\gamma \nabla u) + \lambda u = 0 \end{aligned} \tag{8}$$

Where $k^2 = \omega^2/c^2$, $\gamma = 1/\rho$, and $\lambda = k^2/\rho_0$.

2 Edge Conductivities

2.1 Discretization of Edge Conductivities

We can now discretize (8) for the case of a network with potential u defined at the nodes, and conductivity γ defined for the edges. The discretization of the divergence term parallels that given by Curtis and Morrow [1]:

$$\nabla \cdot (\gamma \nabla u) \quad \rightarrow \quad \sum_{j \sim i} \gamma_{i,j}[u(j) - u(i)] \tag{9}$$

Where i refers to a node of the network, $j \sim i$ refers to the set of nodes connected to i, $\gamma_{i,j}$ is the conductivity between nodes i and j and $u(i)$ is the potential at node i.

Continuing the analogy with an electrical network, we use the definition of a *Kirchhoff matrix* K given by Curtis and Morrow [1]. K is an $m \times m$ matrix (where m is the number of nodes in the network), whose entries are defined as follows:

(1) If $i \neq j$ then $K_{i,j} = -\gamma_{i,j}$

(2) $K_{i,i} = \sum_{j \neq i} \gamma_{i,j}$

If we now let $\vec{u}$ be the $m \times 1$ column vector of node potentials, the right-hand expression in (9) becomes equivalent to $-K\vec{u}$, so that the left-hand side of (8) becomes $(-K + I\lambda)\vec{u}$. We make a distinction between boundary nodes and interior nodes by insisting that the discretization of (8) be satisfied at interior nodes. This does not have to be true for boundary nodes since, in the case of acoustic waves (for example), we could have net mass flow out of the system at the boundary. For convenience, in labeling the nodes of the network, we label the boundary nodes first. Writing K in block form, the discretization of (8) is:

$$\begin{bmatrix} A - \lambda & B \\ B^T & C - \lambda \end{bmatrix} \vec{u} = \begin{bmatrix} \Phi \\ 0 \end{bmatrix}$$

Here, $A = K(\mathcal{B}; \mathcal{B})$, where $\mathcal{B}$ is the set of boundary nodes. Letting $\vec{u} = \begin{bmatrix} \psi \\ x \end{bmatrix}$ where ψ represents the potentials at the boundary nodes while x represents the potentials at the interior nodes, we have

$$\begin{bmatrix} A - \lambda & B \\ B^T & C - \lambda \end{bmatrix} \begin{bmatrix} \psi \\ x \end{bmatrix} = \begin{bmatrix} \Phi \\ 0 \end{bmatrix}$$

Since the current is conserved at the interior nodes, $B^T\psi + (C - \lambda)x = 0$, so $x = -(C - \lambda)^{-1}B^T\psi$ for all λ such that $(C - \lambda)$ is invertible. Thus,

$$\begin{bmatrix} A - \lambda & B \\ B^T & C - \lambda \end{bmatrix} \begin{bmatrix} \psi \\ x \end{bmatrix} = \begin{bmatrix} (A - \lambda)\psi - B(C - \lambda)^{-1}B^T\psi \\ 0 \end{bmatrix}.$$

We will call the response matrix $\Lambda(\lambda) : \psi \mapsto \Phi$ where

$$\Lambda(\lambda) = (A - \lambda) - B(C - \lambda)^{-1}B^T. \tag{10}$$

Since $\Lambda(\lambda)$ is a meromorphic function, for large values of $|\lambda|$ ($|\lambda| > \|C\|$) we can expand $(C - \lambda)^{-1}$ and write $\Lambda(\lambda)$ as a power series:

$$\Lambda(\lambda) = (A-\lambda) + \left(\frac{1}{\lambda}\right) B \sum_{k=0}^{\infty} \left(\frac{C^k}{\lambda^k}\right) B^T$$

$$\Lambda(\lambda) = -\lambda + A + \sum_{k=0}^{\infty} BC^k B^T \lambda^{-k-1} \tag{11}$$

If we know $\Lambda(\lambda)$, we know the coefficients of the power series expansion.

2.2 Poles and Zeroes of $\Lambda(\lambda)$

A graph of the determinant of the response function $\Lambda(\lambda)$ can be constructed for an electrical network. Notation is the same as in the introduction, where K is the Kirchhoff matrix and C is the lower right block entry of K. We can determine the locations of the zeroes and poles of det $\Lambda(\lambda)$.

Theorem 2.2.1 *For the Kirchhoff matrix K of an electrical network and the lower right block entry of K, C, the following statements hold*
(1) The poles of det Λ are the eigenvalues of the matrix C.
(2) The zeroes of det Λ are the eigenvalues of the matrix K.

Proof We can write the function $\Lambda(\lambda)$ in terms of the block entries of K as in (10).

$$\Lambda(\lambda) = (A-\lambda) - B(C-\lambda)^{-1}B^T$$

This is the Schur complement of $(C-\lambda)$ within $(K-\lambda)$: $\Lambda(\lambda) = (K-\lambda)/(C-\lambda)$. Taking the determinant of both sides gives

$$\det \Lambda(\lambda) = \frac{\det(K-\lambda)}{\det(C-\lambda)}$$

Thus, the poles of $\det \Lambda(\lambda)$ are where $\det(C-\lambda) = 0$, the eigenvalues of C. The zeroes of $\det \Lambda(\lambda)$ are where $\det(K-\lambda) = 0$, the eigenvalues of K.

2.3 Boundary to Boundary Edge Formula

Manipulating the graph to decrease the number of edges is one method of recovering edge conductivities in networks. This can be done when there is at least one boundary to boundary connection within the graph. In the figures to follow, boundary nodes are emphasized with circles. The two boundary nodes are denoted 1 and 2 with an edge conductivity of a. Node 1 is also connected to n other nodes and node 2 is connected to m other nodes with conductivities denoted $b_1, \ldots, b_n$ and $c_1, \ldots, c_m$, respectively. We start with the response matrix $\Lambda(\lambda)$ and show how to recover the conductivity a and how to obtain $\Lambda'(\lambda)$, the response matrix for the graph with the edge between nodes 1 and 2 removed.

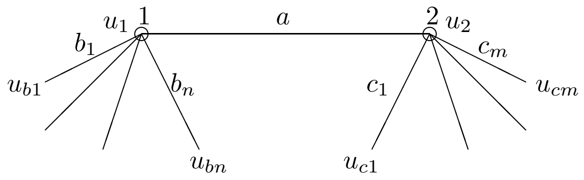


Figure 1: Deletion of a boundary-to-boundary edge

The conductivity a can be determined from the power expansion of the response matrix (11). By definition, the (1;2) entry of matrix A (boundary sub-block of the Kirchhoff matrix) is given by: $A(1;2) = -a$. Since A is a known term in the power expansion, the conductivity a can be directly recovered as: $a = -A(1;2)$.

To find the currents at the boundary nodes, we solve the Dirichlet Problem for the matrix $K - \lambda I$. The potentials, which are denoted by $\vec{u}$, include $u_{b1}, \ldots, u_{bn}, u_{c1}, \ldots, u_{cm}$ where the subscripts correspond to the boundary and interior nodes connected to 1 and 2. $\vec{\Phi}$ denotes the net currents, taken to be zero at the interior nodes.

$$(K - \lambda I)\vec{u} = \vec{\Phi}$$

Using the notation

$$\alpha = \sum_{k=1}^{n} -b_k u_{bk} \qquad \beta = \sum_{k=1}^{m} -c_k u_{ck},$$

$$\Phi_1 = (a_1 + b_1 + \cdots + b_n - \lambda)u_1 - au_2 + \alpha$$
$$\Phi_2 = -au_1 + (a + c_1 + \cdots + c_m - \lambda)u_2 + \beta$$

The next step is to delete the edge between nodes 1 and 2. The rest of the network remains the same. Writing the currents for the modified network, Φ_1' and Φ_2' we have:

$$\Phi_1' = (b_1 + \cdots + b_n - \lambda)u_1 + \alpha$$
$$\Phi_2' = (c_1 + \cdots + c_m - \lambda)u_2 + \beta$$

Φ_1' can be written in terms of Φ_1 and Φ_2' can be written in terms of Φ_2 as seen below.

$$\begin{aligned}
\Phi_1' &= \Phi_1 - a(u_1 - u_2)\\
\Phi_2' &= \Phi_2 + a(u_1 - u_2)\\
\Phi_j' &= \Phi_j \qquad \text{for } j \neq 1,2
\end{aligned}$$

Thus the result only depends on the u_1 and u_2 terms and the rest of the network is not affected by breaking the connection between nodes 1 and 2. We can now write the response matrices for the two networks. The response matrix of the original network is denoted

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1j}\\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2j}\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_{j1} & \lambda_{j2} & \cdots & \lambda_{jj} \end{bmatrix}$$

The response matrix for the new network, Λ', can be written as

$$\mathbf{\Lambda}' = \begin{bmatrix} \lambda_{11} - a & \lambda_{12} + a & \lambda_{13} & \cdots & \lambda_{1j}\\ \lambda_{21} + a & \lambda_{22} - a & \lambda_{23} & \cdots & \lambda_{2j}\\ \lambda_{31} & \lambda_{32} & \lambda_{33} & \cdots & \lambda_{3j}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \lambda_{j1} & \lambda_{j2} & \lambda_{j3} & \cdots & \lambda_{jj} \end{bmatrix}$$

Since we know the beginning conductivity a, the original network can be simplified to the modified network. Calculations can be continued with this new network.

Note that in the above derivation, the requirement that the graph remains connected following the edge deletion was not used. Indeed, the result holds in the case of a disconnected final graph, and the new response matrix (Λ') can be partitioned as follows:

$$\Lambda' = \begin{bmatrix} \Lambda_1 & 0\\ 0 & \Lambda_2 \end{bmatrix}$$

Here, Λ_1 corresponds to the response matrix for the graph connected to boundary node 1 (graph 1) and Λ_2 corresponds to the response matrix for the graph connected to boundary node 2 (graph 2)(see Figure 1). This block-partitioning can be understood intuitively by noting that when two graphs are not connected, potentials on the boundary nodes of one graph do not influence the currents on the boundary nodes of the other graph. Mathematically, this result is shown using the Schur complement formula:

$$\Lambda = (A - \lambda) - B^T(C - \lambda)^{-1}B \tag{12}$$

Let P_1 and P_2 denote the set of boundary nodes in graph 1 and graph 2, respectively. We need to show that $\lambda'_{ij} = 0$ for $i \in P_1$ and $j \in P_2$. Since $i \neq j$, and since the only connection between graphs 1 and 2 is through boundary nodes 1 and 2, we have:

$$(A-\lambda)_{ij} = A_{ij} = \begin{cases} -a & \text{when } i=1, j=2 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, since none of the boundary nodes in graph 1 can be connected to interior nodes of graph 2 (and vice-versa), we have: $B_{ik} = 0$ for k an interior node of graph 2 and $B_{jk} = 0$ for k an interior node of graph 1. Thus, we can write:

$$\begin{aligned} (B^T(C-\lambda)^{-1}B)_{ij} &= \sum_{k,\ell} B_{ik}((C-\lambda)^{-1})_{k\ell}(B^T)_{lj} \\ &= \sum_{k,\ell} B_{ik}((C-\lambda)^{-1})_{k\ell}B_{j\ell} \qquad (13) \end{aligned}$$

In the above sums, k and ℓ range over all the interior nodes. When k is in graph 2, $B_{ik} = 0$; when ℓ is in graph 1, $B_{il} = 0$; when k is in graph 1 and ℓ is in graph 2, $((C-\lambda)^{-1})_{k\ell} = 0$. Thus, the term given by (13) is also zero. Substituting these results into (12), we see that:

$$\lambda_{ij} = \begin{cases} -a & \text{when } i=1, j=2 \\ 0 & \text{otherwise} \end{cases}$$

Using the boundary edge formula, $\lambda'_{12} = -a + a = 0$, and hence $\lambda'_{ij} = 0$ for $i \in P_1$ and $j \in P_2$. This justifies the decomposition given.

By partitioning Λ as shown, we have effectively created two separate problems, each one with its own response matrix. These problems can then be approached independently. In particular, in the case where node 1 is isolated after the edge removal, $\Lambda_1 = 0$, and Λ_2 is used to further recover the graph. This approach will be used in conjunction with the boundary spike formula in recovering general tree graphs(section 5.1).

2.4 Boundary Spike Formula

A boundary spike is an edge of a graph connecting an isolated boundary node to the rest of the graph, as shown in Figure 2a. We label the boundary node 1, the interior node to which it is connected n, and the connecting conductivity a. Λ denotes the response matrix for this graph. Following the edge contraction, node n becomes a boundary node, and node 1 and edge a are deleted (Figure 2b). Λ' is the response matrix for this new graph. Given Λ, we wish to determine a and Λ', and thereby reduce the problem.

To determine a, we use the power expansion for $\Lambda(\lambda)$ (11). Noting that a is the conductivity of the only edge connected to node 1, $K(1;1) = A(1;1) = a$. Since knowing Λ is equivalent to knowing each term in the power expansion, we know A, and hence $a = A(1;1)$. Thus, recovery of a follows from the power expansion of Λ.

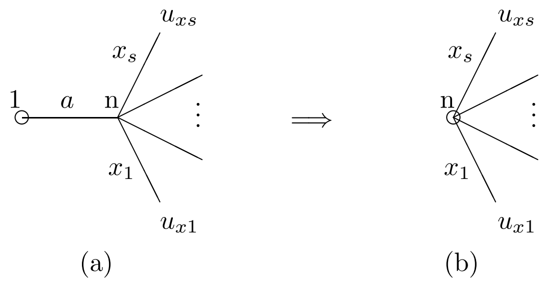


Figure 2: Deletion of a boundary spike

Let u_1 and u_n denote the potentials of nodes 1 and n, respectively. Let x_i denote the conductivities of the edges connected to node n, and u_{xi} the potentials at the corresponding nodes connected to node n. By the problem statement, $(K-\lambda)\vec{u} = \vec{\Phi}$. Writing out the nth equation of this system for both problems,

$$u_n(a-\lambda) - u_1 a + \sum_{i=1}^{s} u_n x_i - \sum_{i=1}^{s} u_{xi} x_i = \Phi_n = 0 \tag{14}$$

$$u_n(-\lambda) + \sum_{i=1}^{s} u_n x_i - \sum_{i=1}^{s} u_{xi} x_i = \Phi'_n \tag{15}$$

Subtracting (14) from (15), we have:

$$\Phi'_n = a(u_1 - u_n) \tag{16}$$

The current at node 1 of the original graph can be found from the Kirchhoff matrix and from the response matrix:

$$\Phi_1 = (a-\lambda)u_1 - a u_n = \sum_i \lambda_{1i} u_i = \lambda_{11} u_1 + \sum_{j \neq 1} \lambda_{1j} u_j \tag{17}$$

The above sums are carried out over all boundary nodes of the original graph with the restrictions shown. Solving (17) for u_1:

$$u_1 = \frac{a}{a-\lambda-\lambda_{11}} u_n + \sum_{j \neq 1} \frac{\lambda_{1j}}{a-\lambda-\lambda_{11}} u_j \tag{18}$$

We use this expression for u_1 to write the currents of the new graph in terms of u_n and u_j. This will then yield the response matrix for the new graph. By (16):

$$\Phi'_n = \frac{a(\lambda+\lambda_{11})}{a-\lambda-\lambda_{11}} u_n + \sum_{j\neq 1} \frac{a}{a-\lambda-\lambda_{11}} \lambda_{1j} u_j \tag{19}$$

Since node 1 is not connected to any other boundary node in the original graph, the edge contraction does not change the expression for the currents at boundary nodes k, where $k \neq 1$. This can be verified by writing out the expressions for Φ_k and Φ'_k using the Kirchhoff matrix. Substituting for u_i, the result is:

$$\begin{aligned} \Phi'_k &= \Phi_k = \sum_i \lambda_{ki} u_i = \lambda_{k1} u_1 + \sum_{j\neq 1} \lambda_{kj} u_j \\ &= \frac{a}{a-\lambda-\lambda_{11}} \lambda_{k1} u_n + \sum_{j \neq 1} \left(\frac{\lambda_{k1}\lambda_{1j}}{a-\lambda-\lambda_{11}} + \lambda_{kj} \right) u_j \end{aligned} \tag{20}$$

Now, let J be the set of all boundary node indices excluding the index 1. In the equations that follow, u_J refers to the vector of potentials at the nodes in J, and $\Lambda(J;J)$ refers to the submatrix of Λ where the rows and columns are referenced by the elements of J. Combined with the expression for Φ'_1, this result can be used to write Λ':

$$\Lambda' \begin{bmatrix} u_n \\ u_J \end{bmatrix} = \begin{bmatrix} \Phi'_n \\ \Phi_J \end{bmatrix}$$

$$\Lambda' = \begin{bmatrix} \frac{a(\lambda+\lambda_{11})}{\delta} & \frac{a}{\delta}\Lambda(1;J) \\ \frac{a}{\delta}\Lambda(J;1) & \frac{1}{\delta}\Lambda(J;1)\Lambda(1;J) + \Lambda(J;J) \end{bmatrix}, \quad \delta = a - \lambda - \lambda_{11} \tag{21}$$

2.5 Numerical Example: One Boundary Node Chain

A chain network is a sequence of nodes 1 ... n in which node 1 is the only boundary node and node i is connected to node $i+1$ by an edge of conductivity a_i, for $1 \leq i <$ n (see Figure 3).

Recovery of this network follows by repeated application of the boundary spike formula. Since there is only one boundary node, the response matrix is a scalar function. Let $\Lambda(\lambda)$ be the response matrix at the step when node k is the boundary node. Given $\Lambda(\lambda)$, a_k is computed from the power series expansion (11), and the response function following the next contraction, $\Lambda'(\lambda)$ is computed from (21):

$$a_k = \lim_{\lambda\to\infty} (\lambda + \Lambda(\lambda)) \tag{22}$$

$$\Lambda' = \frac{a_k(\lambda+\Lambda)}{a_k - (\lambda+\Lambda)} \tag{23}$$

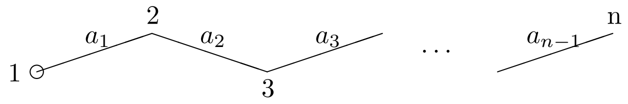


Figure 3: A one-boundary-node chain

For computational ease, it is desirable to characterize $\Lambda(\lambda)$ by the location of its zeroes and poles. This is done by Theorem 1.1. Since both K and C are symmetric matrices, they each have real eigenvalues whose number, including multiplicities, corresponds to the size of the matrix. Noting that K is singular (rows sum to 0), 0 is always an eigenvalue of K. We label the eigenvalues of K by $0, z_1, z_2, ...z_m$, and the eigenvalues of C by $s_1, s_2, ...s_m$. K has only one more eigenvalue than C because the network has only one boundary node at each step. Since $\Lambda(\lambda)$ is a rational function, we can write it as a quotient of two polynomials: $\Lambda(\lambda) = P(\lambda)/Q(\lambda)$. By Theorem 1.1, the roots of $P(\lambda)$ are the eigenvalues of K, and the roots of $Q(\lambda)$ are the eigenvalues of C. Thus, we can write:

$$\Lambda(\lambda) \quad = \quad -\frac{\lambda(\lambda - z_1)(\lambda - z_2)\ldots(\lambda - z_m)}{(\lambda - s_1)(\lambda - s_2)\ldots(\lambda - s_m)} \tag{24}$$

The minus sign in equation (24) arises because $\Lambda(\lambda)$ asymptotes to $-\lambda$ as $\lambda \to \infty$ by the power expansion. Thus, a numerical recovery program only needs to store the sets $\{z_1 \ldots z_m\}$ and $\{s_1 \ldots s_m\}$ at each step. By equation(23) the new zeroes and singularities are determined as follows:

(1) λ is a zero of $\Lambda'(\lambda)$ when $\lambda + \Lambda(\lambda) = 0$

(2) λ is a singularity of $\Lambda'(\lambda)$ when $-a_k + \lambda + \Lambda(\lambda) = 0$

These two linear equations (indicated by the dashed lines) are plotted on a $\Lambda - \lambda$ plot, along with $\Lambda(\lambda)$, in Figure 4. In this example, the starting conductivities $(a_1, a_2, a_3, a_4) = (1, 2, 3, 4)$ are used in the initial forward problem. The new zeros and poles are determined by the intersections of these lines with $\Lambda(\lambda)$. As the graph shows, the intersections for finding the new singularities occur in regions where the two curves are almost tangent. This situation becomes more pronounced with the addition of more conductivities. As a result, this inverse problem becomes ill-conditioned as the number of initial conductivities grows. Some of the initial and recovered conductivities for n = 11 are shown in Table 1.

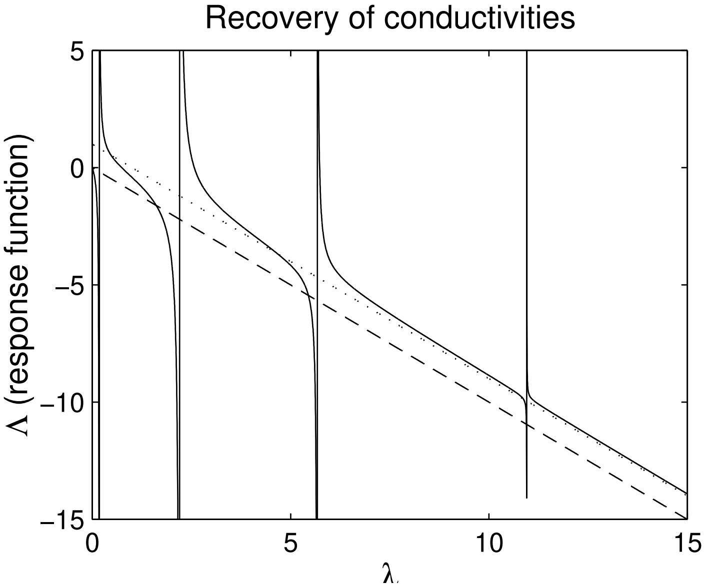


Figure 4: $\Lambda - \lambda$ plot for a 4-edge chain before any reductions (k = 1)

Edge	Initial	Recovered
1	1.0	1.0000
2	2.0	2.0000
3	3.0	3.0000
⋮	⋮	⋮
9	9.0	8.6810
10	10.0	9.6159
11	11.0	11.5701

Table 1

The loss of accuracy is clearly evident at the inner edges (the ones furthest from the initial boundary node).

3 Vertex Conductivities

3.1 Discretization for Vertex Conductivities

We can now discretize (8) for the case of a network where both the potential u and the conductivity γ are defined at the vertices. The discretization parallels that of the edge conductivity case. The main difference is in our new choice for the discretization of the divergence operator:

$$\nabla \cdot (\gamma \nabla u) \quad \rightarrow \quad \sum_{j \sim i} \gamma(j)[u(j) - u(i)] \tag{25}$$

The index i refers to a node of the network, $j \sim i$ refers to the set of nodes connected to i, $\gamma(j)$ is the conductivity at node j and $u(i)$ is the potential at node i.

The analogous $m \times m$ Kirchoff matrix K (where m is the number of nodes in the network) is no longer symmetric. The entries of K are defined as follows:

(1) If $i \neq j$ and there is an edge joining i to j , then $K(i;j) = \gamma(j)$

(2) If $i \neq j$ and there is no edge joining i to j , then $K(i;j) = 0$

(3) $K(i;i) =$ -$\sum_{j \neq i} K(i;j)$

As in the edge conductivity case, let $\vec{u}$ be the $m \times 1$ column vector of node potentials, the right-hand expression in (25) becomes equivalent to $K\vec{u}$, so that the left-hand side of (8) becomes $(K + I\lambda)\vec{u}$. The same convention for labeling the boundary nodes first holds with K. Thus, writing K in block form, the discretization of (8) is:

$$\begin{bmatrix} A+\lambda & B \\ C & D+\lambda \end{bmatrix} \vec{u} = \begin{bmatrix} \Phi \\ 0 \end{bmatrix}$$

Letting $\vec{u} = \begin{bmatrix} \psi \\ x \end{bmatrix}$ where ψ represents the potentials at the boundary nodes while x represents the potentials at the interior nodes, we have

$$\begin{bmatrix} A+\lambda & B \\ C & D+\lambda \end{bmatrix} \begin{bmatrix} \psi \\ x \end{bmatrix} = \begin{bmatrix} \Phi \\ 0 \end{bmatrix}$$

Since the current is conserved at the interior nodes, $C\psi + (D + \lambda)x = 0$, so $x = -(D + \lambda)^{-1}C\psi$ for λ such that $(D + \lambda)$ is invertible. Thus,

$$\begin{bmatrix} A+\lambda & B \\ C & D+\lambda \end{bmatrix} \begin{bmatrix} \psi \\ x \end{bmatrix} = \begin{bmatrix} (A+\lambda)\psi - B(D+\lambda)^{-1}C\psi \\ 0 \end{bmatrix}.$$

Now, we will call the response matrix for the vertex conductivity case $\Lambda(\lambda)$: $\psi \mapsto \Phi$ where

$$\Lambda(\lambda) = (A + \lambda) - B(D + \lambda)^{-1}C. \tag{26}$$

So the power series expansion for $\Lambda(\lambda)$ is

$$\Lambda(\lambda) \quad = \quad \lambda + A - \sum_{k=0}^{\infty} B(-D)^k C \lambda^{-k-1} \tag{27}$$

3.2 Boundary to Boundary Edge Formula

As seen in the edge conductivity section, the graph can be manipulated to decrease the number of edges present. For the vertex conductivity case, the argument if similar. The two boundary nodes are labeled 1 and 2 with vertex conductivities a and b respectively. Node 1 is connected to n other nodes denoted $c_1, \ldots, c_n$ and node 2 is connected to m other nodes denoted $d_1, \ldots, d_m$.

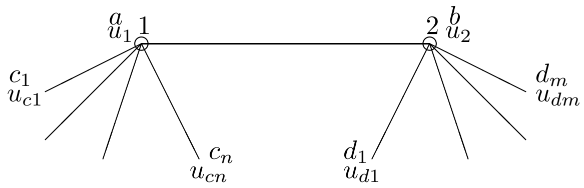


Figure 5: Deleting a boundary-to-boundary edge

The conductivities a and b can be determined from the power expansion of the response matrix (27). By definition, $A(2;1) = a$, and $A(1;2) = b$. Since A is a known term in the power expansion, a and b can be directly recovered.

To find the currents at the boundary nodes, we solve the Dirichlet Problem for the matrix $K + \lambda I$. The potentials, which are denoted by $\vec{u}$, include $u_{b1}, \ldots, u_{bn}, u_{c1}, \ldots, u_{cm}$ where the subscripts correspond to the boundary and interior nodes connected to 1 and 2. $\vec{u}$ represents the vector of all the potentials in the network, while $\vec{\Phi}$ denotes the net currents, taken to be zero at the interior nodes.

$$(K + \lambda I)\vec{u} = \vec{\Phi}$$

Using the notation

$$\alpha = \sum_{k=1}^{n} -c_k u_{ck} \qquad \beta = \sum_{k=1}^{m} -d_k u_{dk},$$

$$\Phi_1 = -(b_1 + b_1 + \cdots + b_n - \lambda)u_1 - bu_2 + \alpha$$
$$\Phi_2 = au_1 - (a + d_1 + \cdots + d_m - \lambda)u_2 + \beta$$

The next step is to delete the edge between nodes 1 and 2. The rest of the network remains the same. We determine the currents for the modified network, Φ_1' and Φ_2':

$$\Phi_1' = (c_1 + \cdots + c_n - \lambda)u_1 + \alpha$$
$$\Phi_2' = (d_1 + \cdots + d_m - \lambda)u_2 + \beta$$

Φ'_1 can be written in terms of Φ_1 and Φ'_2 can be written in terms of Φ_2:

$$\begin{aligned}
\Phi'_1 &= \Phi_1 + b(u_1 - u_2)\\
\Phi'_2 &= \Phi_2 - a(u_1 - u_2)\\
\Phi'_j &= \Phi_j \qquad \text{for } j \neq 1, 2
\end{aligned}$$

Thus the result only depends on the u_1 and u_2 terms and the rest of the network is not affected by breaking the connection between nodes 1 and 2. We can now write the response matrices for the two networks. The response matrix of the original network takes the form:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1j} \\ \lambda_{21} & \lambda_{22} & \dots & \lambda_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j1} & \lambda_{j2} & \dots & \lambda_{jj} \end{bmatrix}$$

The response matrix for the new network, Λ', is then written as:

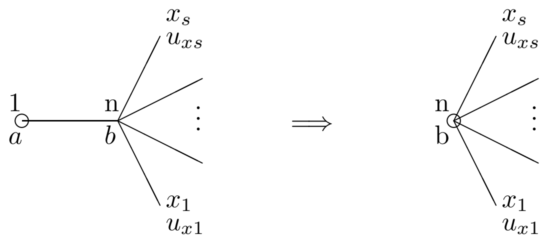
$$\mathbf{\Lambda}' = \begin{bmatrix} \lambda_{11} + b & \lambda_{12} - b & \dots & \lambda_{1j} \\ \lambda_{21} - a & \lambda_{22} + a & \dots & \lambda_{2j} \\ \lambda_{31} & \lambda_{32} & \dots & \lambda_{3j} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{j1} & \lambda_{j2} & \dots & \lambda_{jj} \end{bmatrix}$$

Since we know a and b from the power series expansion, the original network can be simplified to the modified network. Calculations can be continued with this new network to recover the remaining edges.

3.3 Boundary Spike Formula

The derivation of the boundary spike formula for a vertex conductivity function parallels that of the edge conductivity function. The graph before and after the spike contraction is shown in Figure 6(a,b). Node 1 is a boundary node with conductivity a and potential u_1, and node n is an interior node with conductivity b and potential u_n. Let $x_1 \dots x_s$ and $u_{x1} \dots u_{xs}$ denote the conductivities and potentials, respectively, of all other vertices connected to node n. Given the response matrix Λ for the original graph, we wish to determine the conductivity a and the new response matrix Λ', and thereby reduce the problem.

We can, in fact, determine both a and b, using the power expansion for $\Lambda(\lambda)$ (27). We partition K as usual into A, B, C, and D. Noting that b is the conductivity of the only vertex connected to node 1, $K(1;1) = A(1;1) = -b$, $K(1;n) = B(1;1) = b$, and $K(n;1) = C(n;1) = a$ (node n is labeled as the first interior node). All of the other entries in the first row of B and the first column of C are 0, since node 1 is not connected to any other interior nodes. Since



(a) (b)

Figure 6: Deleting a boundary sipke

we know each term in the power expansion of Λ, we know A and BC. Thus, $b = -A(1;1)$. The (1;1) entry of (BC) is ab by the above discussion. Thus, $a = (BC)(1;1)/b$.

By the problem statement, $(K+\lambda)\vec{u} = \vec{\Phi}$. Writing out the nth equation of this system for both problems,

$$u_n(-a+\lambda) + u_1 a - \sum_{i=1}^{s} u_n x_i + \sum_{i=1}^{s} u_{xi} x_i = \Phi_n = 0 \tag{28}$$

$$u_n \lambda - \sum_{i=1}^{s} u_n x_i + \sum_{i=1}^{s} u_{xi} x_i = \Phi'_n \tag{29}$$

Subtracting (28) from (29), we have:

$$\Phi'_n = a(u_n - u_1) \tag{30}$$

The current at node 1 of the original graph can be found from the Kirchhoff matrix and from the response matrix:

$$\Phi_1 = (-b+\lambda)u_1 + b u_n = \sum_i \lambda_{1i} u_i = \lambda_{11} u_1 + \sum_{j \neq 1} \lambda_{1j} u_j \tag{31}$$

The above sums are carried out over all boundary nodes of the original graph with the restrictions shown. Solving (31) for u_1:

$$u_1 = \frac{b}{b - \lambda + \lambda_{11}} u_n - \sum_{j \neq 1} \frac{\lambda_{1j}}{b - \lambda - \lambda_{11}} u_j \tag{32}$$

We use this expression for u_1 to write the currents of the new graph in terms of u_n and u_j. This will then yield the response matrix for the new graph. By (30):

$$\Phi'_n = \frac{a(\lambda_{11} - \lambda)}{b + \lambda_{11} - \lambda} u_n + \sum_{j \neq 1} \frac{a\lambda_{1j}}{b + \lambda_{11} - \lambda} u_j \tag{33}$$

Since node 1 is not connected to any other boundary node in the original graph, the edge contraction does not change the expression for the currents at boundary nodes k, where $k \neq 1$. This can be verified by writing out the expressions for Φ_k and Φ'_k using the Kirchhoff matrix. Using the response matrix and substituting for u_i, the result is:

$$\begin{aligned}
\Phi'_k &= \Phi_k = \sum_i \lambda_{ki} u_i = \lambda_{k1} u_1 + \sum_{j \neq 1} \lambda_{kj} u_j \\
&= \frac{b}{b + \lambda_{11} - \lambda} \lambda_{k1} u_n + \sum_{j \neq 1} \left(-\frac{\lambda_{k1}\lambda_{1j}}{b + \lambda_{11} - \lambda} + \lambda_{kj} \right) u_j
\end{aligned} \tag{34}$$

Now, let J be the set of all boundary node indices excluding the index 1. In the equations that follow, u_J refers to the vector of potentials at the nodes in J, and $\Lambda(J;J)$ refers to the submatrix of Λ where the rows and columns are referenced by the elements of J. Combined with the expression for Φ'_1, this result can be used to write Λ':

$$\Lambda' \begin{bmatrix} u_n \\ u_J \end{bmatrix} = \begin{bmatrix} \Phi'_n \\ \Phi_J \end{bmatrix}$$

$$\Lambda' = \begin{bmatrix} \frac{a(\lambda_{11} - \lambda)}{\delta} & \frac{a}{\delta}\Lambda(1;J) \\ \frac{b}{\delta}\Lambda(J;1) & -\frac{1}{\delta}\Lambda(J;1)\Lambda(1;J) + \Lambda(J;J) \end{bmatrix}, \quad \delta = b + \lambda_{11} - \lambda$$

4 Eigenvalues

4.1 Finding Real Eigenvalues

The following theorem, taken from Wilkinson (page 355), is the motivation behind Theorem 4.3.1.

Theorem 4.1.1 *A general tridiagonal matrix can be transformed into a real symmetric matrix.*

Proof Let M be a general tri-diagonal matrix with the following entries, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n-1$

$$m_{i,i} = s_i \qquad m_{j+1,j} = a_{j+1} \qquad m_{j,j+1} = b_{j+1}$$

where the entries above and below the main diagonal are of the same sign, that is $a_i b_i > 0$.

Then there exists a diagonal matrix, G, and its inverse defined by

$$g_{1,1} = 1 \qquad g_{i,i} = \left(\frac{a_2 a_3 \ldots a_i}{b_2 b_3 \ldots b_i}\right)^{\frac{1}{2}}$$

Multiplying M by G and G^{-1} the following results

$$G^{-1}MG = T$$

where T is a symmetric tri-diagonal matrix with the following entries

$$t_{i,i} = s_i \qquad t_{j,j+1} = t_{j+1,j} = (a_{j+1}b_{j+1})^{\frac{1}{2}}.$$

4.2 The Block Analogy

This result can be applied to a matrix M with the following block form

$$M_{i,i} = S_i \qquad M_{j+1,j} = A_{j+1} \qquad M_{j,j+1} = B_{j+1}$$

Let $\mathcal{A}$ denote the set of all matrices A_{j+1} and $\mathcal{B}$ denote the set of all matrices B_{j+1} and $\mathcal{S}$ denote the set of all the matrices S_i. We are assuming that all the elements of $\mathcal{S}$ are symmetric and commute with all the elements of $\mathcal{A} \cup \mathcal{B}$. We are also assuming that all the elements of $\mathcal{A}$ and $\mathcal{B}$ are symmetric, positive-definite and that all the elements of $\mathcal{A} \cup \mathcal{B}$ commute with each other. The analogous block diagonal matrix G is defined as

$$G_{1,1} = 1 \qquad G_{i,i} = \big((A_2 A_3 \ldots A_n)(B_2 B_3 \ldots B_n)^{-1}\big)^{\frac{1}{2}}$$

Following the same steps as above,

$$G^{-1}MG = T$$

where

$$T_{i,i} = S_i \qquad T_{i,i+1} = T_{i+1,i} = (A_{i+1}B_{i+1})^{\frac{1}{2}}.$$

4.3 Application to Vertex Conductivity Networks

We can apply a transformation similar to the one in Theorem 4.1.1 to symmetrize a vertex conductivity K or D matrix - where D is the interior node sub-matrix of K defined in the discretization section. The K and D matrices are not necessarily tri-diagonal, but have the property that the off-diagonal terms are positive and that the zero entries are symmetric about the main diagonal.

Theorem 4.3.1 *All the eigenvalues of the K matrix of a vertex conductivity network are real.*

Proof The proof that follows is for a complete graph, for which K has no zero entries. The generalization to other graphs follows readily by the above observation that zeros are symmetric about the diagonal of K. With a_j as the conductivity at vertex j, the K matrix is given by:

$$k_{i,i} = \sigma_i = -\sum_{j \neq i}^{n} a_j \qquad k_{i,j} = a_j \qquad \text{for } i \neq j$$

Since $a_i > 0$, $(a_1/a_i) > 0$, and we can define the diagonal matrix G by

$$g_{1,1} = 1 \qquad g_{i,i} = \left(\frac{a_1}{a_i}\right)^{\frac{1}{2}}$$

Here we take the positive root of each radical. G^{-1} exists because all the diagonal entries are positive. When we multiply K on the left by G^{-1} and on the right by G, we obtain:

$$G^{-1}KG = S$$

where

$$s_{i,i} = \sigma_i \qquad s_{i,j} = s_{j,i} = (a_i a_j)^{\frac{1}{2}} \qquad \text{for } i \neq j$$

Again we take the positive roots of each radical in the last equation - the radical exists since $a_i > 0$. Thus, S is symmetric, and hence has real eigenvalues. Note that if K contained zero elements, S would remain symmetric. Since K and S share the same eigenvalues, we conclude that the K matrix for a vertex conductivity network has real eigenvalues.

Corollary 4.3.1 *All the eigenvalues of the D matrix (interior node sub-matrix of K) of a vertex conductivity network are real.*

Proof The proof is the same as in the case of the K matrix, since the zero row sum property was not used in the diagonalization.

The principal minors of an $n \times n$ matrix A are given by $A(J;J)$, where $J \subset \{1, \ldots, n\}$, $J \neq \emptyset$). The following lemma applies in particular to $K' = -K$ for vertex conductivity networks, but more generally to K matrices for directed graphs (see Section 6).

Lemma 4.3.1 *The determinants of the principal minors of a matrix K_n of the following form are all nonnegative:*

$$K_n = \begin{bmatrix} \sigma_1 + \epsilon_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sigma_2 + \epsilon_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sigma_n + \epsilon_n \end{bmatrix} \qquad \begin{array}{c} \sigma_i = \sum_{j \neq i} a_{ij} \\ a_{ij} \geq 0 \\ \epsilon_i \geq 0 \end{array} \tag{35}$$

Proof We proceed by induction on n, where n is the size of the principal minor being considered. For the base case $n = 1$, the determinant is nonnegative since $\sigma_1 + \epsilon_1 = \epsilon_1 \geq 0$. Now assume that all principal minors of size $m - 1$ have nonnegative determinant, for some integer $m > 1$. Consider a principal minor of size m: K_m. If $(\sigma_m + \epsilon_m) = 0$ then $a_{mj} = 0$ and so det $K_m = 0$ (ie. the determinant is nonnegative). Now assume that $(\sigma_m + \epsilon_m) \neq 0$. Since row operations do not change the determinant, we can perform Gaussian elimination on K_m to obtain $\overline{K}_m$ with the same determinant. In particular, we add appropriate multiples of row m to rows 1 through $m - 1$ so that the last entry in rows 1 through $m - 1$ is zero.

$$\overline{K}_m = \begin{bmatrix} \overline{\sigma_1} + \overline{\epsilon}_1 & -\overline{a}_{12} & \cdots & 0 \\ -\overline{a}_{21} & \overline{\sigma}_2 + \overline{\epsilon}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & \cdots & \sigma_m + \epsilon_m \end{bmatrix} \tag{36}$$

The off-diagonal entries of the resulting $\overline{K}_m$ are:

$$-\overline{a}_{ij} = -a_{ij} - \frac{a_{in}a_{nj}}{\sigma_n + \epsilon_n} \leq 0 \tag{37}$$

The new row sums remain nonnegative:

$$\overline{\epsilon}_i = \epsilon_i + \frac{a_{in}\epsilon_{1n}}{\sigma_n + \epsilon_n} \geq 0 \tag{38}$$

The determinant of $\overline{K}_m$ (and K_m) is then found by a cofactor expansion about the last column, which has only one nonzero entry. By (37) and (38), $\overline{K}_m(\{1, \ldots, m-1\}; \{1, \ldots, m-1\})$ is a principal minor of size $m-1$, its determinant is nonnegative; call it δ. Thus, $\det K_m = \det \overline{K}_m = (\sigma_m + \epsilon_m)\delta \geq 0$.

Theorem 4.3.2 *The eigenvalues of a vertex conductivity K matrix are all nonpositive.*

Proof Let $K' = -K$. We show that the eigenvalues of K' are nonnegative by showing that K' is similar to a symmetric, positive semi-definite matrix under the transformation given by Theorem 4.3.1:

$$G^{-1}K'G = S'$$

S' is positive semi-definite if and only if the determinants of its principal minors are all nonnegative. The principal minors of S' and K' are similar in this case because G is diagonal. Hence, the determinant of each principal minor of S' is equal to the determinant of each principal minor of K'. By Lemma 4.3.1, the principal minors of K' have nonnegative determinants. Thus, S' is positive semi-definite, and its eigenvalues, which equal the eigenvalues of K', are nonnegative.

Corollary 4.3.2 *The eigenvalues of the D matrix corresponding to a connected network are all negative.*

Proof Let $D' = -D$. D' is positive semi-definite by Theorem 4.3.2. To show that D' is positive definite, it is sufficient to show that it is nonsingular. To this end we consider the system $C\vec{u}_b + D\vec{u}_i = 0$; i refers to the interior nodes, and b to the boundary nodes. This system is the current conservation statement for the interior nodes (see the discretization section). The uniqueness of the solution to the Dirichlet problem then requires D to be invertible.[1] Thus D' is positive definite, which means that the eigenvalues of D are all negative.

5 Recoverable Networks

5.1 Tree Graphs

In previous sections, we discussed ways to manipulate graphs in order to simplify them so that the conductivities can be recovered. As a direct result, tree graphs are recoverable with repeated application of the aforementioned procedures to delete edges.

A tree graph consists of a graph with no closed paths and with all single valence vertices designated as boundary nodes. An example of such a graph is given in Figure 7.

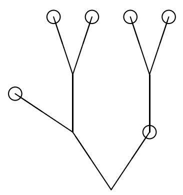


Figure 7: An example of a tree graph

Theorem 5.1.1 *All tree graphs are recoverable.*

Proof The following argument is for either edge or vertex conductivities. Each spike is either a boundary spike or boundary to boundary edge connection. For either type of connection, after removing the associated edge, the modified graph has a boundary node where the corresponding edge is removed. The remaining graph is still a tree graph by definition. Thus, our method can be repeated and every conductivity can be recovered down to the case where a single boundary node remains. At this point, all the conductivities will be known.

[1] The uniqueness of the Dirichlet problem solution can be proven by assuming that two solutions exist, and then using the maximum principle on their difference, which is also a solution. The result is that the difference between the solutions has to be zero.

5.2 Ring Networks

Ring networks are recoverable in a similar way to which tree graphs are recoverable. A ring network is denoted $R(r, \ell)$, where r is the number of rays and ℓ is the number of layers. The rays are evenly distributed around the circles with $\frac{2\pi}{r}$ radians between each ray. The outer most layer consists of boundary spikes. $R(5, 2)$ is shown in figure 8.

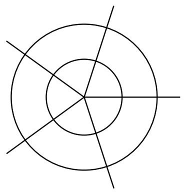


Figure 8: Ring Network R(5,2)

Theorem 5.2.1 *Ring networks are recoverable.*

Proof The following argument is for either edge or vertex conductivities. Starting with the boundary spikes, each edge can be removed using the boundary spike formula. This yields a graph with the outermost ring consisting of all boundary nodes. The boundary to boundary edge formula can now be applied. The resulting graph is an $R(r, \ell - 1)$ network. The next step is to remove the edges using the boundary spike formula again. This process continues until a single boundary node remains. Thus, the whole graph is recoverable.

6 Non-recoverable Networks

6.1 Double Interior Spikes

By inspection and working with the K matrix generally, we found patterns to graphs that were not recoverable. One such graph is one with two interior spikes joined to one boundary node. We are defining an interior spike as an edge of a graph connecting an isolated interior node to the rest of the graph.

Theorem 6.1.1 *A graph that includes two interior spikes joined to one boundary node is not recoverable.*

Proof From the K matrix of the network, a power series expansion can be derived. The following matrix shows the key entries in the K matrix.

$$K = \begin{bmatrix} -(b+c)-\sigma & * & \cdots & * & b & c & * & \cdots & * \\ * & * & \cdots & * & 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * & 0 & 0 & * & \cdots & * \\ a & 0 & \cdots & 0 & -a & 0 & 0 & \cdots & 0 \\ a & 0 & \cdots & 0 & 0 & -a & 0 & \cdots & 0 \\ * & \cdots & \cdots & * & 0 & 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * & 0 & 0 & * & \cdots & * \end{bmatrix}$$

a is the conductivity of the single boundary node (labeled first among the other boundary nodes) while b and c are the conductivities of the interior nodes of the spikes (labeled first among the other interior nodes). The K matrix is divided into four submatrices.

$$B = \begin{bmatrix} b & c & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & * & \cdots & * \end{bmatrix} \qquad C = \begin{bmatrix} a & 0 & \cdots & 0 \\ a & 0 & \cdots & 0 \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}$$

A and D are the remaining upper left and lower right submatrices of the K matrix. The following is a representation of the power series

$$\Lambda(\lambda) = \lambda + A - \sum_{k=0}^{\infty} B(-D)^k C \lambda^{-k-1}$$

$$\Lambda(\lambda) = \lambda + \begin{bmatrix} -(b+c)-\sigma & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix} - \sum_{k=0}^{\infty} \begin{bmatrix} -a^{k+1}(b+c)+\epsilon_k & * & \ldots & * \\ * & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & * \end{bmatrix} \lambda^{-k-1}$$

where σ and ϵ are sums that are determined by the remainder of the network, and do not involve b and c. The network can be composed of any number of boundary and interior nodes and edges that join them.

Since $D^k C$ has the first two rows equal, we can conclude that the conductivities b and c will always appear as a sum $(b+c)$ in the coefficients of the power series. Thus, the two conductivities cannot be distinguished from one another. Therefore, the network is not recoverable.

6.2 An Edge Conductivity Example

An example of a non-recoverable network for the edge conductivity case is shown in Figure 9. It consists of four interior nodes, one boundary node found in the center of the graph, and nine edges.

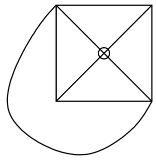


Figure 9: A non-recoverable edge conductivity graph

From Theorem 3.2.2, we know that that the poles and zeroes of det Λ are found from the eigenvalues of the K and C matrices. Since K is 5×5, we know that there are four non-trivial zeroes of det Λ (by non-trivial we mean all zeroes except $\lambda = 0$, and we are including multiplicities). Since C is a 4×4 matrix, there are four eigenvalues, none of which are zero (since C is nonsingular). Referring to (24), $\Lambda(\lambda)$ is fully determined by $s_1, \ldots, s_4, z_1, \ldots, z_4$. This means that there are eight givens and nine unknowns for this network. Thus, the system in not solvable. Therefore, the network in not recoverable.

7 Miscellaneous Leads and Ends

The following is a small collection of results, ideas, and counter-examples. We did not have time to consider most of these questions in depth.

7.1 Directed Networks

Directed networks are networks in which the conductivity from node i to j is not necessarily the same as from node j to i. This is difficult to conceive physically, as the difference in conductivity is not related to the direction of the current. Rather, the conductivity of an edge assumes one value when one writes the current conservation equation for node i and another value when one writes a similar equation for node j.

The Kirchhoff matrix, K, for a directed graph is asymmetric. A vertex conductivity network is an example of a directed graph, but it is special in that its K matrix has all off-diagonal column entries equal, and zeros symmetric about the main diagonal. For a general directed graph, K can be written as:

$$K = \begin{bmatrix} \sigma_1 & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sigma_2 & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sigma_n \end{bmatrix}$$

The row sums of K are still zero, and the determinants of the principal minors of K are nonnegative by Lemma 4.2.1. However, it turns out that the eigenvalues of K for directed graphs do not have to be real, as in the following example:

$$K = \begin{bmatrix} 4 & -3 & -1 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix} \quad \lambda = \text{eigenvalues} = \begin{bmatrix} 0 \\ (1/2)(11+\sqrt{3}i) \\ (1/2)(11-\sqrt{3}i) \end{bmatrix}$$

Moreover, K is not necessarily diagonable (in $\mathbb{C}$):

$$K = \begin{bmatrix} 5 & -2 & -3 \\ -2 & 6 & -4 \\ -2 & -1 & 3 \end{bmatrix} \quad \lambda = \begin{bmatrix} 0 \\ 7 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2.5 \\ -1 \\ -1 \end{bmatrix}$$

We have not looked at recoverability criteria for these graphs with respect to the scattering problem. The above properties of K suggest that quite different results are possible.

7.2 Recoverable Asymmetric Double Interior Spike

In section 6.2, a network with two interior spikes connected to a boundary node was determined to be non-recoverable for the vertex conductivity scattering problem. However, the following network (Figure 10) is recoverable:

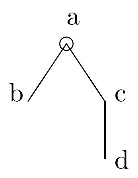


Figure 10: A recoverable double spike arrangement

The following is the K matrix for the network.

$$\begin{bmatrix} -(b+c) & b & c & 0 \\ a & -a & 0 & 0 \\ a & 0 & -(a+d) & d \\ 0 & 0 & c & -c \end{bmatrix}$$

Using the power series expansion, we recover the conductivities as follows. From the A matrix, we obtain the quantity $(b+c)$. Then from expanding BC, we can recover the conductivity a. From BDC we can recover dc. Then by expanding BD^2C, we can recover $(d+c)$. Thus we can determine d and c, and hence b. So the whole network is recoverable. This seems counterintuitive at first because the case with two edges, two interior spikes, and one boundary node is not recoverable, but buy adding one more interior node and edge to the network, it is recoverable.

7.3 The Schroedinger Equation

The continuous (time independent) Schroedinger equation for scattering off a spatially-dependent potential $q(x)$, at constant wavelength, is given by:

$$\triangle u = qu$$

If we are speaking of scattering in quantum mechanics, u is the wave function that governs the scattered particle. More realistically, for scattering using different wavelengths (designated by λ), we can write:

$$\triangle u = (q - \lambda)u$$

Discretizing this equation, we have:

$$K_1 u = (q - \lambda)u \tag{39}$$

Here, K_1 is the discretization of the Laplace operator. K_1 corresponds to a network in which all the conductivities are 1; it is the same up to sign for edge conductivities as for vertex conductivities.

Making the distinction between boundary nodes and interior nodes, we write (39) in block form:

$$\begin{bmatrix} A + \lambda - I_q(\mathcal{B};\mathcal{B}) & B \\ B^T & C + \lambda - I_q(\mathcal{I};\mathcal{I}) \end{bmatrix} \vec{u} = \begin{bmatrix} \Phi \\ 0 \end{bmatrix}$$

I_q is a diagonal matrix formed from the vector q, while $\mathcal{B}$ and $\mathcal{I}$ refer to the boundary and interior nodes, respectively. As in the edge and vertex conductivity case, we can write a power expansion for the response matrix:

$$\Lambda(\lambda) = \lambda + A - I_q(\mathcal{B};\mathcal{B}) - \sum_{k=0}^{\infty} B(-C + I_q(\mathcal{I};\mathcal{I}))^k B^T \lambda^{-k-1}$$

We assume that the network is known, which means that A, B, and C are known. Given $\Lambda(\lambda)$, it is desired to determine $\vec{q}$, and hence I_q. Clearly, $I_q(\mathcal{B};\mathcal{B})$ is recovered from the λ^0 term. It is not clear whether $I_q(\mathcal{I};\mathcal{I})$ can be recovered simply from the expansion terms, the difficulty being the surrounding B and B^T, which have the effect of selecting only a portion of $(-C + I_q(\mathcal{I};\mathcal{I}))^k$, or summing its entries.

Boundary spike and boundary edge formulas also exist for the Schroedinger formulation. Derivation of these formulas follows closely the derivations for the edge and vertex conductivity cases. Only the results are summarized below.

In contracting a boundary spike, we use the response function for the original network, $\Lambda(\lambda)$, to derive the potentials at nodes 1 and n (see Figure 6), q_1 and q_n, and the new response function, $\Lambda'(\lambda)$. q_1 is recovered from the λ_0, and q_n is recovered from λ_{-2}:

$$
\begin{aligned}
q_1 &= A(1;1) - \lambda_0(1;1) = -1 - \lambda_0(1;1) \\
q_n &= C(n;n) - \lambda_{-2}(1;1)
\end{aligned}
$$

$\Lambda'(\lambda)$ is obtained by writing the equations corresponding to $(K_1 - I_q + \lambda)\vec{u} = \vec{\Phi}$ for nodes 1 and n before the contraction and for node n after the contraction. u_1 is eliminated from these equations, and the new currents are written as functions of the new boundary potentials to give $\Lambda'(\lambda)$:

$$
\Lambda'(\lambda) = \begin{bmatrix} \dfrac{(q_1 + \lambda_{11} - \lambda)}{\delta} & \dfrac{1}{\delta}\Lambda(1;J) \\ \dfrac{1}{\delta}\Lambda(J;1) & -\dfrac{1}{\delta}\Lambda(J;1)\Lambda(1;J) + \Lambda(J;J) \end{bmatrix}, \quad \delta = 1 + q_1 + \lambda_{11} - \lambda
$$

In the above expression, J is the set of all boundary nodes excluding nodes 1 and n. In the boundary edge case (see Figure 5), q_1 and q_2, corresponding to nodes 1 and 2, are recovered directly from λ_0 of the power expansion:

$$
\begin{aligned}
q_1 &= A(1;1) - \lambda_0(1;1) = -1 - \lambda_0(1;1) \\
q_2 &= A(2;2) - \lambda_0(2;2) = -1 - \lambda_0(2;2)
\end{aligned}
$$

Current conservation is written for boundary nodes 1 and 2 before and after the edge deletion. These equations lead to the following new response matrix:

$$
\Lambda'(\lambda) = \begin{bmatrix} \lambda_{11} + 1 & \lambda_{12} - 1 & \cdots & \lambda_{1n} \\ \lambda_{21} - 1 & \lambda_{22} + 1 & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix}
$$

The above boundary spike and boundary edge formulas can then be used to recover potentials for entire graphs when at each step there is always a boundary spike or boundary edge.

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