Vertex Conductivity Networks

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Abstract

Inverse problems for conductivity networks are discussed. Small networks and circular networks are considered. As in [2], but unlike [1], conductors in the network are considered to be on the vertices of the graph.

1 Introduction

A conductivity network, $G = (\Gamma, \gamma)$, is represented by a graph Γ, and a set of positive conductivities γ defined over the vertices of the graph. The graph is assumed to be connected. The vertices are divided into boundary ($\mathcal{B}$) and interior ($\mathcal{I}$) nodes, and are ordered with the boundary nodes listed first.

The current into node p due to some potentials u on the vertices is defined by the following discretization of the γ-Laplacian operator $\nabla \cdot (\gamma \nabla u)$, as described in [2].

$$I_p(u) = \sum_{q \in \mathcal{N}(p)} (u_q - u_p)\gamma_q \tag{1}$$

Potentials on G must satisfy the conductivity equation at every interior node, that is,

$$I_p(u) = 0 \quad \text{for } p \in \mathcal{I} \tag{2}$$

The Kirchhoff matrix K is defined so that $K \cdot u = I(u)$. The information contained in this matrix is equivalent to G. The entries of K are determined by following conditions.

$$K_{ij} = \begin{cases} 0 & j \neq i,\ j \notin \mathcal{N}(i) \\ \gamma_j & j \neq i,\ j \in \mathcal{N}(i) \\ -\sum_{n \neq i} K_{in} & j = i \end{cases}$$

For convenience, the Kirchhoff matrix may be divided into four blocks by splitting the boundary and interior rows and columns. Thus,

$$K = \begin{array}{c} \\ \mathcal{B} \\ \mathcal{I} \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{B} & \mathcal{I} \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$$

The response matrix Λ is defined so that $\Lambda \cdot u|_{\mathcal{B}} = I(u)|_{\mathcal{B}}$. Λ is thus the Schur complement of D in K, that is,

$$\Lambda = K/D = A - BD^{-1}C$$

It is important that we assume to assume D is invertible. It can in fact be shown to be invertable under certain hypotheses, as discussed in [2], Lemma 3.4 and also in [3].

The inverse problem is this: given a graph Γ, and the response matrix Λ, find the set of conductivities γ that generate Λ.

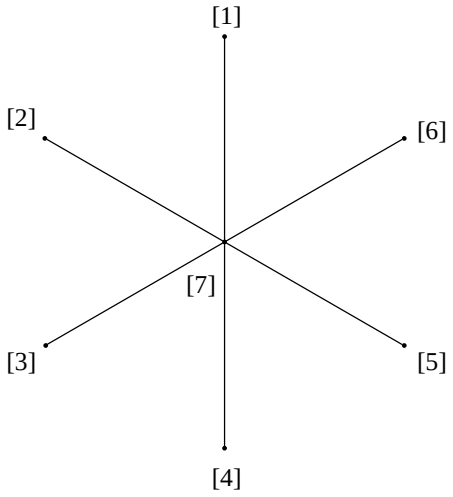

Figure 1: A 6-star graph

2 Simple Networks

2.1 Graphs with no interior nodes

Networks with no interior nodes are trivial to recover. In this case, the response matrix is identical to the Kirchhoff matrix, so the values of γ may be read off of the response matrix.

2.2 Stars

An n-star graph is a graph with one interior node and n boundary nodes, each of which is joined by an edge to the interior node. For such a network, $\mathcal{B} = \{1, \ldots, n\}$, and $\mathcal{I} = \{n+1\}$. Note that n is assumed to be larger than 2.

Theorem 2.1. *An n-star network is not recoverable.*

Proof. Let γ be a set of conductivities that generate Λ. Define γ' by multiplying each boundary conductivity by a constant t, that is,

$$\gamma' = (t\gamma_1, t\gamma_2, \ldots, t\gamma_n, \gamma_{n+1})$$

Now, this γ' generates a new Kirchhoff matrix, K', and

$$K' = \begin{pmatrix} A & B \\ tC & tD \end{pmatrix}$$

Finally,

$$\Lambda' = A - B(tD)^{-1}(tC) = A - Bt^{-1}D^{-1}tC = A - BD^{-1}C = \Lambda$$

Thus there is no unique set of conductivities that generates Λ. □

Theorem 2.2. *The conductivity at the central node of an n-star network may be recovered.*

Proof. The conductivity at the central node may be recovered by the following method. Set $u_1 = -1$, $u_2 = x$, and u_3 through u_n equal to 0. We ask, what value of x will produce a current of 0 at node 3? (Note that x exists and is unique, because it can be determined by γ-harmonic continuation of u.) To find x, we write the conductivity equation (2) at node 3:

$$I_3(u) = 0 = x\lambda_{32} - \lambda_{31}$$

Next we obtain the current at node 1, first from the response matrix, and then from the definition of current. Note that the potential at the central node is equal to 0.

$$I_1(u) = x\lambda_{12} - \lambda_{11} = (0 - (-1))\gamma_{n+1}$$

Solving this gives a value for γ_{n+1}. □

Interestingly, direct algebraic manipulation of the response matrix allows us to compute the conductivity at the central node more simply. For an n-star graph,

$$K = \begin{bmatrix} -\gamma_{n+1} & 0 & \cdots & \gamma_{n+1} \\ 0 & -\gamma_{n+1} & \cdots & \gamma_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1 & \gamma_2 & \cdots & -\sum_1^n \gamma_i \end{bmatrix}$$

This Kirchhoff matrix generates the the following response matrix:

$$\Lambda = \begin{bmatrix} -\gamma_{n+1} + \frac{\gamma_{n+1}\gamma_1}{\sum_1^n \gamma_i} & \frac{\gamma_{n+1}\gamma_2}{\sum_1^n \gamma_i} & \cdots & \frac{\gamma_{n+1}\gamma_n}{\sum_1^n \gamma_i} \\ \frac{\gamma_{n+1}\gamma_1}{\sum_1^n \gamma_i} & -\gamma_{n+1} + \frac{\gamma_{n+1}\gamma_2}{\sum_1^n \gamma_i} & \cdots & \frac{\gamma_{n+1}\gamma_n}{\sum_1^n \gamma_i} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma_{n+1}\gamma_1}{\sum_1^n \gamma_i} & \frac{\gamma_{n+1}\gamma_2}{\sum_1^n \gamma_i} & \cdots & -\gamma_{n+1} + \frac{\gamma_{n+1}\gamma_n}{\sum_1^n \gamma_i} \end{bmatrix} \tag{3}$$

It is transparent that the value of γ_{n+1} is equal to $\Lambda(2;1) - \Lambda(1;1)$. This fact can be used to determine the central node conductivity for 2-star networks as well.

Theorem 2.3. *Given an n-star network, inserting at least one, but less than $n(n-1)/2$ boundary-to-boundary edges gives a recoverable network.*

Proof. For every i, j with $i \neq j$, λ_{ij} will either be $\frac{\gamma_j\gamma_{n+1}}{\sum_1^n \gamma_k}$ (if node j is not connected to node i), or $\gamma_j(1+\frac{\gamma_{n+1}}{\sum_1^n \gamma_k})$ (if node j is joined by an edge to node i). The hypothesis of this theorem ensures that Λ will contain at least one of each type of entry.

Suppose that column j is a column that contains at east one entry of the first form and at least one entry of the second form. Suppose that $\lambda_{i_1 j}$ is the entry in this column that has the first form, and that $\lambda_{i_2 j}$ is the entry that has the second form. Then $\gamma_k = \lambda_{i_2 j} - \lambda_{i_1 j}$. Now, for $k : k \leq n, k \neq j$, the ratio γ_j/γ_k may be found by dividing either $\lambda_{i_1 j}$ or $\lambda_{i_2 j}$ by an entry in column k. Thus, every boundary conductivity may be recovered. The central conductivity may be recovered by subtracting the diagonal entry λ_{jj} from $\lambda_{i_2 j}$. □

2.3 Capped delta

The capped delta network, shown in Figure 2, is a simple example of a 3-star graph with one inserted edge. While this network is not recoverable in the edge conductivity case, it can be recovered in the vertex conductivity case. This network is given by the Kirchhoff matrix

$$K = \begin{bmatrix} -d & 0 & 0 & d \\ 0 & -(d+c) & c & d \\ 0 & b & -(d+b) & d \\ a & b & c & -(a+b+c) \end{bmatrix}$$

The Schur complement of $K(\mathcal{I};\mathcal{I}) = [-(a+b+c)]$ in K produces

$$\Lambda = \begin{bmatrix} -d + \frac{da}{a+b+c} & \frac{db}{a+b+c} & \frac{dc}{a+b+c} \\ \frac{da}{a+b+c} & \frac{db}{a+b+c} - (d+c) & \frac{dc}{a+b+c} + c \\ \frac{da}{a+b+c} & \frac{db}{a+b+c} + b & \frac{dc}{a+b+c} - (d+b) \end{bmatrix}$$

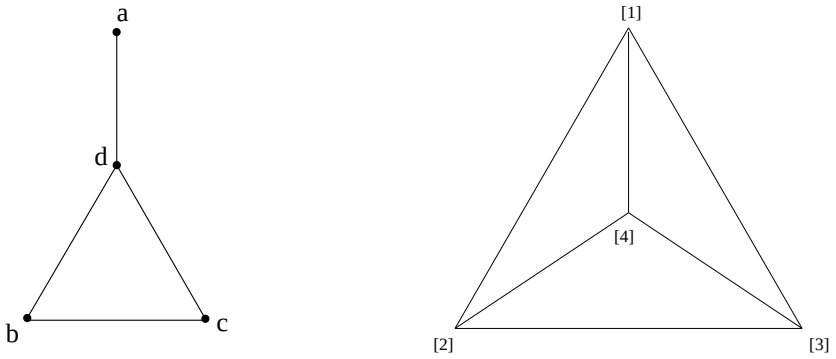

Figure 2: The capped delta graph (left), with conductivities labelled and; a complete graph (right) of 3 boundary nodes and one interior node

It easy to see that we can derive γ from the response matrix.

$$b = \Lambda(3,2) - \Lambda(1,2)$$
$$c = \Lambda(2,3) - \Lambda(1,3)$$
$$d = \Lambda(2,1) - \Lambda(1,1)$$

Once we have derived b, c, and d, we can compute a by noticing that

$$\frac{\Lambda(2,1)}{\Lambda(1,2)} = \frac{a}{b} \implies a = \frac{\Lambda(2,1)}{\Lambda(1,2)} b$$

2.4 Complete graphs

A graph is called complete if each node is joined by a single edge to every other node.

Theorem 2.4. *A network with a complete graph and at least one interior node is not recoverable.*

Proof outline. Because Γ is complete, each entry of K is nonzero. Also, in each column, every entry is identical, except for the diagonal entry. Interestingly, Λ also has this property, because the Schur complement operation acts in the same way on each entry in a single column. The graph Γ has $|\mathcal{B}|$ boundary nodes and $|\mathcal{I}|$ interior nodes. Then the response matrix has only $|\mathcal{B}|$ unique off-diagonal entries. (The diagonal entries are not particularly useful either—they are merely the negative of the row-sum.) $|\mathcal{B}|$ unique pieces of data is not sufficient to determine $|\mathcal{B}| + |\mathcal{I}|$ conductivities. □

3 Circular Networks

3.1 Definitions

A circular network, denoted $C(r, s)$, consists of r concentric rings, and s rays from a central node. There are s edges connecting any ring with either of its neighbouring rings. The central node has valence s. The boundary vertices are labelled v_1 through v_s in anti-clockwise order, beginning with the top-most vertex. On the outermost circle, the node adjacent to v_1 is labelled v_{s+1}, and the remaining vertices on the circle are labelled in anti-clockwise order. This labelling convention is repeated on every circle. The central node is therefore labelled $v_{s(r+1)+1}$.

Every interior node, except the central node, has four neighbours—an inward neighbour, an outward neighbor, a clockwise neighbour, and an anti-clockwise neighbour.

Let s be greater than or equal to $4r + 3$. Let η be the set of boundary nodes $(2, \ldots, 2r + 2)$. Let χ be the set of boundary nodes $(s - 2r, \ldots, s)$. The cardinalities of these two sets are equal.

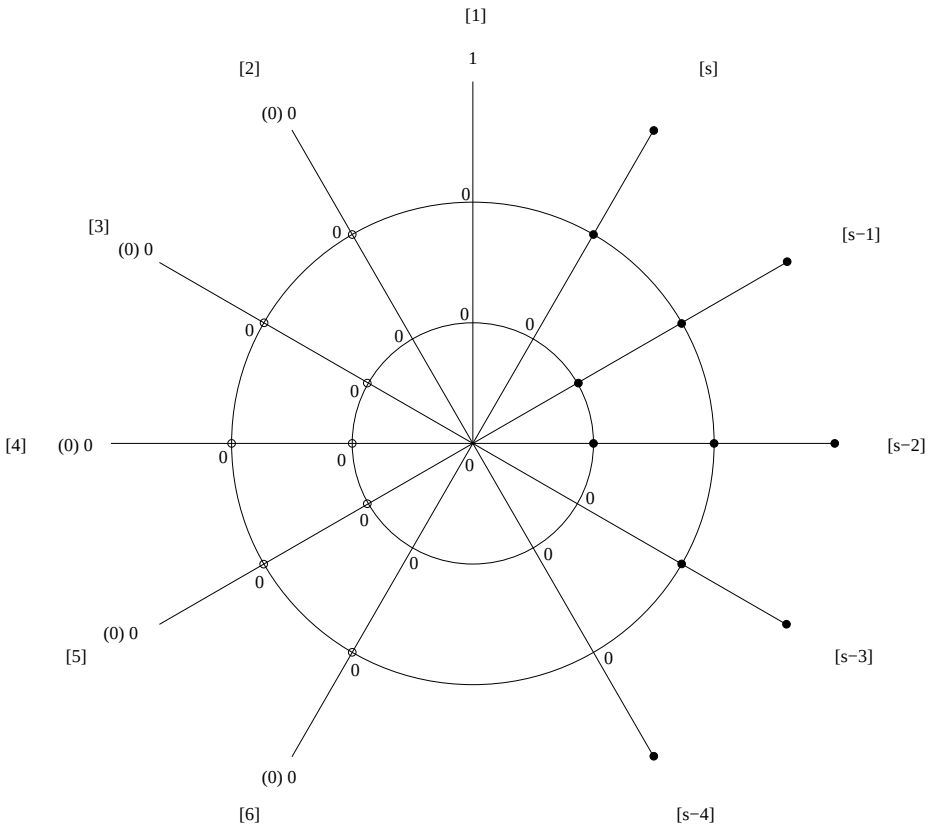

Figure 3: A circular 2-ring network, with vertices numbered in brackets

Theorem 3.1. *Given a network $C(r, s)$ with $s \geq 4r+3$, with current zero on η, and potential zero on $\mathcal{B}-\chi$, then the potential on χ must be zero. This is equivalent to the statement that $\Lambda(\chi;\eta)$ is nonsingular (see [3]).*

Proof outline. γ-harmonic extension to the interior of the imposed boundary potentials results in zero potential, and therefore zero current, everywhere on the network. □

While we only need a minimum of $4r+2$ boundary spikes to determine that a unique set of γ-harmonic potentials exist, this minimum can only determine the solution where $u_i = 0$ for all i. In order to achieve a non-trivial solution, we choose to include one additional boundary node, where we will set a nonzero potential. Hence, we need s to consist of at least $2(2r+1)+1 = 4r+3$ elements in order to uniquely determine a nontrivial set of potentials on the network.

We now impose boundary conditions that will allow us to solve for the γ in the network. Let the potential at v_1 be set to 1. We ask, what potentials may be imposed on χ that will result in zero current on η?

Notice that in this boundary setup, there are $2r+1$ consecutive boundary nodes with zero current and zero potential. These are the η. The $2r+1$ nodes adjacent to η (the 5 vertices that are on the outer circle and marked with open dots in Figure 3) thus also have zero potential. There are then $2r-1$ nodes (the 3 vertices marked with open dots) with zero potential on the second outermost circle. Thus, this boundary setup results in a zero potential on the central node.

3.2 The solution

Outer circle Like the lattice in [2], we first find γ on the outer circle. Solving

$$\Lambda(\eta;\chi)\cdot u = -\Lambda(\eta;1)$$

gives us the potentials on χ. Next, we find γ on the node adjacent to v_1. To do this, we write equations for the current into v_1, first using the response matrix, and then using the definition of current into v_1.

$$I_1(u) = (u_{s+1} - u_1)\gamma_{s+1} = \Lambda(1;1) \cdot u_1 + \Lambda(1;\chi) \cdot u_\chi$$

Since $u_1 = 1$ and $u_{s+1} = 0$, this equality reduces to

$$\gamma_{s+1} = -(\Lambda(1;1) + \Lambda(1;\chi) \cdot u_\chi)$$

Because the graph is rotationally symmetric, it is possible to compute γ on all the nodes on the outermost circle.

Boundary The conductivity may be computed on the boundary by observing the current flowing into v_{s+1}. This is an interior node, and so satisfies (2).

$$I_{s+1}(u) = 0 = (u_1 - u_{s+1})\gamma_1 + (u_{2s} - u_{s+1})\gamma_{2s}$$

Now v_{2s} is an interior node on the outer circle, so the value of γ_{2s} was computed in the previous step. All that remains is to find u_{2s} before we are able to determine γ_1 from the above equation. We can find u_{2s} by computing the current into the boundary node adjacent to it, v_s. Again, this current is represented by two separate expressions.

$$I_s(u) = (u_{2s} - u_s)\gamma_{2s} = \Lambda(s;1) \cdot u_1 + \Lambda(s;\chi) \cdot u_\chi$$

The value of u_s is known—it is part of χ, and was computed in the previous step. After solving this equation for u_{2s}, γ_1 may be found. Again, exploiting the symmetry of the graph, we may rotate the network and obtain γ on every boundary node.

At this point, we may also compute the potential everywhere on the outermost circle (we will need it for the next step). The potentials at each node on this circle may be computed by observing the current flow on the neighbouring boundary vertex.

Inner circles The conductivity on the inner circles must be computed from the outside towards the center. Suppose we are on the j^{th} circle from the outermost circle. We have already computed γ and the potentials for the $(j-1)^{\text{th}}$ circle. Let v_i be the first vertex with nonzero potential on the j^{th} circle. Write the conductivity equation on the clockwise and outwards neighbours of v_i. Every value that appears in these two equations is known, except the conductivity and the potential on v_i. By exploiting the symmetry of the graph, we may determine the value of γ on this circle. Having done that, we may find every nonzero potential at every vertex v_i the j^{th} circle from the conductivity equation on the outward neighbor of v_i. We now have enough information to compute the conductivity on the next inner circle.

Central node There are only two nodes adjacent to the central node with nonzero potential, both of which have been computed in the last step. We have also computed the conductivity on these two nodes. Now, write the conductivity equation on either of these nodes. All values that appear in this equation, except for γ on the central node, were found in earlier steps. We may thus solve for the conductivity at the central node.

4 Accuracy of Calculations

In this section, we discuss the recovery of circular networks, as described in the section above. We use the algorithm `createRringK` to formulate response matrices. We use the algorithm `Rring` to recover γ from the response matrices. The `matlab` code is at the end of this document.

We consider the 34 vertex 2-ring network, shown in figure (3). We arbitrarily assign the conductivity 1 to each vertex. By default, `matlab` computes the response matrix to 16 digit accuracy. We attempted to recover the conductivites four times, incrementally decreasing the accuracy of the entries in the response matrix. Table 1 summarizes the results.

It is unreasonable to expect hand measurements to produce a response matrix that is accurate to more than a few decimal places. Clearly, 900% inaccuracy is unacceptable. Therefore, we must conclude that even in the highly symmetric case (all conductivities equal), our algorithm is not a useful computational tool.

Accuracy of Λ	Mean boundary value	Central node value
16 digits	0.99999999999990	1.00000000000115
10 digits	0.99995736805399	0.99929900738288
6 digits	0.81779375000000	−6.73788884298711
5 digits	0.79792733564014	−8.51713642498162

Table 1: Recovered values for the 2-ring network

5 Code

The matlab code to create the Kirchhoff matrix from conductivities for a circular network, and then to recover conductivities from the response matrix, is given below.

```
function K = createRringK (gamma, r)
%given a vector of conductivities, create the Kirchhoff
%matrix for an r-ring conductivity network

s = 4*r+3;                                          % = number of spikes
K = zeros(4*r^2 + 7*r + 4);                         % size is the total number of nodes

%boundary
for i=1:s
  K(i, i+s) = gamma(i+s);
end

%interior circles
for i=1:r  % each circle
  for j=1:s  % each spike on circle i
    n = [];
    for k=0:r n = [n [j:s 1:(j-1)]+(k*s)]; end     % set up rotating machine
    if i==r                                         % if at the innermost circle
      last = 4*r^2 + 7*r + 4;                       % refer inner neighbor to central node
    else
      last = n(s*(i+1)+1);
    end
    neighbors = [n([s*i+s s*i-s+1 s*i+2]) last];   % vector of the neighbors of i
    K(n(s*i+1), neighbors) = gamma(neighbors);     % assign values of gamma in row j(i+1)
    end
end

%central node
for i=1:s
  K(4*r^2 + 7*r + 4, r*s+i) = gamma(r*s+i);
end

%diagonal entries
for i=1:(4*r^2 + 7*r + 4)
  K(i,i) = -sum( K(i, 1:(4*r^2 + 7*r + 4)) );      % diagonal value = negative row sum
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function gamma = Rring(L)
%calculate conductivities for circular network from the response matrix

s = size(L,1);                                      % = num spikes
r = (s-3)/4;                                        % = rings
```

```
numVars = 2*r+1;                                    % = number of variable potential spikes
zeros = 2:(numVars+1);                              % = set of zero current spikes
vars = (s-numVars+1):s;                             % = set of variable potential spikes

%outermost circle
for i=1:s
  n = [];
  for k=0:r n = [n [i:s 1:(i-1)]+(k*s)]; end       % set up rotating machine
  % solve u(vars)*L(zeros;vars) = L(zeros;1), for unknown boundary potentials
  u(n(vars),i) = L(n(zeros), n(vars))^-1 * -L(n(zeros), n(1));
  gamma(n(s+1)) = -( L(n(1),n(vars))*u(n(vars),i) + L(n(1),n(1)) );
end

%boundary
for i=1:s
  n = [];
  for k=0:r n = [n [i:s 1:(i-1)]+(k*s)]; end       % set up rotating machine
  for j=n(vars)+s        %nodes adjacent to the vars
    u(j,i) = ( L(j-s,n(1)) + L(j-s,n(vars))*u(n(vars),i) )/gamma(j) + u(j-s,i);
  end
  gamma(n(1)) = -u(n(2*s),i)*gamma(n(2*s));
end

%inner circles
for circle=2:r
  for i=1:s
    n = [];
    for k=0:r n = [n [i:s 1:(i-1)]+(k*s)]; end     % set up rotating machine
    node = (circle+1)*s - 2; % wrong
    gamma(n(node)) = -sum(gamma(n([node-s-1 node-2*s node-s+1])))                    ...
      + gamma(n([node-s+1 node-2*s])) * u(n([node-s+1 node-2*s]), i) / u(n(node-s),i);
  end

  % compute the potential on this circle for the first iteration for i=n(vars)+circle*s
  for i=(s-2*r):s
    n = [];
    for k=0:r n = [n [1:s]+(k*s)]; end             % set up rotating machine
    node = i+circle*s;
    u(node,1) = ( u(node-s,1)*sum(gamma(n([node node-s-1 node-2*s node-s+1])))       ...
      - gamma(n([node-s-1 node-2*s node-s+1]))* u(n([node-s-1 node-2*s node-s+1]), 1) ) ...
      /gamma(node);
  end
end

%center
nz = 4*r^2+7*r+4 - (numVars+1)/2;
gamma(4*r^2+7*r+4) = -sum(gamma([nz+1 nz-s nz-1]))                                   ...
  + gamma([nz+1 nz-s])*u([nz+1 nz-s],1)/u(nz,1);
```

References

[1] Curtis, Edward B., and James A. Morrow. *Inverse Problems for Electrical Networks.*

[2] Oberlin, Richard. *Discrete inverse problems for Schrödinger and Resistor networks.*

[3] Blunk, Mark, and Sam Coskey. *Connections and Determinants.*