Eigenvalues and Eigenvectors for the Layered Square Lattice Networks

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Summer 2000

ABSTRACT.

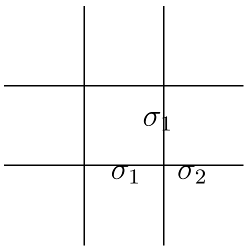
In this paper, the diagonalization of response matrices for layered square lattice networks is investigated. The utility of the eigenvalues is then investigated, revealing a nice relationship between the eigenvalues and the values of the conductances on layers.

Chapter 1

Introduction

1.1 Background

An n *edged* square lattice network Γ (sometimes called a tic-tac-toe network) is one which consists of n lines laid across n lines such that the result is a grid, similar to a tic-tac-toe grid, shown below.



The nodes of the network can be broken into two groups: *int* Γ and $\partial\Gamma$. *Int* Γ refers to the set of nodes within the interior of Γ, while $\partial\Gamma$ refers to the set of boundary nodes. Using the traditional Graph Theory notion of edge, σ is the function that assigns a conductivity to each edge in Γ. The networks we look at are considered to obey Kirchhoff's Law at each node $p \in$ *int* Γ. Kirchhoff's Law states that the sum of currents out of a node is 0.

$$\sum_{j=1}^{m} I_j = 0$$

where I_j refers to the current flow from node p into edge j. A function that obeys Kirchhoff's law is said to be γ-harmonic.

1.2 The Problem

In order to completely describe the problem, a few definitions are in order: First, we define the Kirchhoff Matrix. The Kirchhoff matrix is used to store all of the conductivities in the network.

The Kirchhoff Matrix, $\mathbf{K}$, is defined as the matrix of conductivities, σ_{ij}. The off-diagonal entries of $\mathbf{K}$, σ_{ij} = the conductivity on edge i-j. If nodes i and j are not adjacent, then σ_{ij} is defined to be zero. The diagonal entries, σ_{ii} are defined as the sum of the off diagonal entries:

$$\sigma_{ii} = \sum_{j=1}^{n} \sigma_{ij}$$

With that, we are ready to give a visual representation of the Kirchhoff matrix.

$$\mathbf{K} = \begin{pmatrix} \sum_{j=1}^{n} \sigma_{1j} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sum_{j=1}^{n} \sigma_{2j} & \dots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

Next, we give a suitable definition for the response matrix $\mathbf{\Lambda}$. It is not necessary for our purposes to know the actual entries of $\mathbf{\Lambda}$. For the record, the off-diagonal entries (λ_{ij}) of $\mathbf{\Lambda}$ are defined as the current at node i due to a voltage of 1 at node j and zero everywhere else on $\partial\Gamma$. For our purposes, we must think of the definition of $\mathbf{\Lambda}$ in terms of a linear map. The map $\mathbf{\Lambda}$: V $\longrightarrow$ C is the map from the space of possible boundary voltages, V to possible boundary currents, C.

With these definitions, it is possible to define some problems involving electrical networks. The first is what is known as the dirichlet problem. Simply stated, the dirichlet problem is
Given: $\vec{\varphi}$ on $\partial\Gamma$
Find: $\vec{u}$ such that $\vec{u} = \vec{\varphi}$ on $\partial\Gamma$ and $\vec{u}$ γ-harmonic on *int* Γ. James Morrow and Edward Curtis showed that if we partition $\mathbf{K}$ into blocks as shown $\begin{array}{|c|c|}\hline \mathbf{A} & \mathbf{B} \\ \hline \mathbf{B}^T & \mathbf{C} \\ \hline \end{array}$. Then the solution to the Dirichlet problem, $\vec{u}$, is given in block form as $\begin{array}{|c|}\hline \vec{\varphi} \\ \hline -\mathbf{C^{-1}B^T}\vec{\varphi} \\ \hline \end{array}$.

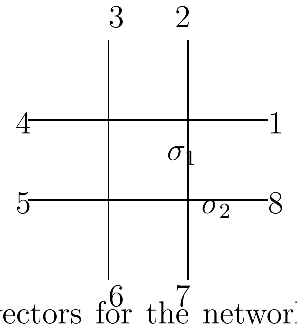
Another problem of interest, and one that we will be investigating in detail is what is known as the inverse problem. The inverse problem is stated very simply as
Given: $\mathbf{\Lambda}$
Find: $\mathbf{K}$
The solution to this problem is not nearly as simply found as the dirichlet problem. In the remainder of this paper, we will be investigating this inverse problem on square lattice networks, as defined earlier, and as shown below. Morrow and Curtis showed that we can express $\mathbf{\Lambda}$ in terms of the blocks of $\mathbf{K}$. Specifically,
$\mathbf{\Lambda} = \mathbf{A} - \mathbf{B}\mathbf{C}^{\mathbf{-1}}\mathbf{B}^{\mathbf{T}}$. In the inverse problem, however, this formula is of little help since it would be impossible to derive the blocks of $\mathbf{K}$ with only the response matrix.

Chapter 2

The simplest case:true tic-tac-toe network

2.1 Finding the eigenvectors

The first step in considering the networks is numbering. For the 2-spiked case, we will obey the numbering scheme shown below.



In order to fnd the eigenvectors for the network (which we call Γ), we must observe and make use of the symmetries unique to the network. Obviously, this includes the fact that all conductivities(σ_i) are equal on layers. The best way to do this is to impose a voltage of 1 at both nodes 1 and 2. Then, we rotate the nodes, and multiply by a number. In one case, we rotate through multiplying by -1. To illustrate this, then nodes 3 and 4 have voltages of -1, nodes 5 and 6 have voltages of 1, and nodes 7 and 8 have voltages of -1. To verify that this creates an eigenvector, we must recall the definition of the $\mathbf{\Lambda}$ map. This is the map from voltages to currents. In order for the vector $\mathbf{x} = (1, 1, -1, -1, 1, 1, -1, -1)$ to be an eigenvector, it must

satisfy the equation
$\mathbf{\Lambda x} = \xi \mathbf{x}$ for some constant ξ. In our case, this means that the resulting currents on $\partial\Gamma$ must be a multiple of $\mathbf{x}$. In order to verify this, some extrapolation is required. It is clear that if we let the voltage at node 9 be v, then the voltage at node 10 will just be -v, due to the way we rotated the picture when creating $\mathbf{x}$. Similarly, the voltages at nodes 11 and 12 will be v and -v, respectively. Next, we use the fact that node 10 must obey Kirchhoff's law.

$\sum_{j=1}^{4} I_j = 0$ We then obtain the equation$(v-1)2\sigma_2 + (v+v)2\sigma_1 = 0$. Solving for v yields

$$v = \frac{\sigma_2}{\sigma_2 + 2\sigma_1}$$

Next, we determine the value of the current at node 1, I_1. By Ohm's Law, it must be

$$I_1 = (1-v)\sigma_2 = (1 - \frac{\sigma_2}{\sigma_2 + 2\sigma_1})\sigma_2 = \frac{2\sigma_2\sigma_1}{\sigma_2 + 2\sigma_1}.$$

This value for I_1 is a multiple of the voltage value at node. Specifically, $1 * \frac{2\sigma_2\sigma_1}{\sigma_2+2\sigma_1} = I_1$. Verification of the other seven $\partial\Gamma$ nodes is similar, and the vector $\mathbf{x} = (1, 1, -1, -1, 1, 1, -1, -1)$ is an eigenvector with eigenvalue $\xi = \frac{2\sigma_2\sigma_1}{a+2\sigma_1}$. The other seven eigenvectors and eigenvalues are found through a similar process of rotation by a constant, usually -1 or the imaginary number i. To summarize the results,

$$\text{for } \xi_1 = 0, \text{ there is(are) eigenvector(s)}(1, 1, 1, 1, 1, 1, 1, 1)$$

$$\text{for } \xi_2 = \sigma_2, \text{ there is(are) eigenvector(s)}(1, -1, 0, 0, 0, 0, 0, 0), (0, 0, 1, -1, 0, 0, 0, 00$$
$$(0, 0, 0, 0, 1, -1, 0, 0), (0, 0, 0, 0, 0, 0, 1, -1)$$

$$\text{for } \xi_3 = \frac{\sigma_2\sigma_1}{\sigma_2 + \sigma_1}, \text{ there is(are) eigenvector(s)}(1, 1, i, i, -1, -1, -i, -i), (1, 1, -i, -i, -1, -1, i, i).$$

$$\text{for } \xi_4 = \frac{2\sigma_2\sigma_1}{\sigma_2 + 2\sigma_1}, \text{ there is(are) eigenvector(s)}(1, 1, -1, -1, 1, 1, -1, -1)$$

2.2 Expressing conductivities as eigenvalues

Using some results from linear algebra, we can summarize this all very nicely. From Linear Algebra, we know that two matrices, $\mathbf{A}$ and $\mathbf{B}$ are similar ($\mathbf{A} \sim \mathbf{B}$)if and only if $\exists$ some matrix $\mathbf{P}$ such that $\mathbf{A} = \mathbf{P^{-1}AP}$ A theorem in linear algebra says that the matrix with the eigenvalues of $\mathbf{A}$ on the diagonal and zeros everywhere else ($\mathbf{A_D}$)is similar to A. In this case, the matrix of eigenvectors, ($\mathbf{E}$) takes the place of $\mathbf{P}$. In other words
$\mathbf{A} = \mathbf{E^{-1}A_DE}$ Thus we can summarize our eigenvectors and eigenvalues above with one simple formula.

$$\mathbf{\Lambda} = \mathbf{E^{-1}\Lambda_DE} \text{ where } \mathbf{E} \text{ is the matrix of eigenvectors and}$$

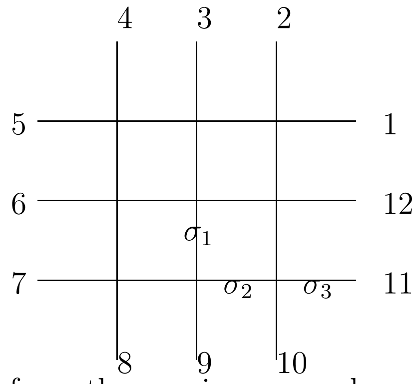
$\mathbf{\Lambda_D}$ is the matrix with eigenvalues on the diagonal. Since we know how to express the eigenvalues of Λ in terms of the conductivities, we can do the reverse and express the σ_i in terms of the eigenvalues. I have done that, and the results are shown below.

$$\sigma_2 = \xi_2 \qquad \sigma_1 = \frac{\xi_3}{1 - \frac{\xi_3}{\xi_2}} = \frac{\xi_3 \lambda_2}{\xi_2 - \xi_3}$$

Chapter 3

The next case up:3 Layered Network

In this chapter, I will avoid giving the steps and definitions required to find the eigenvectors. I will say that we label the network as shown and follow a similar method of rotation to find the eigenvectors.



We note that eigenvalues from the previous case show up again, as do eigenvectors of very similar form. That being said, let me reveal the eigenvalues and vectors for the three layered network:

$$\text{for } \xi_1 = 0, \text{ there is(are) eigenvector(s)}(1,1,1,1,1,1,1,1,1,1,1,1)$$

$$\text{for } \xi_2 = \sigma_3, \text{ there is(are) eigenvector(s)}(1,-1,0,0,0,0,0,0,0,0,0,0)$$
$$(0,0,0,1,-1,0,0,0,0,0,0,0)(0,0,0,0,0,0,1,-1,0,0,0,0)$$
$$(0,0,0,0,0,0,0,0,0,1,-1,0)$$

for $\xi_3 = \dfrac{\sigma_3\sigma_2}{\sigma_3+\sigma_2}$, there is(are) eigenvector(s)$(1,1,0,-1,-1,0,1,1,0,-1,-1,0)$

$$(1,1,0,i,i,0,-1,-1,0,-i,-i,0)(1,1,0,-i,-i,0,-1,-1,0,i,i,0)$$

for $\xi_4 = \dfrac{2\sigma_3\sigma_2+\sigma_3\sigma_1}{\sigma_3+2\sigma_2+\sigma_1}$, there is(are) eigenvector(s)$(0,0,1,0,0,i,0,0,-1,0,0,-i)$

$$(0,0,1,0,0,-i,0,0,-1,0,0,i)(0,0,1,0,0,-1,0,0,-1,0,0,1)$$

$$(0,0,-1,0,0,1,0,0,-1,0,0,1)$$

After some difficulty, the value of the conductivities can be obtained in terms of the eigenvalues

$$\sigma_3 = \xi_2 \qquad\qquad \sigma_2 = \frac{\xi_3}{1-\frac{\xi_3}{\xi_2}} = \frac{\xi_3\xi_2}{\xi_2-\xi_3}$$

$$\sigma_1 = \frac{\xi_4}{1-\frac{\xi_4}{\xi_2}} - \frac{2\xi_3}{1-\frac{\xi_3}{\xi_2}}$$

Chapter 4

Generalizing to the larger cases

Here, we will attempt to make an argument for the form of the eigenvalues and eigenvectors in the larger lattice networks, hopefully revealing an efficient way to solve the inverse problem. First, we reveal a new way of drawing Γ in hopes that it will reveal more of the symmetries. We will attempt to exploit those symmetries, and possibly achieve some results. The two previous chapters have given some feel for what the eigenvalues and eigenvectors will look like.

In order to get deep into the general problem, we must exploit some symmetries inherent to our picture. To do this, we introduce a group of operations that can be performed on Γ. This group will be referred to as $\mathbf{G}$. Any one of its components will be called $\mathbf{g}$. $\mathbf{G}$ has five components total. Specifically,

$$\mathbf{G} = \begin{cases} \mathbf{R} & \text{Rotating the picture by 90 degrees} \\ \mathbf{f_1} & \text{Flipping the picture over diagonally} \\ \mathbf{f_2} & \text{Flipping the picture over diagonally the other way} \\ \mathbf{f_3} & \text{Flipping the picture over horizontally} \\ \mathbf{f_4} & \text{Flipping the picture over vertically} \end{cases}$$

With these definitions, we are able to see a theorem that may help.

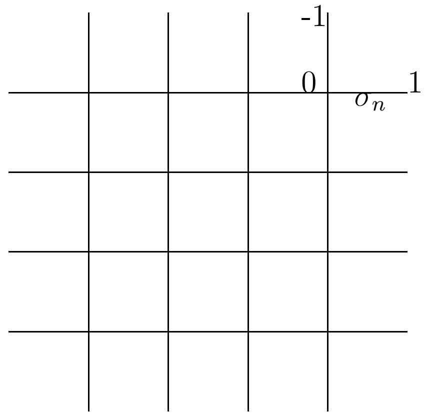
Theorem 1 : If $\vec{\varphi}$ is an eigenvector for $\mathbf{\Lambda}$, then $\mathbf{g}(\vec{\varphi})$ is also an eigenvector with the same eigenvalue.

Proof The proof of this theorem is quite simple to see, as any operation, $\mathbf{g}$ will simply have the effect of rotating the picture, maintaining the same symmetries, therefore any eigenvector remains in tact.

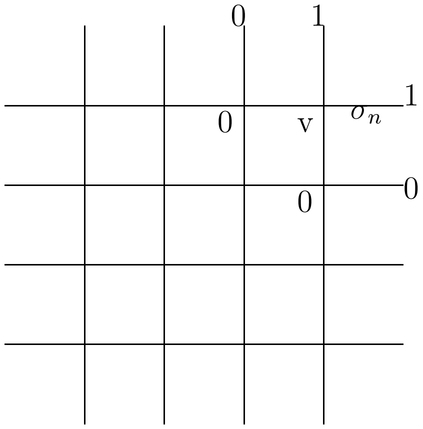
This theorem gives way to another interesting theorem.

Theorem 2 : $\mathbf{\Lambda g} = \mathbf{g\Lambda}$

Proof Let $\vec{\varphi}$ be an eigenvector of $\mathbf{\Lambda}$. Then, by definition of eigenvector:
$\mathbf{\Lambda}\vec{\varphi} = \lambda\vec{\varphi}$ By Theorem 1, we know that any operation will still give us an eigenvector, so
$\mathbf{\Lambda g}(\vec{\varphi}) = \lambda\mathbf{g}(\vec{\varphi}) = \mathbf{g}(\lambda\vec{\varphi}) = \mathbf{g}(\mathbf{\Lambda}\vec{\varphi})$
The result follows.
This theorem is actually somewhat useful because it allows us to use a theorem from linear algebra about commutative maps. That theorem states that if two maps,$\mathbf{A}$ and $\mathbf{B}$ are commutative(i.e. $\mathbf{AB} = \mathbf{BA}$), then they share the same eigenvectors.
Well, if we think of $\mathbf{g}$ as a map, then we can conclude that the eigenvectors of $\mathbf{\Lambda}$ are also going to be eigenvectors for $\mathbf{g}$. An eigenvector of $\mathbf{g}$ is easy to find. For example, the vector (1,1,-1,-1,1,1,-1,-1) is an eigenvector under several operations in $\mathbf{G}$ such as
$\mathbf{R}$(1,1,-1,-1,1,1,-1,-1) = (-1,-1,1,1,-1,-1,1,1) = -1*(1,1,-1,-1,1,1,-1,-1)
Another tool that we can make use of in the general case is an equation for the eigenvectors. We define the vector $\vec{\psi}$ to be the vector of boundary potentials. We define the vector $\vec{u}$ to be the vector of potentials on the next layer, so that each component of $\vec{\psi}$ matches up with its neighboring node on $\vec{u}$. If we assume that $\vec{\psi}$ is to be an eigenvector, then we can make the following statement:
$(\vec{\psi}$ - $\vec{u})\sigma_n = \lambda\vec{\psi}$. Solving this relation for $\vec{u}$ yields $\vec{u} = \vec{\psi}(1$ - $\frac{\lambda}{\sigma_n})$.
We can make use of this equation in a few ways. We first notice that $\lambda = 0$ will always be an eigenvalue of multiplicity 1. We notice that if we make $\lambda = \sigma_n$, then the value of $\vec{u}$ is $\vec{0}$. This is the case where there are potentials in the corners of Γ of 1 and -1 and zero everywhere else, as shown below.

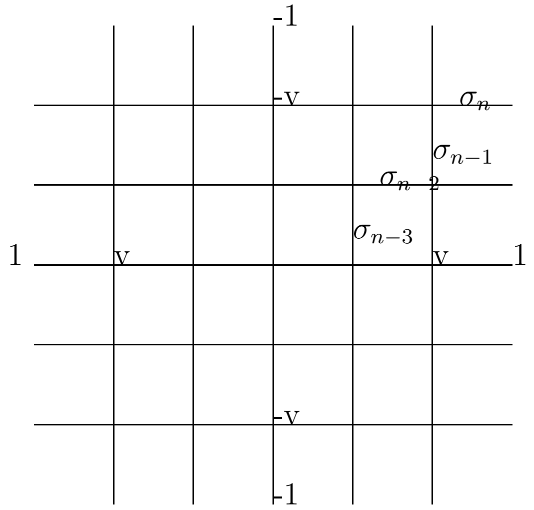


There are always going to be four eigenvectors for the eigenvalue $\lambda = \sigma_n$, regardless of the size of the network because all sqaure lattice networks have four corners.
Another eigenvalue that presents itself in every case is $\lambda = \frac{\sigma_n \sigma_{n-1}}{\sigma_n + \sigma_{n-1}}$. The picture that corresponds to this eigenvalue is shown below.



We are not sure of the multiplicity of the eigenvalue. In the n=2 case, it is of multiplicity 2. However, in the next case up it was of multiplicity 3. Yet another eigenvalue of interest is $\lambda = \frac{2\sigma_n \sigma_{n-1}}{\sigma_n + 2\sigma_{n-1}}$. It turns out that this eigenvalue occurs in all networks as well, but not as clearly as the last two eigenvalues. $\lambda = \frac{2\sigma_n \sigma_{n-1}}{\sigma_n + 2\sigma_{n-1}}$ occurs in all networks where n is an even number. If n is odd, then this eigenvalue adds the term $\lambda = \frac{2\sigma_n \sigma_{n-1} + \sigma_n \sigma_{n-2}}{\sigma_n + 2\sigma_{n-1} + \sigma_{n-2}}$. The picture corresponding to this network is shown on the following page.

The recurrence of these eigenvalues in every case seems to suggest a possible theorem about the eigenvalues.
Theorem3 If λ is an eigenvalue for the n = i case, then it is an eigenvalue for all cases where n $\geq$ i.
The proof of this theorem is unknown, but inspection of our eigenvalues found so far seems to suggest it.



Γ corresponding to $\lambda = \frac{2\sigma_n\sigma_{n-1}+\sigma_n\sigma_{n-2}}{\sigma_n+2\sigma_{n-1}+\sigma_{n-2}}$

References [1] Curtis, Edward B. and Morrow, James A. *Inverse Problems for Electrical Networks.* Singapore: World Scientific, 2000.