Notes on Schrödinger and conductivity networks with Tower of Hanoi graphs

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The even and odd Tower of Hanoi graphs are constructed and numbered as shown (for the 8 and 7 boundary node cases) in Figures 1 and 2.

Lemma 0.1 Given Schrödinger and conductivity networks \((\Gamma, \gamma)\) and \((\Gamma, q)\) on a T.H. graph with \(2n\) boundary nodes, the submatrices \(\Lambda_\gamma(1..n; n+1..2n)\) and \(\Psi_q(1..n; n+1..2n)\) are nonsingular.

Sketch of Proof. Setting the potential and Neumann data on nodes \(v_1..v_n\) equal to 0, processes of harmonic continuation give us potentials of 0 on nodes \(v_{n+1}..v_{2n}\).

Corollary 0.2 Given Schrödinger and conductivity networks \((\Gamma, \gamma)\) and \((\Gamma, q)\) on a T.H. graph with \(2n\) boundary nodes, and potentials and Neumann data \(g\) and \(p\) on nodes \(v_1..v_n\), there are unique \(\gamma\) and \(q\) harmonic functions \(u\) and \(w\) such that \(u\) \(|_{v_1..v_n} = w\) \(|_{v_1..v_n} = g\) with corresponding Neumann data \(p\).

Lemma 0.3 Given Schrödinger and conductivity networks \((\Gamma, \gamma)\) and \((\Gamma, q)\) on a T.H. graph with \(2n+1\) boundary nodes, the submatrices \(\Lambda_\gamma(1..n; n+2..2n+1)\) and \(\Psi_q(1..n; n+2..2n+1)\) are nonsingular.

Sketch of Proof. Setting the potential on nodes \(v_1..v_{n+1}\) and Neumann data on nodes \(v_1..v_n\) equal to 0, processes of harmonic continuation give us potentials of 0 on nodes \((n + 2..2n + 1)\).

Corollary 0.4 Given Schrödinger and conductivity networks \((\Gamma, \gamma)\) and \((\Gamma, q)\) on a T.H. graph with \(2n+1\) boundary nodes, potentials \(g\) on nodes \(v_1..v_{n+1}\) and Neumann data \(p\) on nodes \(v_1..v_n\), there are unique \(\gamma\) and \(q\) harmonic functions \(u\) and \(w\) such that \(u\) \(|_{v_1..v_{n+1}} = w\) \(|_{v_1..v_{n+1}} = g\) with corresponding Neumann data \(p\).
Figure 1:

Figure 2:
1 Recovery of $q$

Let $(\Gamma, q)$ be a Schrödinger network on a T.H. graph with $2n + 1$ boundary nodes. Set the potentials on nodes $v_1..v_n$ equal to 0. Set the Neumann data equal to 0 on $v_1..v_{n-1}$ and 1 on $v_n$. The potentials below the diagonal extending to the South-East of the interior node neighboring node $v_{n+1}$ are 0. The potentials on this diagonal alternate between 1 and $-1$. Let node $i$ be a node on this diagonal. The neighbors of $i$ either have potential of 0 or are boundary nodes. Arbitrarily picking the potential at node $v_{n+1}$, we may recover the potentials on the rest of the boundary nodes, by inverting $\Psi_q(1...n; n+2..2n+1)$. This allows us to recover the value of $q$ at $i$. Similarly we may recover all the values of $q$ on this diagonal.

Set the potential at node $v_n$ equal to 1. Set the rest of the potentials and the Neumann data on nodes $v_1..v_n$ equal to 0. Potentials below the diagonal extending to the South-East of node $v_n$ are 0. The potential on $v_n+1$ is nonzero, and may be found by inverting $\Lambda_q(1..n; n+1..2n)$. From this information we may calculate $\gamma$ at $v_{n+1}$. Let $i$ be the interior node neighboring $v_{n+1}$ to the South. the potential at $i$, and its neighbors to the South and West is 0. The potential of $i$’s neighbor to the east, $v_{n+2}$ can be calculated, and is nonzero. As we know the value of $\gamma$ on $v_{n+1}$, we may now calculate the value of $\gamma$ at $v_{n+2}$. Similarly, we may calculate $\gamma$ at $v_{n+1}..v_{2n}$. Using a symmetric argument, we may calculate $\gamma$ at $v_1..v_n$.

Set the potentials at $v_1..v_{n-1}$ equal to 0, the potential at $v_n$ equal to 1 and the Neumann data at $v_1..v_n$ equal to 0. Let $j$ be the interior node

2 Recovery of $\gamma$

Let $(\Gamma, \gamma)$ be a conductivity network on a T.H. graph with $2n$ boundary nodes. Set the potentials on $v_1..v_n$ equal to 0, the Neumann data on $v_n$ equal to 1, and the Neumann data on $v_1..v_{n-1}$ equal to 0. The potentials on the interior nodes on or below the diagonal extending to the South-East from $v_n$ are 0. The potential on $v_{n+1}$ is nonzero, and may be found by inverting $\Lambda_\gamma(1..n; n+1..2n)$. From this information we may calculate $\gamma$ at $v_{n+1}$. Let $i$ be the interior node neighboring $v_{n+1}$ to the South. the potential at $i$, and its neighbors to the South and West is 0. The potential of $i$’s neighbor to the east, $v_{n+2}$ can be calculated, and is nonzero. As we know the value of $\gamma$ on $v_{n+1}$, we may now calculate the value of $\gamma$ at $v_{n+2}$. Similarly, we may calculate $\gamma$ at $v_{n+1}..v_{2n}$. Using a symmetric argument, we may calculate $\gamma$ at $v_1..v_n$. 

Set the potentials at $v_1..v_{n-1}$ equal to 0, the potential at $v_n$ equal to 1 and the Neumann data at $v_1..v_n$ equal to 0. Let $j$ be the interior node
neighboring \( v_n \) to the South. The potential at \( j \) is 0. Using the potential and Neumann data at \( v_n \), and the potential and value of \( \gamma \) at \( v_{n+1} \) we may calculate the value of \( \gamma \) at \( j \). Set the potentials at \( v_1..v_{n-2} \) equal to 0, the potential at \( v_{n-1} \) equal to 1 and the Neumann data at \( v_1..v_{n-1} \) equal to 0. Pick arbitrary potential and Neumann data at \( v_n \) and calculate the potentials on \( v_{n+1}..v_{2n} \). From the Neumann data at \( v_n \), the potentials at \( v_n \) and \( v_{n+1} \) and the values of \( \gamma \) at \( j \) and \( v_{n+1} \), we may calculate the potential at \( j \). Let \( k \) be the interior node neighboring \( v_{n-1} \) to the South. The potential at \( k \) is 0. From the potential at \( v_{n-1} \) and \( j \), the Neumann data at \( v_{n-1} \), and \( \gamma \) at \( j \), we may calculate \( \gamma \) at \( k \). Similarly we may calculate the values of \( \gamma \) on every interior node neighboring \( v_1..v_n \). Using a symmetric argument, we may calculate the values of \( \gamma \) at the rest of the interior nodes neighboring boundary nodes.

Let \((\Gamma, \gamma)\) be a conductivity network on a T.H. graph with \( 2n+1 \) boundary nodes. Set potential on \( v_1..v_n \) equal to 0, the potential on \( v_{n+1} \) equal to 1, and the Neumann data on \( v_1..v_n \) equal to 0. Calculate the potentials on \( v_{n+2}..v_{2n+1} \). Now let node \( i \) be the interior node neighboring \( v_{n+1} \) to the South. The potential at \( i \) is 0. The value of \( \gamma \) at \( i \) can be calculated using the Neumann data and potential at \( v_{n+1} \). The calculation of \( \gamma \) at the remaining interior nodes neighboring boundary nodes proceeds similarly to the even case.

**Conjecture 2.1** For a conductivity network \((\Gamma, \gamma)\) on a Tower of Hanoi graph with an odd number of boundary nodes, the value of \( \gamma \) on the boundary is not determined by \( \Lambda_\gamma \).

The simplest nontrivial odd T.H. graph has three boundary nodes (see Figure 3). Assigning conductivities \( a \) to \( v_1 \) \( b \) to \( v_2 \) \( c \) to \( v_3 \) and \( d \) to the interior node, the response matrix is easily calculated.

\[
\Lambda_\gamma = \begin{bmatrix}
\frac{da}{a+b+c} & -d & \frac{db}{a+b+c} & -d \\
\frac{da}{a+b+c} & \frac{dc}{a+b+c} & \frac{da}{a+b+c} \frac{dc}{a+b+c} & \frac{dc}{a+b+c} - d
\end{bmatrix}
\]
Clearly, if we scale \( a \), \( b \), and \( c \) by a common value, \( \Lambda_\gamma \) will be unaffected. Thus, a recovery of \( a \), \( b \), and \( c \) from \( \Lambda_\gamma \) is impossible (although we can recover their relative magnitudes).