Discrete inverse problems for Schrödinger and Resistor networks

Richard Oberlin

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1 Abstract

Sylvester and Uhlmann related solutions of the conductivity equation to corresponding solutions of the Schrödinger equation. This allowed them to solve the inverse conductivity problem by translating it into an inverse Schrödinger problem, whose method of solution was known. This paper deals with the relationship between discrete analogs of the conductivity and Schrödinger equations.

2 Introduction

For each positive integer n, construct a square graph with boundary $\Gamma = (V, V_B, E)$ as follows. V is the set of vertices in the graph and consists of the integer lattice points (x, y) where $0 \leq x \leq n+1$ and $0 \leq y \leq n+1$ excluding the four corner points $(0,0)$, $(0, n+1)$, $(n+1, 0)$, and $(n+1, n+1)$. $V_B \subseteq V$ is the set of boundary vertices and consists of the vertices in V where x or y is equal to 0 or $n+1$. The interior vertices are denoted int$V$ and consist of $V - V_B$. E is the set of edges. Every interior vertex is connected by exactly one edge to each of the four vertices at unit distance from it. Every boundary vertex is connected by exactly one edge to the interior vertex unit distance away. These edges are the only edges in E. Given any two vertices p and q, if there is an edge in E connecting p and q we say that p neighbors q. Given a vertex p, $\mathcal{N}(p)$ is the set of all vertices q such that q neighbors p.

A conductivity network is a graph with boundary $\Gamma = (V, V_B, E)$ together with a positive real-valued function γ defined on V. A Schrödinger network is a graph with boundary $\Gamma = (V, V_B, E)$ together with a real valued function q defined on V.

The continuous conductivity equation with a positive conductivity gamma and a real valued potential u defined on a domain Υ is:

$$L_\gamma u = \mathrm{div}(\gamma \nabla u) = \gamma \Delta u + \nabla \gamma \cdot \nabla u = 0 \quad \text{in } \Upsilon.$$

The continuous Schrödinger equation with real valued q is:

$$S_q u = \Delta u - qu = 0 \quad \text{in } \Upsilon.$$

Take u, γ, and q to be defined on V. Choosing discrete representations of the Laplacian and dot product of gradients:

$$\Delta_d u(i) = \sum_{j \in \mathcal{N}(i)} u(j) - u(i)$$

$$\nabla_d \gamma \cdot \nabla_d u(i) = \sum_{j \in \mathcal{N}(i)} (u(j) - u(i))(\gamma(j) - \gamma(i))$$

we have discretizations of the conductivity and Schrödinger equations:

$$L_{\gamma_d} u(i) = \gamma(i)\Delta_d u(i) + \nabla_d \gamma \cdot \nabla_d u(i) = \sum_{j \in \mathcal{N}(i)} \gamma(j)(u(j) - u(i))$$

$$S_{q_d} u(i) = \Delta_d u(i) - q(i)u(i) = \left(\sum_{j \in \mathcal{N}(i)} u(j) - u(i) \right) - q(i)u(i).$$

In the continuous case if u is a solution to $L_\gamma u = 0$ then $w = \sqrt{\gamma} u$ is a solution to $S_q w = 0$ with $q = \frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}$ ([S-U]). Using the chosen discretizations if u is a solution to $L_{\gamma_d} = 0$ then $w = \gamma u$ is a solution to $S_{q_d} = 0$ with $q = \dfrac{\Delta_d \gamma}{\gamma}$.

Section one and two establish the basic properties of conductivity and Schrödinger networks. Section two also gives a method of solution for the inverse Schrödinger problem on square networks. Section three uses the solution of the inverse problem for the Schrödinger network to solve the inverse problem for the conductivity network.

Curtis and Morrow have done extensive work with a different discretization of the conductivity problem. Many of the properties established in this paper are adaptations of results in one of their early papers on the subject([C-M]).

3 Conductivity Networks

A function u is said to be γ-harmonic on (Γ, γ) if $L_{\gamma_d} u = 0$ for every interior vertex. A γ-harmonic function in the discrete case is analogous to a function which solves the Dirichlet problem with the conductivity equation in the continuous case. Conductivity networks can be thought of as an approximations of conductors; because of this, the values of γ-harmonic functions and their restrictions are often referred to as potentials.

Lemma 3.1 *Let $L_{\gamma_d} u(i) = 0$. Either $u(j) = u(i)$ for all $j \in \mathcal{N}(i)$ or there exist $j, k \in \mathcal{N}(i)$ such that $u(j) > u(i)$ and $u(k) < u(i)$.*

Proof. $L_{\gamma_d} u(i) = 0$ may be written:

$$\sum_{j \in \mathcal{N}(i)} \gamma_j u_j = u_i \left(\sum_{j \in \mathcal{N}(i)} \gamma_j \right)$$

Thus the value of u at i is the weighted average of the values at the neighboring vertices.□

Corollary 3.2 *Let u be a γ-harmonic function on a conductivity network (Γ, γ). Then the maximum and minimum values of u occur on the boundary of Γ.*

Corollary 3.3 *Let u be a γ-harmonic function on a conductivity network (Γ, γ) such that $u \mid_{V_B} = 0$. Then $u = 0$ on all vertices.*

Given a graph Γ with d vertices numbered $v_1 ... v_d$ construct the $d \times d$ matrix K_1 as follows.

(1) For $i \neq j$ $K_{1_{i,j}} = 1$ if $v_j \in \mathcal{N}(v_i)$ and $K_{1_{i,j}} = 0$ if $v_j \notin \mathcal{N}(v_i)$.

(2) $K_{1_{i,i}} = -\sum_{j : j \neq i} K_{1_{i,j}}$

Now, given a function defined on V we may identify it with a vector and by multiplying K_1 by the vector, we get the discrete Laplacian of the function. For example defining a column vector u such that $u_i = u(v_i)$ we have $\Delta_d u(v_i) = (K_1 u)_i$. Unless stated otherwise, functions and vectors will be treated interchangeably in this manner (with an understood ordering of vertices).

Given a row or column vector w with j entries, let I_w be the $j \times j$ diagonal matrix with $I_{w_{i,i}} = w_i$ and let $I_{\frac{1}{w}}$ be the $j \times j$ diagonal matrix with $I_{\frac{1}{w}\,i,i} = \frac{1}{w_i}$. Given a matrix M with j rows and k columns, and subsets A, B

of $\{1...j\}$ and $\{1...k\}$, let $M(A;B)$ be the submatrix consisting of the rows A and columns B of M. Given a vector v with j entries and a subset E of $\{1...j\}$ let $v(E)$ be the subvector of v consisting of the entries E.

Let (Γ, γ) be a conductivity network with k boundary vertices and a total of d vertices. Pick an ordering of the vertices with boundary nodes $v_1...v_k$ and interior nodes $v_{k+1}...v_d$. Let $N = \{1...k\}$ and $B = \{(k+1)...d\}$. Let $K_\gamma = (K_1 - I_q)I_\gamma$ where $q = I_{\frac{1}{\gamma}} K_1 \gamma$. Now:

$$(K_\gamma u)_i = \left(\sum_{j \in \mathcal{N}(i)} \gamma_j u_j - \gamma_i u_i \right) - \frac{\sum_{j \in \mathcal{N}(i)} \gamma_j - \gamma_i}{\gamma_i} \cdot \gamma_i u_i = \sum_{j \in \mathcal{N}(i)} \gamma_j (u_j - u_i) = L_{\gamma_d} u(i)$$

Divide the matrix K_γ into interior and boundary columns and rows, and divide the vector u into interior and boundary entries. If u is γ-harmonic the following holds:

$$\begin{bmatrix} K_\gamma(B;B) & K_\gamma(B;N) \\ K_\gamma(N;B) & K_\gamma(N;N) \end{bmatrix} \begin{bmatrix} u(B) \\ u(N) \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix}$$

or restated: $K_\gamma(N;N)u(N) = -K_\gamma(N;B)u(B)$.

Lemma 3.4 *Submatrix $K_\gamma(N;N)$ is nonsingular*

Proof. Submatrix $K_\gamma(N;N)$ has the following interpretation: given a vector of interior potentials g, $(K_\gamma(N;N)g)_i = L_{\gamma_d} u(i)$ where u is the function satisfying $u(B) = 0$ and $u(N) = g$. Thus, if $K_\gamma(N;N)g = 0$ then u is γ-harmonic, but by Corollary 3.3 this implies $g = 0$.□

Theorem 3.5 *Let (Γ, γ) be a conductivity network with boundary potential f. There exists a unique γ-harmonic function u such that $u \mid_{V_B} = f$.*

Proof. This follow immediately from Lemma 3.4 and the observation that the corresponding interior potential $g = -K_\gamma(N;N)^{-1} K_\gamma(N;B) f$.□

If we take our discrete Dirichlet data to be boundary potentials, and our discrete Neumann data to be $\sum_{j \in \mathcal{N}(i)} \gamma(j)(u(j) - u(i))$ at each boundary vertex i then we may define the discrete Dirichlet to Neumann Map Λ_γ in terms of K_γ:

$$\Lambda_\gamma f = (K_\gamma(B;B) - K_\gamma(B;N) K_\gamma(N;N)^{-1} K_\gamma(N;B)) f. \qquad (1)$$

The square graph with boundary has four faces: North, West, South, and East. Label the boundary vertices in counterclockwise order, starting

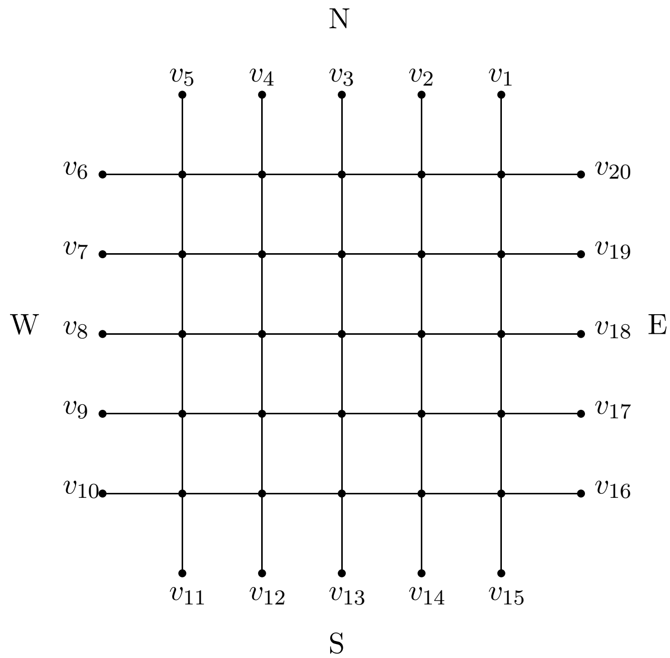

Figure 1:

with v_1 at the rightmost position of the North face and v_{4n} at the topmost position on the East face (the 5×5 graph is labeled as an example in Figure 1). The subsets N, W, S, E of $\{1...4n\}$ correspond to the sets of boundary nodes on the North, West, South, and East respectively.

Lemma 3.6 *Given a square conductivity network with boundary potential u defined on the North West and South Faces and Neumann data defined on the West face, there is a uniquely determined γ-harmonic extension of u to the boundary vertices on the East face and the interior vertices.*

Proof. Let column j be the column of interior vertices connected by edges to the West face. Let i be a vertex in column j connected by an edge to a boundary node on the West face b. The potential on i is determined by the potential on b, the value of the Neumann data on b, and the value of γ on i. Similarly all the potentials in column j are determined. Now the potentials and values of γ on the vertices neighboring i to the North West and South are known, the potential of i is known, and γ is known on the vertex neighboring i to the East, so there is only one choice for the potential of the vertex to the East satisfying $L_{\gamma_d} u = 0$ at i. Similarly the potentials of all the interior vertices in the column directly East of column j are determined. The potentials on the remaining vertices follow by induction.□

Corollary 3.7 *Let* (Γ, γ) *be a square conductivity network. Let* u *be a* γ*-harmonic function on* (Γ, γ) *which is* 0 *on the North West and South faces, with corresponding Neumann data which is* 0 *on the West face. For each remaining vertex* i*,* $u(i)$ *is also* 0*.*

Lemma 3.8 *The submatrix of* Λ_γ *consisting of the rows corresponding to the boundary vertices on the West face and the columns corresponding to the boundary vertices on the East face is nonsingular.*

Proof. The submatrix $\Lambda_\gamma(W; E)$ has the following interpretation: given a boundary potential u which is 0 on the North, West, and South faces, and equal to a function g on the East face, $\Lambda_\gamma(W; E)g$ is the resulting Neumann data on the West face from a γ-harmonic extension of u. By Corollary 3.7 if $\Lambda_\gamma(W; E)g = 0$ then $g = 0$.□

Corollary 3.9 *Given* Λ_γ *and a vector of potentials* u *defined on the North, West, and South faces and corresponding Neumann data* p *on the West face, there is a unique* γ*-harmonic extension of* u *to the East face.*

Proof. Let g be the boundary potential on the East face. $\Lambda_\gamma(W; N + W + S)u + \Lambda_\gamma(W; E)g = p$. Rewritten: $g = \Lambda_\gamma(W; E)^{-1}(p - \Lambda_\gamma(W; N + W + S)u)$.□

Theorem 3.10 *Given a square conductivity network* (Γ, γ) *we can recover* γ *on the boundary vertices and interior vertices adjacent to the boundary vertices.*

Proof. Let Λ_γ be the Dirichlet to Neumann map for an $n \times n$ conductivity network. Let v_j be a boundary vertex on the North face. By Corollary 3.9 there is a unique set of potentials on the East face, which together with a potential of 1 on vertex v_j and potentials of 0 on every other vertex of the North, West, and South faces, will give corresponding Neumann data of 0 on the West face. Using the same method presented in the proof of Lemma 3.6 we may determine that the potential of every interior vertex below the diagonal connecting v_j to v_{4n-j+1} is 0. For k such that $1 \leq k \leq 4n$ let i_k be the interior vertex connected by an edge to boundary vertex v_k. The region of interior nodes determined to have a potential of 0 includes i_j. Thus, we may calculate the value of γ at i_j using the Neumann data and the potential of 1 at v_j. Similarly, we may calculate the value of γ at every interior vertex connected to a boundary vertex. Take the value of γ at these vertices to be known; we may calculate their potentials using the Neumann data.

Assume $1 < j < n$. $L_\gamma = 0$ at i_j, and we know the potential on i_j and its four neighbors. The potential is the same at i_j and its neighbors to the South and West, we know the value of γ on the Eastern neighbor, and the potential of i_j is not equal to the potential of v_j, so we may determine the value of γ at v_j. Similarly, we may calculate the values of γ at every boundary vertex not in a corner. Take these values of γ to be known.

Assume $j = 2$. The values of γ are known at i_j and its neighbors to the North, West, and East. The potential of v_j is 1, the potential of i_j and its neighbors to the South and West are 0, and $L_{\gamma_d} = 0$ at i_j, so we may calculate the potential at its neighbor to the East, i_1. Call this potential p. Using the Dirichlet to Neumann map, we may find the potentials g on the East face, which together with potentials of 1 at v_j, p at v_1, and 0 at every other boundary vertex on the North, West, and South faces result in Neumann data of 0 on the West face:

$$g = -\Lambda_\gamma(W;E)^{-1}(\Lambda_\gamma(W;2) + p\Lambda_\gamma(W;1))$$

The potentials at i_j and its Neighbors to the North, West, and South remain the same, so the potential at i_1 is still p. Figure 2 illustrates the state of the network at this point, with interior potentials of 0 indicated by circles, and boundary potentials on the East face $\alpha ... \epsilon$. The potentials are known at i_1 and its four neighbors. The values of γ are known at vertices to the West and South of i_1 and the potential at the Neighbor to the North, v_1 is the same as the potential at i_1. Applying Lemma 3.1 at vertex i_2 we see that $p \neq 0$. Applying the Lemma at i_1 we see that the potential at $v_{4n} \neq p$. This information, together with the fact that $L_{\gamma_d} = 0$ at i_1 allows us to calculate the value of γ at the Eastern neighbor of i_1, v_{4n}. Through symmetrical arguments, we may calculate the values of γ at the remaining corner vertices.□

4 Schrödinger networks

Given a Schrödinger network (Γ, q) and a function u defined on its vertices, u is said to be q-harmonic if $S_{q_d}u(i) = 0$ for every interior vertex i. The values of q-harmonic functions and their restrictions are often referred to as potentials. A q-harmonic function is analogous to a function which solves

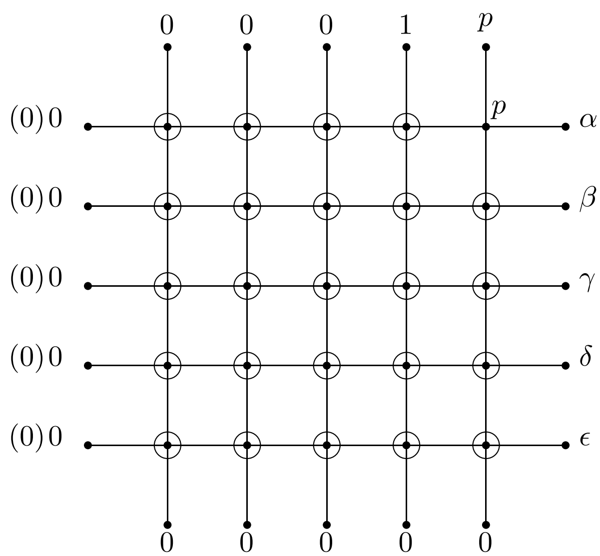

Figure 2:

the Dirichlet problem with the Schrödinger equation. Given a boundary function f, and Schrödinger network (Γ, q) it is not always true that there exists a unique q-harmonic function u with $u\,|_{V_B} = f$ unless we place certain restrictions on q.

Assign the same numbering of vertices to the Schrödinger network that we assigned previously to the conductivity network. Identify the function q with a vector, and let $H_q = K_1 - I_q$. Now, $(H_q u)_i = S_{q_d} u(i)$.

Lemma 4.1 *Given a Schrödinger network with q strictly positive on the interior, and a boundary potential f, there is a unique q-harmonic function u such that $u\,|_{V_B} = f$.*

Proof. With q strictly positive on the interior, submatrix $H_q(N; N)$ is diagonally dominant and thus invertible.□

Lemma 4.2 *Let (Γ, γ) be a conductivity network. Let (Γ, q) be the Schrödinger network with $q = \dfrac{\Delta_d \gamma}{\gamma}$. Given a boundary potential f, there is a unique q-harmonic function u with $u\,|_{V_B} = f$.*

Proof. Observe that

$$K_\gamma = H_q I_\gamma. \tag{2}$$

Noting that $I_\gamma(B;N) = 0$ we see that $K_\gamma(N;N) = H_q(N;N)I_\gamma(N;N)$. By Lemma 3.4 $K_\gamma(N;N)$ is invertible. $I_\gamma(N;N)$ is diagonal with strictly positive entries on the diagonal, and thus invertible, so $H_q(N;N)^{-1} = I_\gamma(N;N)K_\gamma(N;N)^{-1}$.□

Now, given a Schrödinger network satisfying the hypothesis of either of the two previous lemmas, we may construct a Dirichlet to Neumann map Ψ_q taking $\sum_{j\in\mathcal{N}(i)} u(j) - u(i)$ at each boundary vertex i to be our Neumann data:

$$\Psi_q f = (H_q(B;B) - I_q(B;B) - H_q(B;N)H_q(N;N)^{-1}H_q(N;B))f \tag{3}$$

The inverse problem is to recover q from Ψ_q and the geometry of the network.

Lemma 4.3 *Given a square Schrödinger network with boundary potential u defined on the North West and South Faces and Neumann data defined on the West face, there is a uniquely determined q-harmonic extension of u to the boundary vertices on the East face and the interior vertices.*

Proof. Let column j be the column of interior vertices connected by edges to the West face. The potential on each vertex in column j is determined by the Neumann data and the potentials on the West face. Let i be a vertex in column j. The potentials of the Neighboring vertices to the North West and South are known, the potential of i is known, and $q(i)$ is known, so there is only one choice for the potential of the vertex to the East satisfying $S_{q_d}u = 0$ at i. Similarly all the interior vertices in the column directly east of column j are determined. The potentials on the remaining vertices follow by induction.□

Corollary 4.4 *Let (Γ, q) be a square Schrödinger network. Let u be a q-harmonic function on (Γ, q) which is 0 on the North West and South faces, with corresponding Neumann data which is 0 on the West face. For each remaining vertex i, $u(i)$ is also 0.*

Lemma 4.5 *The submatrix of Ψ_q consisting of the rows corresponding to the boundary vertices on the West face and the columns corresponding to the boundary vertices on the East face is nonsingular.*

Proof. Making the appropriate substitutions, the proof is identical to that of Lemma 3.8.□

Corollary 4.6 *Given Ψ_q and a vector of potentials u defined on the North, West, and South faces and corresponding Neumann data p on the West face, there is a unique q-harmonic extension of u to the East face.*

Proof. Making the appropriate substitutions, the proof is identical to that of Corollary 3.9.□

Theorem 4.7 *Given a square Schrödinger network (Γ, q) we can recover q on the interior vertices.*

Proof. Let Ψ_q be the Dirichlet to Neumann map for an $n \times n$ square network. By Corollary 4.6 there is a unique set of potentials on the East face, which together with a potential of 1 on v_2 and 0 on the rest of the vertices of the North, West, and South faces extend to a q-harmonic function with Neumann data of 0 on the West face. Let vertex i be the interior vertex connected by edges to boundary vertices v_1 and v_{4n}. If the potential at a vertex p is 0 then the condition $S_{q_d}u(p) = 0$ becomes $\sum_{j \in \mathcal{N}(p)} u(j) - u(p) = 0$ and knowing the potentials at three of the neighboring vertices allows us to calculate the potential at the fourth without knowing $q(p)$. Thus, despite the fact that q is unknown on the interior vertices, we may use the same method presented in the proof of Lemma 4.3 to determine that the potential on every interior vertex except i is 0 and that the potential on vertex i is -1. The potentials on the East face may be calculated by inverting[1] $\Psi_q(W; E)$. Now, because the potential on i is nonzero, $q(i)$ may be calculated using the condition that $S_{q_d}u(i) = 0$ and the potentials on i and the four neighboring vertices.

For each boundary vertex v_k, let diagonal k be the diagonal extending from v_k to v_{4n-k+1}. For $1 < k < n$ if we know the values of q for each interior vertex on or above diagonal k, we may calculate the values of q on diagonal $k+1$. Let the potential on boundary v_{k+1} equal 1 and the potential on the rest of the boundary vertices of the North, West, and South faces equal 0. By inverting $\Psi_q(W; E)$ find the boundary potentials needed on the East face to give Neumann data of 0 on the West face. Using the boundary potentials, and Ψ_q, we may calculate the Neumann data on the North face. Using the

[1] The boundary potentials are easily seen to be 0 on $v_{3n+1} \ldots v_{4n-2}$ and 1 on v_{4n-1}. This yields a more efficient method of calculating the potential at v_{4n}, but it is omitted in the interest of brevity.

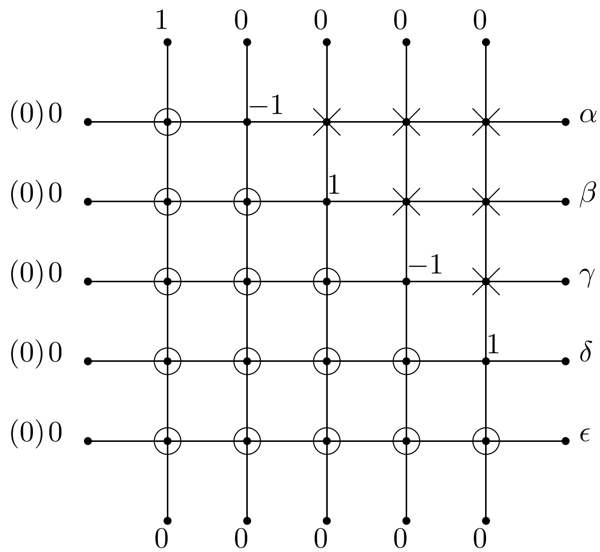

Figure 3:

Neumann data, the known values of q, and the boundary potentials on the North and East faces, we may calculate the potentials on the interior vertices above diagonal $k+1$. On interior vertices below diagonal $k+1$ the potentials are 0. On diagonal $k+1$ the potentials alternate between 1 and -1. All the potentials are known, and the potentials on diagonal $k+1$ are nonzero, thus q on this diagonal may be calculated using the fact that $S_{q_d} = 0$ on interior vertices. This is illustrated in Figure 3 for $k = 4$ and $n = 5$. Each known q is indicated by an $\times$. Interior potentials of 0 are indicated by circles.

For $k = n$, the same process is used, but the diagonal extends from the interior vertex adjacent to v_{k+1} to the interior vertex adjacent to v_{4n-k} instead of from v_{k+1} to v_{4n-k}.

By induction, we may calculate the values of q on or above the main diagonal. Using a symmetrical argument, the same process can be used to calculate the values of q below the main diagonal. □

5 Solution of the Conductivity Inverse Problem by use of the Schrödinger network

Let (Γ, γ) be a square conductivity network. Let (Γ, q) be the Schrödinger network with $q = \frac{\Delta_d \gamma}{\gamma}$. Using equations 1, 2, and 3 we have:

$$\Psi_q = \Lambda_\gamma I_\gamma(B; B)^{-1} - I_q(B; B).$$

In Section 3 we showed that given Λ_γ we can recover γ on the boundary vertices and the interior vertices adjacent to boundary vertices. This means we may calculate I_γ and I_q, and thus Ψ_q from Λ_γ. From Ψ_q we may recover q. Using q, γ on the boundary, and Lemma 4.2 we may recover γ on the interior.

References

[C-M] E. B. Curtis and J. A. Morrow, *The Dirichlet to Neumann map for a resistor network,* SIAM J. Appl. Math., 51 (1991) pp. 1011-1029.

[S-U] J. Sylvester and G. Uhlmann, *The Dirichlet to Neumann map and applications*