Notes on Schrödinger and conductivity networks with Tower of Hanoi graphs

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June 18, 2003

The even and odd Tower of Hanoi graphs are constructed and numbered as shown (for the 8 and 7 boundary node cases) in Figures 1 and 2.

Lemma 0.1 *Given Schrödinger and conductivity networks* (Γ, γ) *and* (Γ, q) *on a T.H. graph with* $2n$ *boundary nodes, the submatrices* $\Lambda_\gamma(1..n; n+1..2n)$ *and* $\Psi_q(1..n; n+1..2n)$ *are nonsingular.*

Sketch of Proof. Setting the potential and Neumann data on nodes $v_1..v_n$ equal to 0, processes of harmonic continuation give us potentials of 0 on nodes $v_{n+1}..v_{2n}$.$\square$

Corollary 0.2 *Given Schrödinger and conductivity networks* (Γ, γ) *and* (Γ, q) *on a T.H. graph with* $2n$ *boundary nodes, and potentials and Neumann data* g *and* p *on nodes* $v_1..v_n$*, there are unique* γ *and* q *harmonic functions* u *and* w *such that* $u\mid_{v_1..v_n} = w\mid_{v_1..v_n} = g$ *with corresponding Neumann data* p.

Lemma 0.3 *Given Schrödinger and conductivity networks* (Γ, γ) *and* (Γ, q) *on a T.H. graph with* $2n+1$ *boundary nodes, the submatrices* $\Lambda_\gamma(1..n; n+2..2n+1)$ *and* $\Psi_q(1..n; n+2..2n+1)$ *are nonsingular.*

Sketch of Proof. Setting the potential on nodes $v_1..v_{n+1}$ and Neumann data on nodes $v_1..v_n$ equal to 0 , processes of harmonic continuation give us potentials of 0 on nodes $(n+2..2n+1)$.$\square$

Corollary 0.4 *Given Schrödinger and conductivity networks* (Γ, γ) *and* (Γ, q) *on a T.H. graph with* $2n+1$ *boundary nodes, potentials* g *on nodes* $v_1..v_{n+1}$ *and Neumann data* p *on nodes* $v_1..v_n$*, there are unique* γ *and* q *harmonic functions* u *and* w *such that* $u\mid_{v_1..v_{n+1}} = w\mid_{v_1..v_{n+1}} = g$ *with corresponding Neumann data* p.

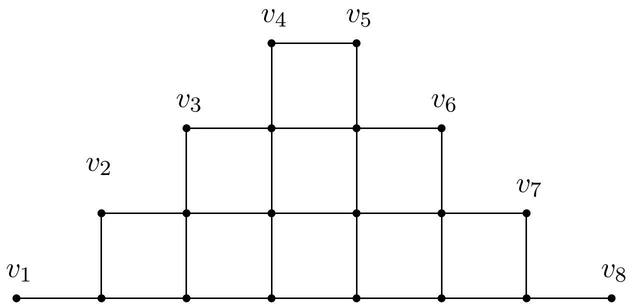

Figure 1:

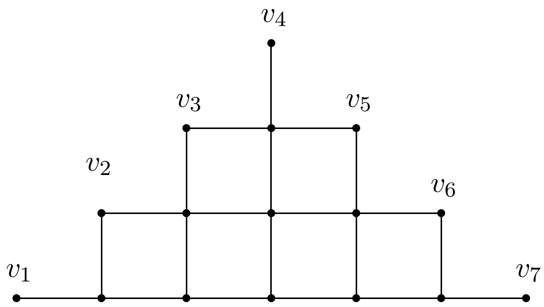

Figure 2:

1 Recovery of q

Let (Γ, q) be a Schrödinger network on a T.H. graph with $2n+1$ boundary nodes. Set the potentials on nodes $v_1..v_n$ equal to 0. Set the Neumann data equal to 0 on $v_1..v_{n-1}$ and 1 on v_n. The potentials below the diagonal extending to the South-East of the interior node neighboring node v_{n+1} are 0. The potentials on this diagonal alternate between 1 and -1. Let node i be a node on this diagonal. The neighbors of i either have potential of 0 or are boundary nodes. Arbitrarily picking the potential at node v_{n+1}, we may recover the potentials on the rest of the boundary nodes, by inverting $\Psi_q(1..n; n+2..2n+1)$. This allows us to recover the value of q at i. Similarly we may recover all the values of q on this diagonal.

Set the potential at node v_n equal to 1. Set the rest of the potentials and the Neumann data on nodes $v_1..v_n$ equal to 0. Potentials below the diagonal extending to the South-East of node v_n are 0. The potentials on this diagonal alternate between 1 and -1. Potentials above the diagonal (after picking arbitrary potential at v_{n+1}, and finding corresponding potentials on nodes $v_{n+2}..v_{2n+1}$) may be determined using the Neumann data, and the boundary potentials. Now we may recover q on this diagonal.

For the remaining diagonals a similar procedure is used, with the addition that the known q's must be used to determine potentials above the diagonal.

An analogous procedure is used in the even case.

2 Recovery of γ

Let (Γ, γ) be a conductivity network on a T.H. graph with $2n$ boundary nodes. Set the potentials on $v_1..v_n$ equal to 0, the Neumann data on v_n equal to 1, and the Neumann data on $v_1..v_{n-1}$ equal to 0. The potentials on the interior nodes on or below the diagonal extending to the South-East from v_n are 0. The potential on v_{n+1} is nonzero, and may be found by inverting $\Lambda_\gamma(1..n; n+1..2n)$. From this information we may calculate γ at v_{n+1}. Let i be the interior node neighboring v_{n+1} to the South. the potential at i, and its neighbors to the South and West is 0. The potential of i's neighbor to the east, v_{n+2} can be calculated, and is nonzero. As we know the value of γ on v_{n+1}, we may now calculate the value of γ at v_{n+2}. Similarly, we may calculate γ at $v_{n+1}..v_{2n}$. Using a symmetric argument, we may calculate γ at $v_1..v_n$.

Set the potentials at $v_1..v_{n-1}$ equal to 0, the potential at v_n equal to 1 and the Neumann data at $v_1..v_n$ equal to 0. Let j be the interior node

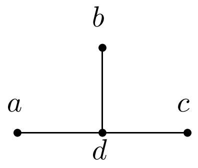

Figure 3:

neighboring v_n to the South. The potential at j is 0. Using the potential and Neumann data at v_n, and the potential and value of γ at v_{n+1} we may calculate the value of γ at j. Set the potentials at $v_1..v_{n-2}$ equal to 0, the potential at v_{n-1} equal to 1 and the Neumann data at $v_1..v_{n-1}$ equal to 0. Pick arbitrary potential and Neumann data at v_n and calculate the potentials on $v_{n+1}...v_{2n}$. From the Neumann data at v_n, the potentials at v_n and v_{n+1} and the values of γ at j and v_{n+1}, we may calculate the potential at j. Let k be the interior node neighboring v_{n-1} to the South. The potential at k is 0. From the potential at v_{n-1} and j, the Neumann data at v_{n-1}, and γ at j, we may calculate γ at k. Similarly we may calculate the values of γ on every interior node neighboring $v_1..v_n$. Using a symmetric argument, we may calculate the values of γ at the rest of the interior nodes neighboring boundary nodes.

Let (Γ, γ) be a conductivity network on a T.H. graph with $2n+1$ boundary nodes. Set potential on $v_1..v_n$ equal to 0, the potential on v_{n+1} equal to 1, and the Neumann data on $v_1..v_n$ equal to 0. Calculate the potentials on $v_{n+2}..v_{2n+1}$. Now let node i be the interior node neighboring v_{n+1} to the South. The potential at i is 0. The value of γ at i can be calculated using the Neumann data and potential at v_{n+1}. The calculation of γ at the remaining interior nodes neighboring boundary nodes proceeds similarly to the even case.

Conjecture 2.1 *For a conductivity network* (Γ, γ) *on a Tower of Hanoi graph with an odd number of boundary nodes, the value of* γ *on the boundary is not determined by* Λ_γ*.*

The simplest nontrivial odd T.H. graph has three boundary nodes (see Figure 3). Assigning conductivities a to v_1 b to v_2 c to v_3 and d to the interior node, the response matrix is easily calculated.

$$\Lambda_\gamma = \begin{bmatrix} \frac{da}{a+b+c} - d & \frac{db}{a+b+c} & \frac{dc}{a+b+c} \\ \frac{da}{a+b+c} & \frac{db}{a+b+c} - d & \frac{dc}{a+b+c} \\ \frac{da}{a+b+c} & \frac{db}{a+b+c} & \frac{dc}{a+b+c} - d \end{bmatrix}$$

Clearly, if we scale a b and c by a common value, Λ_γ will be unaffected. Thus, a recovery of a b and c from Λ_γ is impossible (although we can recover their relative magnitudes).