Permuted Boundary Nodes of Circular Planar Networks

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1 Introduction

1.1 Circular Planar Networks

An electrical network investigated here can be represented as a two dimensional finite graph $G = (V, V_B, E)$, where V is the set of nodes, V_B is the set of boundary nodes and E is the set of edges. Then a circular planar network is one whose graph G can be circumscribed by a disc with the boundary nodes V_B lying on the circle that bounds the disc. The other nodes lie in the interior of the disc. The boundary nodes are labeled in the clockwise direction around the circle.

Boundary nodes are connected to each other by means of paths. A path is a sequence of edges beginning at boundary node p and ending at boundary node q, using only distinct interior nodes r_i at intermediary steps. Now let P and Q be sets of boundary nodes: $P = (p_1, p_2, \ldots, p_k)$; $Q = (q_1, q_2, \ldots, q_k)$. P and Q are connected through G if there exist disjoint paths from p_1 to q_k, p_2 to q_{k-1}, ..., and p_k to q_1. P and Q are a circular pair, denoted $(P;Q) = (p_1, p_2, \ldots, p_k; q_1, q_2, \ldots, q_k)$, if the sequence $(p_1, p_2, \ldots, p_k, q_k, \ldots, q_2, q_1)$ is in circular order. That is, $(P;Q)$ is a circular pair if

$$p_1 < p_2 < \ldots < p_k < q_k < \ldots < q_2 < q_1.$$

A circular planar network is well-connected if every circular pair $(P;Q)$ has k disjoint paths joining the elements.

1.2 Kirchhoff and Lambda Matrices

Let conductivity on a graph G be defined as a function γ acting on each element in E. A resistor network is the combination of a bounded graph and a conductivity function. The Kirchhoff matrix, K, for such a network with nodes $v_1, v_2, \ldots, v_n$ is an $n \times n$ matrix constructed by taking

$$K_{i,j} = -\gamma_{i,j}$$

where $\gamma_{i,j}$ denotes the sum of $\gamma(e)$ for edges e joining v_i to v_j if $i \neq j$, and

$$K_{i,i} = \Sigma_{i \neq j} \gamma_{i,j}.$$

It follows, then, that the diagonal entries of K are either zero or positive, the non-diagonal entries are zero or negative and the rows and columns sum to zero. A Kirchhoff matrix has the form

$$\mathbf{K} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

for square matrices A, B, and C.

A Λ matrix is defined as $\Lambda = A - BC^{-1}B^T$. This matrix has the same properties for diagonal and non-diagonal entries as well as row and column sums as the Kirchhoff matrix. Let $\Lambda(P;Q)$ denote the submatrix of Λ consisting of rows $p_1, p_2, \ldots, p_k$ and columns $q_1, q_2, \ldots, q_k$.

Theorem 1 *For a circular planar resistor network and a circular pair* $(P;Q)$,
(a) $det\Lambda(P;Q) = 0$ *if* $(P;Q)$ *is not connected through* G
(b) $(-1)^k det\Lambda(P;Q) > 0$ *if* $(P;Q)$ *is connected through* G.

This theorem allows one to determine the original connections of a network by examining the signs of subdeterminants of the Λ matrix. When two boundary nodes of a network are mislabeled, or switched, certain connections are broken. Such an action is reflected in the subdeterminantal signs. This paper begins with a generalization of such determinants for well-connected networks. A specific case of a less-than-well-connected network in which subdeterminantal signs are ambiguous in determining original connections is also considered here. Although this type of network does not yield as much information as the well-connected networks, it can be investigated with moderate success. For further background information, see [1].

2 Well-Connected Networks

2.1 Switching Adjacent Boundary Nodes

In a well-connected network, one is able to determine whether two adjacent boundary nodes were switched by the signs of subdeterminants of a given Λ matrix. Further, it is apparent exactly which two boundary nodes were altered.

2.1.1 Even Number of Boundary Nodes

In this network, there are $2n$ boundary nodes. Let boundary nodes 1 and 2 be switched. Given the Λ matrix for this network, circular pairs of size n are taken. That is, let $(P;Q)$ be the circular pair $[(1,\ldots,n);(n+1,\ldots,2n)]$. It is clear that the determinant of $(P;Q)$ has an incorrect sign because the switched nodes are both elements of P. This same property applies generally if the switched nodes are both included in P or Q. If one of the switched nodes is an element of P and the other is an element of Q, the altered network may or may not be circular planar still. In this situation, one can sometimes say with certainty that the subdeterminant has a specific sign, either correct or incorrect. However, there are times when this is not possible and thus the sign is ambiguous. Moving through the graph, continuing to take circular pairs of size n in the clockwise direction, the signs of the subdeterminants help to reveal which two nodes were switched. Table 2.1.1a is a list of the subdeterminantal signs of these circular pairs.

Table 2.1.1a

$(P;Q)$	$det\Lambda(P;Q)$
$(1,\ldots,n);(n+1,\ldots,2n)$	incorrect
$(2,\ldots,n+1);(n+2,\ldots,1)$	ambiguous
$(3,\ldots,n+2);(n+3,\ldots,2)$	incorrect
$\vdots$	$\vdots$
$(n+1,\ldots,2n);(1,\ldots,n)$	incorrect
$(n+2,\ldots,1);(2,\ldots,n+1)$	ambiguous
$(n+3,\ldots,2);(3,\ldots,n+2)$	incorrect
$\vdots$	$\vdots$
$(2n,\ldots,n-1);(n,\ldots,2n-1)$	incorrect

Since no correct signs are obtained, one cannot easily determine which two nodes were switched and thus caused the series of incorrect signs. It is then necessary to take a smaller sized circular pair. That is, let $(P;Q)$ be of size $n-1$ instead of n. By a deliberate choice, this will exclude two adjacent boundary nodes, (p,q).

Table 2.1.1b

(p,q)	$(P;Q)$	$det\Lambda(P;Q)$
$(2n-1,2n)$	$(1,\ldots,n-1);(n,\ldots,2n-2)$	incorrect
$(2n,1)$	$(2,\ldots,n);(n+1,\ldots,2n-1)$	correct
$(1,2)$	$(3,\ldots,n+1);(n+2,\ldots,2n)$	correct
$(2,3)$	$(4,\ldots,n+2);(n+3,\ldots,1)$	correct
$(3,4)$	$(5,\ldots,n+3);(n+4,\ldots,2)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(n,n+1)$	$(n+2,\ldots,2n);(1,\ldots,n-1)$	incorrect
$(n+1,n+2)$	$(n+3,\ldots,1);(2,\ldots,n)$	ambiguous
$(n+2,n+3)$	$(n+4,\ldots,2);(3,\ldots,n+1)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(2n-2,2n-1)$	$(2n,\ldots,n-1);(n,\ldots,2n-3)$	incorrect

The correct signs appearing in this table suggest that the pairs $[(2,\ldots,n);(n+1,\ldots,2n-1)]$, $[(3,\ldots,n+1);(n+2,\ldots,2n)]$ and $[(4,\ldots,n+2);(n+3,\ldots,1)]$ are in circular order. That is the same as saying the network is in circular planar order when nodes $(2n,1)$, $(1,2)$ and $(2,3)$ are excluded. Upon investigating nodes 1 and 2, one can easily determine that these nodes were switched. A 2×2 subdeterminant of Λ will have the incorrect sign when P or Q are the two switched nodes. If 1 and 2 are switched again, returning to their original circular planar positions, one sees that all the signs are correct.

2.1.2 Odd Number of Boundary Nodes

In this network, there are $2n+1$ boundary nodes. Again, let boundary nodes 1 and 2 be switched and let $(P;Q)$ be a circular pair of size n. To achieve this, boundary node r will be excluded from $(P;Q)$. That is, r will neither be an element of P or Q. When one of the switched nodes is excluded from the circular pair, the subdeterminantal sign will be correct because $(P;Q)$ will be in circular planar order. Table 2.1.2 lists the excluded boundary node r, $(P;Q)$ and the subdeterminantal sign of $(P;Q)$.

Table 2.1.2

r	$(P;Q)$	$det\Lambda(P;Q)$
1	$(2,\ldots,n+1);(n+2,\ldots,2n+1)$	correct
2	$(3,\ldots,n+2);(n+3,\ldots,1)$	correct
3	$(4,\ldots,n+3);(n+4,\ldots,2)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$n+1$	$(n+2,\ldots,2n+1);(1,\ldots,n)$	incorrect
$n+2$	$(n+3,\ldots,1);(2,\ldots,n+1)$	ambiguous
$n+3$	$(n+4,\ldots,2);(3,\ldots,n+2)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$2n+1$	$(1,\ldots,n);(n+1,\ldots,2n)$	incorrect

The only correct signs appear when nodes 1 and 2 are excluded from $(P;Q)$. It is then obvious that nodes 1 and 2 were switched. If they are permuted again, restoring the network to circular planar, the signs of all subdeterminants will be correct.

2.2 Switching Non-Adjacent Boundary Nodes

Two cases are presented here: permuted boundary nodes with one node between them and permuted boundary nodes with two nodes between them. In each situation, one investigates both an odd and even number of total boundary nodes. Although generalizations have not been found, it can be stated that as the number of boundary nodes between the switched nodes increases, it may be necessary to decrease the size of $(P;Q)$ in order to gain pertinent information.

2.2.1 One Node Between Switched Nodes

EVEN CASE:

For a network with $2n$ boundary nodes, let nodes 1 and 3 be switched. To determine which nodes were permuted, the first step is to examine subdeterminants of size n as before.

Table 2.2.1a

$(P;Q)$	$det\Lambda(P;Q)$
$(1,\ldots,n);(n+1,\ldots,2n)$	incorrect
$(2,\ldots,n+1);(n+2,\ldots,1)$	ambiguous
$(3,\ldots,n+2);(n+3,\ldots,2)$	ambiguous
$(4,\ldots,n+3);(n+4,\ldots,3)$	incorrect
$\vdots$	$\vdots$
$(n+1,\ldots,2n);(1,\ldots,n)$	incorrect
$(n+2,\ldots,1);(2,\ldots,n+1)$	ambiguous
$(n+3,\ldots,2);(3,\ldots,n+2)$	ambiguous
$(n+4,\ldots,3);(4,\ldots,n+3)$	incorrect
$\vdots$	$\vdots$
$(2n,\ldots,n-1);(n,\ldots,2n-1)$	incorrect

As in the previous even case (Section 2.1.1), no information is gained by taking pairs of size n. It is then necessary to look at pairs of size $n-1$.

Table 2.2.1b

(p,q)	$(P;Q)$	$det\Lambda(P;Q)$
$(2n-1,2n)$	$(1,\ldots,n-1);(n,\ldots,2n-2)$	incorrect
$(2n,1)$	$(2,\ldots,n);(n+1,\ldots,2n-1)$	incorrect
$(1,2)$	$(3,\ldots,n+1);(n+2,\ldots,2n)$	correct
$(2,3)$	$(4,\ldots,n+2);(n+3,\ldots,1)$	correct
$(3,4)$	$(5,\ldots,n+3);(n+4,\ldots,2)$	incorrect
$(4,5)$	$(6,\ldots,n+4);(n+5,\ldots,3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(n,n+1)$	$(n+2,\ldots,2n);(1,\ldots,n-1)$	incorrect
$(n+1,n+2)$	$(n+3,\ldots,1);(2,\ldots,n)$	ambiguous
$(n+2,n+3)$	$(n+4,\ldots,2);(3,\ldots,n+1)$	ambiguous
$(n+3,n+4)$	$(n+5,\ldots,3);(4,\ldots,n+2)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(2n-2,2n-1)$	$(2n,\ldots,n-1);(n,\ldots,2n-3)$	incorrect

The two correct signs are obtained by excluding the boundary node pairs $(1,2)$ and $(2,3)$. If the ambiguous signs were in fact correct signs, it would be necessary to take one 2×2 subdeterminant of Λ, for example $det\Lambda(1,3;4,5)$ or $det\Lambda(n+1,n+3;n+4,n+5)$, in order to determine exactly which two nodes had been switched. That is, the sign of $det\Lambda(1,3;4,5)$ would be incorrect because the pair is in non-circular order after 1 and 3 have been switched. However, the sign of $det\Lambda(n+1,n+3;n+4,n+5)$ would be correct because no boundary nodes have been permuted and thus $(P;Q)$ is a circular pair. This step makes it clear that nodes 1 and 3 were switched to produce the incorrect subdeterminantal signs.

Excluding two adjacent boundary nodes is deliberate when working with P and Q of size $n-1$; choosing $(P;Q)$ in this fashion yields more information that excluding diametrically opposite nodes. If diametrically opposite nodes p and q had been excluded from the circular pair $(P;Q)$, the following list of signs would have appeared.

Table 2.2.1c

(p,q)	$(P;Q)$	$det\Lambda(P;Q)$
$(n,2n)$	$(1,\ldots,n-1);(n+1,\ldots,2n-1)$	incorrect
$(n+1,1)$	$(2,\ldots,n);(n+2,\ldots,2n)$	incorrect
$(n+2,2)$	$(3,\ldots,n+1);(n+3,\ldots,1)$	ambiguous
$(n+3,3)$	$(4,\ldots,n+2);(n+4,\ldots,2)$	incorrect
$(n+4,4)$	$(5,\ldots,n+3);(n+5,\ldots,3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(2n-1,n-1)$	$(2n,\ldots,n-2);(n,\ldots,2n-2)$	incorrect

This method of circular pairs yields no certainly correct signs and thus does not provide any valuable information, as did the previous method. Therefore, in the following cases one is justified in only investigating circular pairs in which adjacent boundary nodes have been excluded.

ODD CASE:

For a network with $2n+1$ boundary nodes and only one boundary node between the two permuted nodes, it is necessary to take $(P;Q)$ of size $n-1$ as above. The signs obtained from circular pairs of size n are not sufficient to determine which nodes were switched. To understand this, let nodes 1 and 3 be switched again. According to previous notation, let r be the excluded boundary node.

Table 2.2.1d

r	$(P;Q)$	$det\Lambda(P;Q)$
1	$(2,\ldots,n+1);(n+2,\ldots,2n+1)$	incorrect
2	$(3,\ldots,n+2);(n+3,\ldots,1)$	ambiguous
3	$(4,\ldots,n+3);(n+4,\ldots,2)$	incorrect
4	$(5,\ldots,n+4);(n+5,\ldots,3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$n+1$	$(n+2,\ldots,2n+1);(1,\ldots,n)$	incorrect
$n+2$	$(n+3,\ldots,1);(2,\ldots,n+1)$	ambiguous
$n+3$	$(n+4,\ldots,2);(3,\ldots,n+2)$	ambiguous
$n+4$	$(n+5,\ldots,3);(4,\ldots,n+3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$2n+1$	$(1,\ldots,n);(n+1,\ldots,2n)$	incorrect

No correct signs are obtained. It becomes clear that taking subdeterminants of size n never yield pertinent information. Pairs of size n, then, will be disregarded from this point; in the situations of an odd number of boundary nodes, P and Q will immediately be taken to be of size $n-1$. Below is a table of subdeterminantal signs with P and Q of size $n-1$ for boundary nodes 1 and 3 switched.

Table 2.2.1e

(r,s,t)	$(P;Q)$	$det\Lambda(P;Q)$
$(1,2,3)$	$(4,\ldots,n+2);(n+3,\ldots,2n+1)$	correct
$(2,3,4)$	$(5,\ldots,n+3);(n+4,\ldots,1)$	correct
$(3,4,5)$	$(6,\ldots,n+4);(n+5,\ldots,2)$	incorrect
$(4,5,6)$	$(7,\ldots,n+5);(n+6,\ldots,3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(n,n+1,n+2)$	$(n+3,\ldots,2n+1);(1,\ldots,n-1)$	incorrect
$(n+1,n+2,n+3)$	$(n+4,\ldots,1);(2,\ldots,n)$	ambiguous
$(n+2,n+3,n+4)$	$(n+5,\ldots,2);(3,\ldots,n+1)$	ambiguous
$(n+3,n+4,n+5)$	$(n+6,\ldots,3);(4,\ldots,n+2)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(2n-1,2n,2n+1)$	$(1,\ldots,n-1);(n,\ldots,2n-2)$	incorrect
$(2n,2n+1,1)$	$(2,\ldots,n);(n+1\ldots,2n-1)$	incorrect
$(2n+1,1,2)$	$(3,\ldots,n+1);(n+2,\ldots,2n)$	correct

By looking at the sign of $det\Lambda(1,3;4,5)$, it is clear that nodes 1 and 3 were switched since the sign would be incorrect.

2.2.2 Two Nodes Between Switched Nodes

EVEN CASE:

In this example, let nodes 1 and 4 be switched in a network that has $2n$ boundary nodes. Taking circular pairs of size n gives the following list of signs.

Table 2.2.2a

$(P;Q)$	$det\Lambda(P;Q)$
$(1,\ldots,n);(n+1,\ldots,2n)$	incorrect
$(2,\ldots,n+1);(n+2,\ldots,1)$	ambiguous
$(3,\ldots,n+2);(n+3,\ldots,2)$	ambiguous
$(4,\ldots,n+3);(n+4,\ldots,3)$	ambiguous
$(5,\ldots,n+4);(n+5,\ldots,4)$	incorrect
$\vdots$	$\vdots$
$(n+1,\ldots,2n);(1,\ldots,n)$	incorrect
$(n+2,\ldots,1);(2,\ldots,n+1)$	ambiguous
$(n+3,\ldots,2);(3,\ldots,n+2)$	ambiguous
$(n+4,\ldots,3);(4,\ldots,n+3)$	ambiguous
$(n+5,\ldots,4);(5,\ldots,n+4)$	incorrect
$\vdots$	$\vdots$
$(2n,\ldots,n-1);(n,\ldots,2n-1)$	incorrect

No correct signs are given so one must look at smaller circular pairs. When P and Q are of size $n-1$, two correct signs appear.

Table 2.2.2b

(p,q)	$(P;Q)$	$det\Lambda(P;Q)$
$(2n-1,2n)$	$(1,\ldots,n-1);(n,\ldots,2n-2)$	incorrect
$(2n,1)$	$(2,\ldots,n);(n+1,\ldots,2n-1)$	correct
$(1,2)$	$(3,\ldots,n+1);(n+2,\ldots,2n)$	incorrect
$(2,3)$	$(4,\ldots,n+2);(n+3,\ldots,1)$	ambiguous
$(3,4)$	$(5,\ldots,n+3);(n+4,\ldots,2)$	incorrect
$(4,5)$	$(6,\ldots,n+4);(n+5,\ldots,3)$	correct
$(5,6)$	$(7,\ldots,n+5);(n+6,\ldots,4)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(n,n+1)$	$(n+2,\ldots,2n);(1,\ldots,n-1)$	incorrect
$(n+1,n+2)$	$(n+3,\ldots,1);(2,\ldots,n)$	ambiguous
$(n+2,n+3)$	$(n+4,\ldots,2);(3,\ldots,n+1)$	ambiguous
$(n+3,n+4)$	$(n+5,\ldots,3);(4,\ldots,n+2)$	ambiguous
$(n+4,n+5)$	$(n+6,\ldots,4);(5,\ldots,n+3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(2n-2,2n-1)$	$(2n,\ldots,n-1);(n,\ldots,2n-3)$	incorrect

The two correct signs obtained by excluding the boundary node pairs $(2n,1)$ and $(4,5)$ can identify 1 and 4 as the switched nodes if $det\Lambda(1,4;5,6)$ has an incorrect sign.

ODD CASE:

For a network with $2n+1$ boundary nodes and 1 and 4 switched, circular pairs of size $n-1$ will be investigated in order to distinguish which nodes were switched.

Table 2.2.2c

(r,s,t)	$(P;Q)$	$det\Lambda(P;Q)$
$(1,2,3)$	$(4,\ldots,n+2);(n+3,\ldots,2n+1)$	correct
$(2,3,4)$	$(5,\ldots,n+3);(n+4,\ldots,1)$	correct
$(3,4,5)$	$(6,\ldots,n+4);(n+5,\ldots,2)$	incorrect
$(4,5,6)$	$(7,\ldots,n+5);(n+6,\ldots,3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(n,n+1,n+2)$	$(n+3,\ldots,2n+1);(1,\ldots,n-1)$	incorrect
$(n+1,n+2,n+3)$	$(n+4,\ldots,1);(2,\ldots,n)$	ambiguous
$(n+2,n+3,n+4)$	$(n+5,\ldots,2);(3,\ldots,n+1)$	ambiguous
$(n+3,n+4,n+5)$	$(n+6,\ldots,3);(4,\ldots,n+2)$	ambiguous
$(n+4,n+5,n+6)$	$(n+7,\ldots,4);(5,\ldots,n+3)$	incorrect
$\vdots$	$\vdots$	$\vdots$
$(2n-1,2n,2n+1)$	$(1,\ldots,n-1);(n,\ldots,2n-2)$	incorrect
$(2n,2n+1,1)$	$(2,\ldots,n);(n+1\ldots,2n-1)$	incorrect
$(2n+1,1,2)$	$(3,\ldots,n+1);(n+2,\ldots,2n)$	incorrect

As in prior cases, taking a 2×2 subdeterminant can easily determine which two nodes were switched. The nodes involved in the subdeterminant will be those whose exclusion from $(P;Q)$ give a correct sign. Here, $det\Lambda(1,4;5,6)$ has an incorrect sign, signaling that this pair is non-circular. It can then be seen that nodes 1 and 4 were switched.

3 A Less-than-Well-Connected Six Boundary Nodes Network

In this example, a less-than-well-connected circular planar electrical network is established with adjacent boundary nodes 3 relabeled 4 and 4 relabeled 3. This permutation causes the network to be non-circular planar.

Figure 1: Original Network

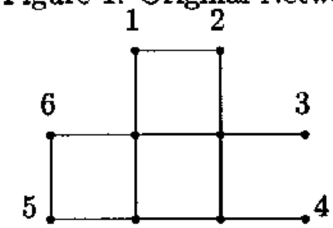
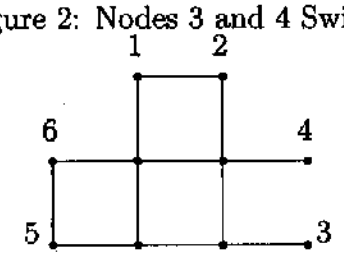


Figure 2: Nodes 3 and 4 Switched



The next step, then, is to investigate the new Λ matrix. Switching the third and fourth boundary nodes corresponds to permuting the respective rows and columns. Let Λ be the original matrix and $\Lambda^{3,4}$ denote the Λ matrix with rows 3 and 4 switched as well as columns 3 and 4 switched. Also, let Σ_q be the sum of the five non-diagonal entries in row q.

$$\Lambda = \begin{pmatrix} \Sigma_1 & -a & -b & -c & -d & -e \\ -a & \Sigma_2 & -f & -g & -h & -i \\ -b & -f & \Sigma_3 & -j & -k & -l \\ -c & -g & -j & \Sigma_4 & -m & -n \\ -d & -h & -k & -m & \Sigma_5 & -p \\ -e & -i & -l & -n & -p & \Sigma_6 \end{pmatrix}$$

$$\Lambda^{3,4} = \begin{pmatrix} \Sigma_1 & -a & -c & -b & -d & -e \\ -a & \Sigma_2 & -g & -f & -h & -i \\ -c & -g & \Sigma_4 & -j & -m & -n \\ -b & -f & -j & \Sigma_3 & -k & -l \\ -d & -h & -m & -k & \Sigma_5 & -p \\ -e & -i & -n & -l & -p & \Sigma_6 \end{pmatrix}$$

By taking subdeterminants of the revised Λ matrix, sign conditions of the original Λ matrix should be violated. That is, connections that existed prior to the relabeling should be broken. This permutation causes 2×2 subdeterminant signs to change from negative values to zero when a connection is broken by Theorem 1. Table 3.1 is a list of some subdeterminantal signs for the original and altered networks.

Since connections not involving nodes 3 and 4 are unaffected by the permutation, the subdeterminantal signs do not change for such $(P; Q)$ in both Λ and $\Lambda^{3,4}$. For example, $det\Lambda(1,2;5,6)$ and $det\Lambda^{3,4}(1,2;5,6)$ are both negative because a connection exists in both networks. However, when a circular pair only includes one of the switched nodes, the sign of $det\Lambda^{3,4}(P;Q)$ is ambiguous when generalizing. One can say with certainty that when both nodes are elements of P or Q, the subdeterminantal sign is changed from negative to positive if a connection existed prior to the switch.

Table 3.1

$(P;Q)$	$det\Lambda(P;Q)$	$det\Lambda^{3,4}(P;Q)$
$(1,2;3,4)$	< 0	0
$(1,2;3,5)$	< 0	< 0
$(1,2;3,6)$	< 0	< 0
$(1,2;4,5)$	< 0	< 0
$(1,2;4,6)$	< 0	< 0
$(2,3;4,5)$	0	0
$(2,3;4,6)$	< 0	< 0
$(2,3;4,1)$	< 0	< 0
$(2,3;5,6)$	0	< 0
$(2,3;5,1)$	< 0	< 0
$(2,3;6,1)$	< 0	< 0
$(3,4;5,6)$	< 0	0
$(3,4;5,1)$	< 0	0
$(3,4;5,2)$	< 0	0
$(3,4;6,1)$	0	0
$(3,4;6,2)$	< 0	0
$(4,5;6,1)$	< 0	< 0
$(4,5;6,2)$	< 0	< 0
$(4,5;6,3)$	< 0	< 0
$(1,3;4,5)$	< 0	< 0
$(1,3;5,6)$	< 0	< 0
$(2,4;5,6)$	< 0	0
$(2,4;6,1)$	< 0	< 0
$(3,5;6,1)$	< 0	< 0
$(1,4;5,6)$	< 0	< 0

The most pertinent information in Table 3.1 involves the subdeterminants that switch from a negative value to zero. In other words, there was a connection before two nodes were switched and after the event that connection is broken. These pairs should be the first clue as to which nodes were switched. Of the pairs listed, $(1,2;3,4)$, $(3,4;5,6)$, $(3,4;5,1)$, $(3,4;5,2)$, $(3,4;6,2)$ and $(2,4;5,6)$ have such sign characteristics. After a brief investigation, one can see that $det\Lambda(3,4;5,6)$ and $det\Lambda^{3,4}(4,3;5,6)$ both have correct signs and thus boundary nodes 3 and 4 were switched.

4 Conclusion

When considering well-connected networks with permuted boundary nodes, it is possible to easily and systematically determine which nodes were switched. As the interval between the switched nodes increases, however, it becomes much more difficult to do so. The number of circular pairs that will lend useful information decreases, meaning that circular pairs of smaller size are needed to find any correct signs. Sometimes unfruitful, the method of taking the largest possible sized circular pair is beneficial because it does indeed cut down the amount of computation needed to find the permuted nodes.

This cannot be said of less-than-well-connected networks, however. The same connection principles do not apply to these networks so it is necessary to look at circular pairs of different sizes and examine which connections were broken due to a switch. There are certain networks in which it is impossible to tell if two particular nodes were switched. Such an example is any corner of a lattice network, consisting of two spikes joined at a common vertex. Some useful information can be gained when the signs of circular pairs are considered; one can see which group of nodes was involved in some switch and which was not. Practically, then, the presence of a broken conductor is somewhat detectable.

References

[1] Curtis, Edward B. and James A. Morrow. Inverse Problems for Electrical Networks. Singapore: World Scientific, 2000.