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DETERMINING CURRENT SOURCES IN A NETWORK

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Abstract. We consider the problem of determining the current sources in a network with known resistances given boundary data. Some restrictions on the values of the sources are generally required for them to be uniquely determined by the boundary data. For certain small networks, an assumption on the spread of the possible values of the sources is sufficient, while for our results which extend to arbitrarily large networks of a certain type, the sources are required to all have the same size. The uniqueness results in this paper which allow there to be multiple sources in a network each require the resistances on each edge to be one, but if we make the assumption that the network contains only one source, the resistances may take on arbitrary values, and only a partial set of boundary data is required to uniquely determine the location of the source. We also briefly explore the properties of the Green's function, which could potentially be employed in algorithms to locate the sources. Results about the general behavior of the Green's function have been difficult to come by, and in some specific cases the Green's function behaves in a surprising fashion, but we conclude by offering conjectures about the Green's function which have held in each of the special cases we have examined and which are analagous to known results about the Green's function in the continuous case.

1. Introduction

For an electrical network with vertex set Γ, boundary $\partial\Gamma$, interior $\text{int}\Gamma = \Gamma \setminus \partial\Gamma$ and with conductivities γ_{pq} on the edge connecting two adjacent points $p, q \in \Gamma$, denote the voltage at any point $p \in \Gamma$ by $v(p)$. Ohm's Law states that, for any edge pq, the current flowing from p to q is $I_{pq} = \gamma_{pq}(v(p) - v(q))$. Where for any point p the set of neighbors of p is denoted $\mathcal{N}(p)$, we may then define the net current at p as

$$I(p) = \sum_{q \in \mathcal{N}(p)} \gamma_{pq}(v(p) - v(q)) \tag{1}$$

In most problems, it is assumed that, for $p \in \partial\Gamma$, $v(p)$ and $I(p)$ are directly measurable quantities, while for $p \in int\Gamma$ the relation $I(p) = 0$, known as Kirchhoff's Law, holds. In the present paper, we consider problems where Kirchhoff's Law is not known to hold at each point in the interior. Ordering the nodes in such a way that the boundary nodes precede the interior nodes, and denoting the voltages and currents on the boundary by, respectively, u_∂ and I_∂ and those on the interior by u_{int} and I_{int}, we have, where

$$K = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$$

is the Kirchhoff matrix,

$$\begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \begin{bmatrix} u_\partial \\ u_{int} \end{bmatrix} = \begin{bmatrix} I_\partial \\ I_{int} \end{bmatrix} \tag{2}$$

Thus $I_\partial = Au_\partial + Bu_{int}$ and $u_{int} = C^{-1}(I_{int} + B^t u_\partial)$, demonstrating that, even when Kirchhoff's Law does not hold, the forward problem of determining the boundary currents and interior voltages given the boundary voltages and interior currents always has a unique solution. This paper considers the inverse problem of finding the interior currents (assumed to be nonnegative and, when positive, referred to as current sources) given boundary measurements. Most of the paper is dedicated to finding sufficient conditions on the values the current sources and in some cases the conductivities can take for the current sources to be uniquely determined by boundary measurements in various types of networks. To prove uniqueness, it is assumed that two networks, Γ and Σ, with identical boundary measurements are given. We then consider the difference network $\Delta = \Gamma - \Sigma$, in which the net current at any point is the difference of the net currents at that point in Γ and Σ, *i.e.*, $I_\Delta = I_\Gamma - I_\Sigma$. So, where the subscripts refer to the network, Kirchhoff's equations in Δ, Γ, and Σ give

$$Ku_\Delta = I_\Gamma - I_\Sigma = Ku_\Gamma - Ku_\Sigma \tag{3}$$

Thus $K[u_\Delta - (u_\Gamma - u_\Sigma)] = 0$. Now any Kirchhoff matrix K has the property that $Kv = 0$ if and only if v is a constant vector. By assumption, $u_\Gamma - u_\Sigma$ is zero on the boundary, so this implies that u_Δ is constant on the boundary; since the voltages in a network are always undetermined up to an additive constant, we may then assume that the boundary voltages in Δ are all zero. Since Γ and Σ have the same boundary currents, the boundary currents in Δ are all zero. Uniqueness of sources may then be shown by proving that a network with boundary voltages and currents zero must have net current zero at each interior node.

2. Uniqueness of multiple sources in 3, 4, and 5-hexagon networks

We prove here that, subject to certain restrictions on the number or on the relative sizes of the current sources in 3, 4, or 5-hexagon networks, the distribution of the sources in the network is uniquely determined by boundary data. We begin with the three-hexagon case. As we will see, the same basic method applies to the proofs for the larger networks, but the mathematical details become more involved, and require us to restrict our attention to networks with identical resistances along each edge.

Uniqueness theorem 1 (3-hexagon networks)**.** *Suppose Γ and Σ are two 3-hexagon networks with the same boundary voltages and boundary currents and no sinks anywhere. Suppose also that it is known that either*

1. *With known arbitrary resistors, Γ and Σ have at most five sources each,* or
2. *Each resistor has the same magnitude, and the ratio of the size of largest source in the difference network of Γ and Σ to that of the smallest source is strictly smaller than six.*

Then the current sources in Γ and Σ are identical.

Proof. Let Δ be the difference network of Γ and Σ, that is, Δ has sources where Γ has sources and sinks where Σ has sources. Then the voltages and currents on the boundary of Δ will be zero, as will the net current at each boundary node. From Kirchhoff's Law at boundary nodes 2, 5, and 8, the voltages at the interior nodes denoted by asterisks (*) (see figure 1) are zero. Knowing this, applying Kirchhoff's Law to nodes 3, 6, and 9, the voltages at the interior nodes denoted by pound signs (#) are zero. Thus the voltage is zero at every node except possibly the node in the center.

Suppose there is a source or a sink of value I at the center node; if it is a sink, we may take $\Delta = \Sigma - \Gamma$ to make it a source, so we may assume without loss of generality that I is positive. Then current flows from the center to at least one of the surrounding nodes, each of which has voltage zero, so the voltage, v, at the center is positive. But then the voltage at the center is greater than each of the surrounding voltages, so current flows from the center into all 6 of the surrounding nodes. Since no current flows in any of the other edges which are connected to these

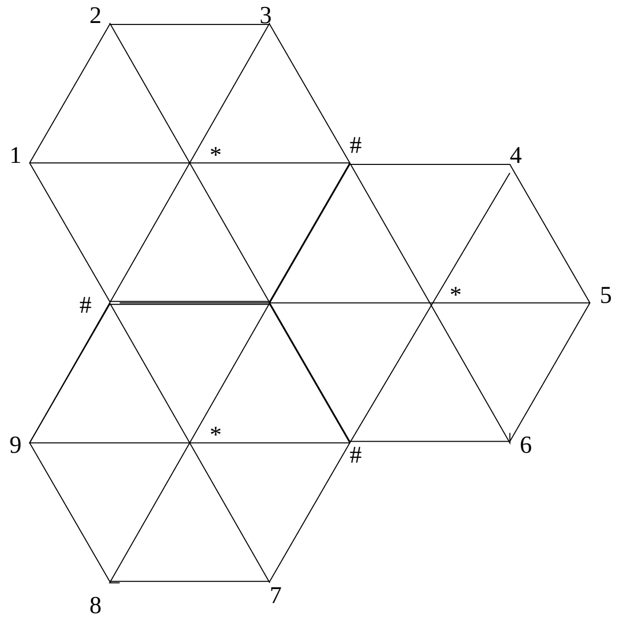

FIGURE 1. Labelling of vertices in the three-hexagon network

surrounding nodes, this implies that each of these nodes is a sink. There are six such nodes, so there must then be at least 6 sources in Σ, contradicting the first assumption in the theorem.

Meanwhile, if all resistances are the same, there will be current $I/6$ flowing into each of the surrounding nodes, so each of these nodes is a sink of magnitude $I/6$, while the center node is a source of magnitude I. This contradicts the second assumption.

Thus, under either of the two assumptions, there must be no source at the center node. Applying Kirchhoff's Law at the center node, we see that the voltage there must be zero. Hence the voltage at every point in Δ is zero, so no current flows in any edge of the graph, and there may be no sources in Δ, implying that all the sources in Γ and Σ are identical. $\square$

Uniqueness theorem 2 (4-hexagon networks)**.** *Let Γ and Σ be two four-hexagon networks with the same boundary voltages and boundary currents and no sinks anywhere, and each having unit conductance on each edge. Suppose also that it is known that the ratio of the size of the largest current source in the difference graph to that of the smallest source is strictly smaller than five. Then the current sources in Γ and Σ are identical.*

Proof. As in the previous proof, let $\Delta = \Gamma - \Sigma$, so that the voltages and currents on $\partial\Delta$, as well as the net currents at each boundary node, are all zero. With the nodes of Δ numbered as in Figure 2, Kirchhoff's Law at nodes 2 and 7 shows that the voltages at nodes A and E are zero. With this information, applying Kirchhoff's Law to nodes 1,3,6, and 8 shows that the voltages at B, D, F, and H are all zero, and then Kirchhoff's Law at nodes 4 and 9 shows that the voltages at C and G are zero. So the voltages at every point in the graph except possibly J and K are zero. Kirchhoff's Law at nodes J and K then yield

$$I(J) = 6v(J) - v(K) \qquad I(K) = 6v(K) - v(J) \tag{4}$$

This implies

$$v(J) = \frac{1}{35}(6I(J) + I(K)) \qquad v(K) = \frac{1}{35}(I(J) + 6I(K)) \tag{5}$$

Now nodes A through H all have zero voltage, and J is the only node with nonzero voltage neighboring any of A, B, or H, and hence is the only node into which current from any of these

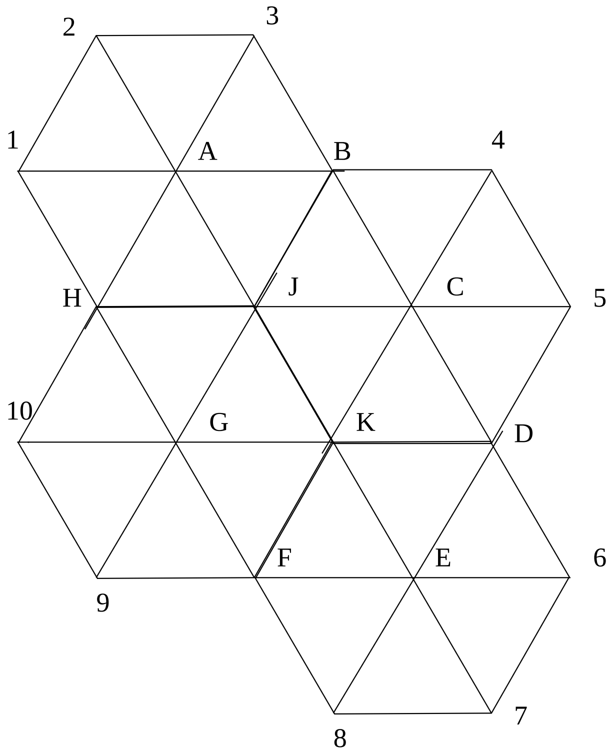

FIGURE 2. Labelling of vertices in the four-hexagon network

nodes may flow. So since the current along any edge XY is just $I(XY) = v(X) - v(Y)$, the currents at A, B, and H are given by

$$I(A) = I(B) = I(H) = -v(J) = -\frac{1}{35}(6I(J) + I(K)) \tag{6}$$

Likewise, since the only node with nonzero voltage neighboring nodes D, E, or F is K, we have

$$I(D) = I(E) = I(F) = -v(K) = -\frac{1}{35}(I(J) + 6I(K)) \tag{7}$$

It hence follows that

$$\min\{|I(A)|, |I(D)|\} \leq \frac{1}{5} max\{|I(J)|, |I(K)|\} \tag{8}$$

So if either $I(J)$ or $I(K)$ is nonzero, the ratio of the magnitude of the largest current source in Δ to that of the smallest is at least five, contradicting our assumption. So both $I(J)$ and $I(K)$ are zero, implying that

$$v(J) = v(K) = 0 \tag{9}$$

Then every voltage in Δ is zero, so no current flows in Δ, so Δ must have no current sources. Hence the current sources in Γ and Σ are identical. $\square$

Uniqueness theorem 3 (5-hexagon networks)**.** *Let Γ and Σ be two five-hexagon networks with the same boundary voltages and boundary currents and no sinks anywhere, and each having unit conductance on each edge. Suppose also that it is known that the ratio of the magnitude of the largest current source in the difference graph to that of the smallest source is strictly smaller than $\frac{34}{7}$. Then the sources in Γ and Σ are identical.*

Proof: The proof is quite similar to the proof for the 4-hexagon network. Where $\Delta = \Gamma - \Sigma$, Kirchhoff's Law first at nodes 2 and 6 (see Figure 3), then at 1,3,5, and 7, then at nodes 4, 8, and

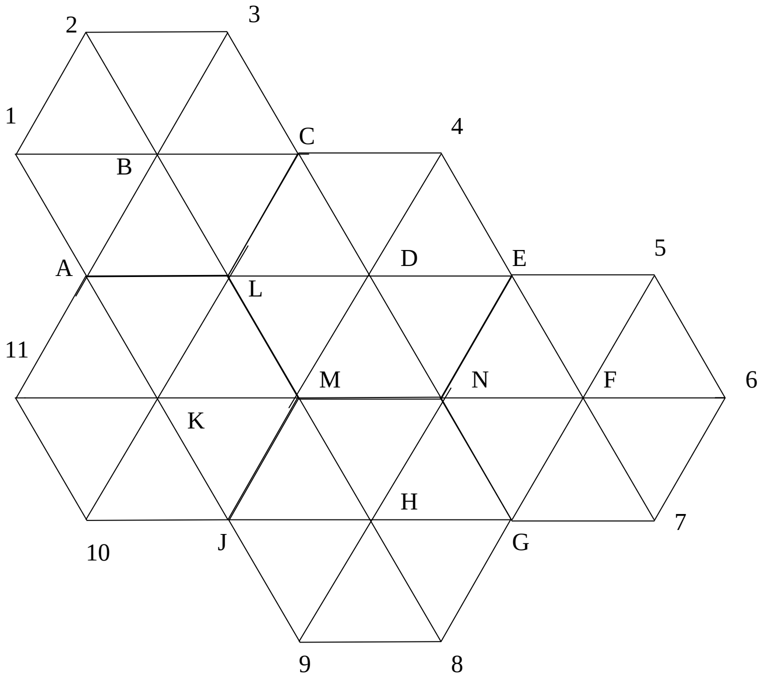

FIGURE 3. Labelling of vertices in the five-hexagon network

11, and finally at node 10 of Δ establish that the voltage is zero at nodes B, F, A, C, E, G, D, H,K, and J, respectively. So nodes L, M, and N are the only nodes at which the voltage might be nonzero, and no current flows on any edge connecting two of nodes A through K. We then see that

$$v(L) = -I(B) \qquad v(M) = -I(J) \qquad v(N) = -I(F) \tag{10}$$

Denote $I_1 = I(B)$, $I_2 = I(J)$, and $I_3 = I(F)$. We find then that the currents at nodes A through H are as follows:

$$I(A) = I(B) = I(C) = I_1 \quad I(D) = I_1 + I_2 + I_3 \quad I(E) = I(F) = I(G) = I_3 \tag{11}$$

$$I(H) = I_2 + I_3 \qquad I(J) = I_2 \qquad I(K) = I_1 + I_2 \tag{12}$$

Kirchhoff's Law at nodes L, M, and N then yields

$$I(L) = -6I_1 + I_2 \qquad I(M) = I_1 - 6I_2 + I_3 \qquad I(N) = I_2 - 6I_3 \tag{13}$$

Solving for I_1, I_2, and I_3 then yields

$$I_1 = -\frac{1}{204}(35I(L) + 6I(M) + I(N)) \qquad I_2 = -\frac{1}{34}(I(L) + 6I(M) + I(N)) \tag{14}$$

$$I_3 = -\frac{1}{204}(I(L) + 6I(M) + 35I(N)) \tag{15}$$

We see from this that

$$\min\{|I_1|, |I_2|, |I_3|\} \leq \frac{7}{34}\max\{|I(L)|, |I(M)|, |I(N)|\} \tag{16}$$

So if any of $I(L)$, $I(M)$, or $I(N)$ is nonzero, the ratio of the magnitude of the largest current source in Δ to that of the smallest current source is at least $\frac{34}{7}$, contradicting our assumption. So each of $I(L)$, $I(M)$, and $I(N)$ is zero, implying $I_1 = I_2 = I_3 = 0$. We've seen that each current source in Δ is a linear combination of I_1, I_2, and I_3, so each current source in Δ is zero. Thus the current sources in Γ and Σ are identical. $\square$

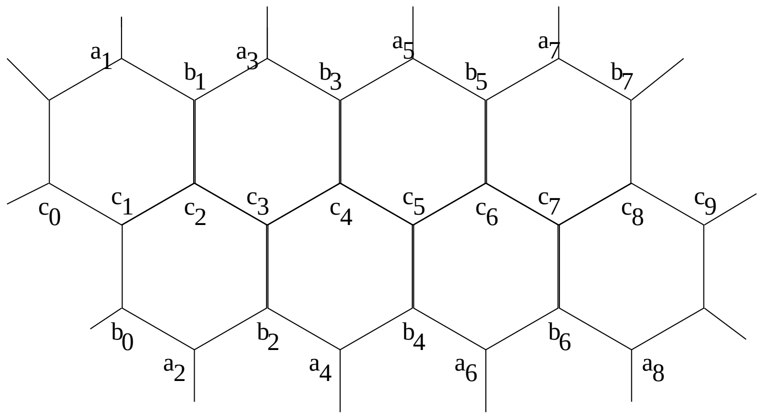

FIGURE 4. A spiked hexagonal network with 8 hexagons in two rows

The hypotheses of each of the above uniqueness theorems contained a requirement that the ratio of the largest source to the smallest source be smaller than a certain number, r. This translates to a requirement on the spread of the possible values of the sources in the original network; if S is the subset of real numbers of which each of the possible magnitudes of the sources in the original network must be a member, the ratio of the largest value in S to the smallest difference between any two members of S must be smaller than r.

In each case above, the proof of uniqueness depended on the fact that all but relatively few nodes in the difference network may immediately be shown to have voltage zero as a consequence of the boundary conditions. As this does not remain true for similar but larger networks, this result is difficult to generalize, at least using the above approach. Similar methods may be used to show uniqueness for small networks of different shapes, but for larger networks, other approaches must be used, and these approaches require assuming that all sources in the network have the same magnitude.

3. Uniqueness of multiple unit sources in spiked and unspiked hexagonal networks

3.1. **Spiked Networks.** We show here that in a network of hexagons which are arranged in two rows, such as in Figure 4, if all the conductances and sources are assumed to have value one, the locations of the sources are uniquely determined by the values of the boundary voltages and currents. To do this, we will consider the difference network Δ of two such networks with identical boundary measurements. The boundary voltages and currents in Δ will then all be zero, and the current at each interior point will be -1, 0, or 1. Where the nodes are denoted as in Figure 4 (so, if the number of hexagons in the network is N, we have nodes $a_1, \ldots, a_N, b_0, \ldots, b_{N-1}, c_0, \ldots, c_{N+1}$), the voltages at all the nodes a_i and also at b_0, b_{N-1}, c_0, c_{N+1} and at all the unlabeled nodes are zero. Uniqueness will be proven if we can show all other voltages to be zero. We will first show that they must all take on values -1, 0, or 1.

Lemma 1. *Every voltage in Δ is either -1, 0, or 1.*

Proof. The bulk of the proof proceeds by induction on the subscripts of the points b_i and c_j. Our inductive hypothesis is as follows:
For each $i < n$ and $j < n-1$, $v(b_i), v(c_j) \in \{-1, 0, 1\}$
Now the claim holds trivially for $n = 1$ (since, as mentioned above, $v(b_0) = 0$). For $n = 2$, the only nodes in question are b_1 and c_0. But $v(c_0) = 0$, and by Kirchhoff's law at a_1, $v(b_1) = -I(a_1) \in \{-1, 0, 1\}$. Thus the claim holds for $n = 2$. Likewise, by Kirchhoff's Law at nodes c_0 and a_2 we have $v(c_1) = -I(c_0) \in \{-1, 0, 1\}$ and $v(b_2) = -I(a_2) \in \{-1, 0, 1\}$, so the claim holds for $n = 3$.
Now suppose the inductive hypothesis holds for some n, where $n \geq 3$. We must show $v(b_n), v(c_{n-1}) \in \{-1, 0, 1\}$. Since

$$I(a_n) = -v(b_{n-2}) - v(b_n) \in \{-1, 0, 1\} \tag{17}$$

and by the inductive hypothesis $v(b_{n-2}) \in \{-1,0,1\}$, we must have $v(b_n) \in \{-2,-1,0,1,2\}$. Suppose, to get a contradiction, that $v(b_n) = 2$. By equation 17 this requires $I(a_n) = -1$ and $v(b_{n-2}) = -1$. Kirchhoff's Law at b_{n-2} then gives

$$I(b_{n-2}) = 3v(b_{n-2}) - v(c_{n-1}) = -3 - v(c_{n-1}) \in \{-1,0,1\} \tag{18}$$

This implies

$$-4 \leq v(c_{n-1}) \leq 2 \tag{19}$$

Applying Kirchhoff's Law at c_{n-1}, in combination with equation 19 and the fact that, by the inductive hypothesis, $v(b_{n-2}), v(c_{n-2}) \in \{-1,0,1\}$, we have

$$-1 \leq I(c_{n-1}) = 3v(c_{n-1}) - v(b_{n-2}) - v(c_{n-2}) - v(c_n) \leq -6+1+1-v(c_n) \tag{20}$$

so that

$$v(c_n) \leq -3 \tag{21}$$

Meanwhile, since $v(a_n) = v(a_{n+2}) = 0$ and by assumption $v(b_n) = 2$, Kirchhoff's Law at b_n gives

$$-1 \leq I(b_n) = 3v(b_n) - v(c_{n+1}) = 6 - v(c_{n+1}) \leq 1 \tag{22}$$

so that

$$5 \leq v(c_{n+1}) \leq 7 \tag{23}$$

By the inductive hypothesis, $v(b_{n-1}) \in \{-1,0,1\}$, so by equations 19, 21, and 23, Kirchhoff's Law at c_n yields

$$I(c_n) = 3v(c_n) - v(c_{n-1}) - v(c_{n+1}) - v(b_{n-1}) \leq -9+4-5+1 = -9 \tag{24}$$

This contradicts the fact that $I(c_n)$ must not have magnitude greater than 1. Hence $v(b_n) \neq 2$. Negating every number in the above argument shows that $v(b_n) \neq -2$. Thus $v(b_n) \in \{-1,0,1\}$. It remains to show that $v(c_{n-1}) \in \{-1,0,1\}$. Now Kirchhoff's Law at node c_{n-1} gives

$$v(c_n) = 3v(c_{n-1}) - v(b_{n-2}) - v(c_{n-2}) - I(c_{n-1}) \tag{25}$$

But $v(b_{n-2}), v(c_{n-2}), I(c_{n-1}) \in \{-1,0,1\}$, so by the reverse triangle inequality this implies

$$|v(c_n)| \geq 3|v(c_{n-1})| - 3 \tag{26}$$

Likewise, Kirchhoff's Law at node c_n gives

$$v(c_{n+1}) = 3v(c_n) - v(c_{n-1}) - I(c_n) - v(b_n) \tag{27}$$

So since $I(c_n), v(b_n) \in \{-1,0,1\}$,

$$|v(c_{n+1})| \geq |3v(c_n) - v(c_{n-1})| - 2 \tag{28}$$

But by equation 26 and the reverse triangle inequality,

$$|3v(c_n) - v(c_{n-1})| \geq 8|v(c_{n-1})| - 9 \tag{29}$$

So equation 28 yields

$$|v(c_{n+1})| \geq 8|v(c_{n-1})| - 11 \tag{30}$$

By Kirchhoff's Law at node b_n,

$$I(b_n) = 3v(b_n) - v(c_{n+1}) \tag{31}$$

Now $I(b_n), v(b_n) \in \{-1,0,1\}$, so this implies $|v(c_{n+1})| \leq 4$. Hence by equation 30,

$$8|v(c_{n-1})| - 11 \leq 4 \tag{32}$$

Since $v(c_{n-1})$ is an integer (this may be seen by applying Kirchhoff's Law at node b_{n-2}), this implies that $v(c_{n-1}) \in \{-1,0,1\}$. Thus the inductive hypothesis holds for $n+1$.

This establishes that the voltages at nodes $b_1, \ldots, b_{N-2}$ and at $c_0, \ldots, c_{N-3}$ are all -1, 0, or 1. The lemma wil be proven when we've shown that this holds for the remaining nodes c_i. A quick examination of the argument that led to equation 30 shows that, since all the nodes b_i have voltage -1, 0, or 1, we have, for all i,

$$|v(c_{i+2})| \geq 8|v(c_i)| - 11 \qquad |v(c_{i-2})| \geq 8|v(c_i)| - 11 \tag{33}$$

(We note both these relations since for some i only one of c_{i-2} and c_{i+2} exists.) Also, for any j Kirchhoff's Law at node b_{j-1} shows that the voltage at c_j is an integer no larger in magnitude than four. Equation 33 then implies that $v(c_i) \in \{-1,0,1\}$. □

We now prove that every voltage in Δ is in fact zero, which establishes that no currents flow in Δ, so there are no sources in the difference network of two networks with identical boundary measurements, implying that any two networks with identical boundary measurements have the same sources.

Uniqueness theorem 4 (Spiked hexagonal networks)**.** *Every voltage in Δ is zero.*

Proof. As noted in the opening paragraph of this section, the voltages at all nodes other than $b_1, \ldots, b_{N-2}$ and $c_1, \ldots, c_N$ are already known to be zero. For all i, Kirchhoff's Law at node b_i gives

$$I(b_i) = 3v(b_i) - v(c_{i+1}) \tag{34}$$

But $I(b_i), v(b_i), v(c_{i+1}) \in \{-1, 0, 1\}$, so the only way the above equation can hold is if $v(b_i) = 0$. Knowing this, for $1 \leq j \leq N$, Kirchhoff's Law at c_j gives

$$I(c_j) = 3v(c_j) - v(c_{j-1}) - v(c_{j+1}) \tag{35}$$

For $j = 1$, since $v(c_0) = 0$ and all voltages and currents are 1, 0, or -1, equation 35 requires $v(c_1) = 0$. Then if, for some j, $v(c_{j-1}) = 0$, again equation 35 implies $v(c_j) = 0$. Hence by induction $v(c_j) = 0$ for all j from 1 to N. Thus every voltage in Δ is zero, and uniqueness is proved. ☐

Remark 1. *Uniqueness fails, in general, if we allow the sources to take on different values. For example, if $I(c_1) = I(c_N) = 2$, $I(c_2) = \cdots = I(c_{N-1}) = 1$, and $I(c_0) = I(c_{N+1}) = I(b_0) = \cdots = I(b_{N-1}) = -1$, all boundary measurements are zero (the voltages are 1 at $c_1, \ldots, c_N$ and zero elsewhere), so a network with sources of value 2 at c_1 and c_N and of value 1 at $c_2, \ldots, c_{N-1}$ cannot be distinguished from a network with sources of value 1 at c_0, c_{N+1} and each of the b_i.*

Remark 2. *Uniqueness also tends to fail in more complicated spiked hexagonal networks, even when all sources are restricted to having unit size. For example, in the network shown in figure 5, the same boundary measurements result if unit sources are placed at points B, C, G, H, J, and K as result if the sources are placed at points 3,4,A, D, E, and F. Note that this network is the simplest spiked hexagonal network in which the hexagons are not arranged in two rows, and in fact a network of this form (call it Γ) is embedded as a subnetwork in any other spiked hexagonal network Σ which is at least two hexagons wide everywhere and in which the hexagons are not arranged in two rows. Since the measurements on $\partial\Gamma$ do not suffice to determine the sources on the interior of Γ, one would expect that measurements on $\partial\Sigma$ would not suffice to determine the sources on int Γ, so since the interior of Γ is contained in the interior of Σ, some sources in Σ would be left undetermined by the measurements on $\partial\Sigma$. As such, one suspects that the only spiked hexagonal networks which are at least two hexagons wide everywhere in which the boundary measurements determine the locations of the sources are those in which the hexagons are arranged in two rows, that is, precisely those networks for which uniqueness has been proven in Theorem 4. To show that this reasoning is valid, we prove the following lemma.*

Lemma 2. *Suppose Γ is a network with nonempty boundary which is embedded as a subnetwork in a network Σ in such a way that $\text{int}\Gamma \subset \text{int}\Sigma$, $\Sigma \setminus \text{int}\Gamma$ is connected, and no point in $\Sigma \setminus \Gamma$ neighbors a point in $\text{int}\Gamma$. Let sources in Γ be arranged so that, when Γ is isolated from the rest of the large network Σ, the voltages and currents on $\partial\Gamma$ are zero, and let there be no sources in $\Sigma \setminus \text{int}\Gamma$. Then all the currents and voltages on $\partial\Sigma$ are zero.*

Proof. Let p be any node in $\Omega = \Sigma \setminus \text{int}\Gamma$ such that $v(p) \geq v(q)$ for all $q \in \Omega$ (such a node exists since Ω is finite). Since there are no sources in Ω (in particular, there is no source at p), Kirchhoff's Law requires that each neighbor of p have voltage $v(p)$ (this is true even if $p \in \partial\Gamma$, since we know the net current from p into all of Σ and the net current from p into $\text{int}\Gamma$ are both zero, so the net current from p into $\Omega = \Sigma \setminus \text{int}\Gamma$ is also zero; for points in $\Sigma \setminus \Gamma = \Omega \setminus \partial\Gamma$, the current into $\text{int}\Gamma$ is not an issue since such points do not have neighbors in $\text{int}\Gamma$.). If p_0 is a node which maximizes v on Ω, let r be any point in Ω. By the hypothesis, there is a path $p_0, p_1, \ldots, p_n = r$ in Ω. Now p_0 maximizes v on Ω, and if p_i maximizes v on Ω then since p_{i+1} is a neighbor of p_i, p_{i+1} maximizes v on Ω. So by induction $p_n = r$ maximizes v on Ω. As r was an arbitrary point in Ω, the voltage is hence constant on Ω. By the assumption on the distribution of the sources, v is zero on $\partial\Gamma$, so v is zero throughout $\Omega = \Sigma \setminus \text{int}\Gamma$; in particular the voltage is zero on $\partial\Sigma$. ☐

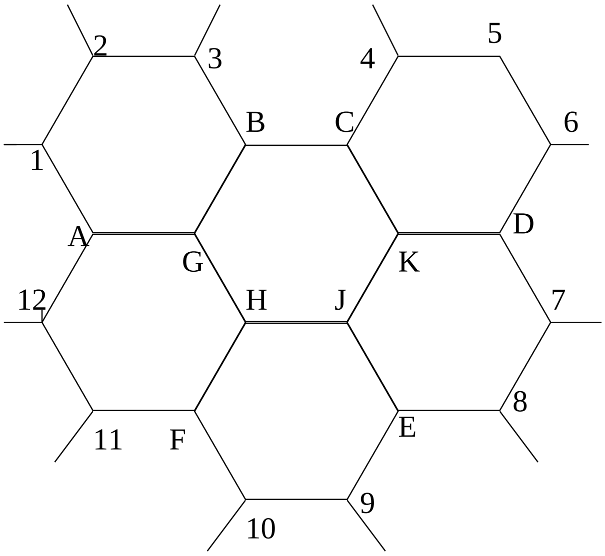

FIGURE 5. A relatively simple spiked network in which uniqueness fails

So if Γ is any network, such as that in figure 5, such that there exists a nontrivial arrangement of unit sources and sinks in intΓ giving zero voltages and currents on $\partial\Gamma$, and Σ is a larger network which has Γ as a subnetwork in such a way that int$\Gamma \subset$ intΣ, $\Sigma \setminus$ intΓ is connected, and no node in $\Sigma \setminus \Gamma$ neighbors a node in intΓ, this arrangement of sources in Γ, along with sources zero in $\Sigma \setminus \Gamma$, will give voltages and currents of zero on $\partial\Sigma$. Viewing Γ and Σ as difference networks, this shows that if sources are not uniquely determined by boundary measurements in a smaller network, and this smaller network is contained in a larger network in the sense described in Lemma 2, sources in the larger network are also not determined by boundary measurements. As mentioned in remark 2, the network shown in figure 5 is contained, in the sense of Lemma 2, in any spiked hexagonal network which is at least two hexagons wide everywhere in which the hexagons are not all arranged in two rows. This, combined with Theorem 4, gives the following characterization of spiked hexagonal networks in which unit sources are uniquely determined by boundary measurements.

Corollary 5. *If all the sources in a spiked hexagonal network with unit resistors are assumed to have value one, and the network is at least two hexagons wide everywhere, the sources are uniquely determined by boundary measurements if and only if the hexagons are arranged in only two rows.*

3.2. **Unspiked Networks.** We now turn to networks similar to those discussed in the previous section, but with no spikes; the boundary nodes will be precisely those nodes which belong to only one hexagon. In the spiked case, the boundary data gave us only the voltage at these nodes (using Ohm's Law), so since in the unspiked case the boundary data consists of both the voltage and the currents at these nodes, one would expect the boundary data to determine the location of unit sources for a larger class of unspiked networks than spiked ones. This is indeed the case, as is shown by the following theorem.

Uniqueness theorem 6 (Unspiked hexagonal networks)**.** *In an unspiked hexagonal network with unit resistors which is exactly two hexagons wide everywhere in which every source has value one, the locations of the sources are uniquely determined by boundary measurements.*

Proof. As in the proof of Theorem 4, we consider the difference network Δ between two networks with identical boundary measurements, so the boundary voltages and boundary currents on Δ will be zero. We label the nodes of Δ as in figure 4 (note that even if the network has "bends" as in figure 5, the same labelling system may be used). Then $c_0, c_{N+1}, b_0, b_{N-1}$, and $a_1, \ldots, a_N$ are all boundary nodes and hence have current and voltage zero. Now no current flows on boundary-to-boundary edges, so since b_1 is the only interior node connected to a_1 and $I(a_1) = v(a_1) = 0$, we have $v(b_1) = 0$. Likewise b_2 is the only interior node connected to the boundary node a_2, so $v(b_2) = 0$. For $3 \leq i \leq N-1$, the node a_i is connected only to the interior nodes b_{i-2} and b_i, so we have

$$0 = I(a_i) = v(b_{i-2}) + v(b_i) \tag{36}$$

Using the fact that $v(b_1) = v(b_2) = 0$, it readily follows by induction that $v(b_i) = 0$ for all i.
We now turn to the nodes c_i. For $1 \leq i \leq N$, c_i has neighbors b_{i-1} (which we've shown to have voltage zero), c_{i-1}, and c_{i+1}, so

$$I(c_i) = 3v(c_i) - v(c_{i-1}) - v(c_{i+1}) \tag{37}$$

For all j, Kirchhoff's Law at node b_{j+1} gives

$$I(b_{j+1}) = -v(c_j) \in \{-1, 0, 1\} \tag{38}$$

Hence each term in equation 37 has value 1, 0, or -1. But then for any i, if $v(c_{i-1}) = 0$, for equation 37 to be satisfied we must have $v(c_i) = 0$. Since c_0 is a boundary node, $v(c_0) = 0$, so it follows by induction that $v(c_i) = 0$ for all i. We have thus shown that each node in Δ has voltage zero, so in the difference network between any two networks with identical boundary measurements, no current flows. Hence the sources in the network are uniquely determined by boundary measurements. $\square$

Remark 3. *Just as in the spiked case, uniqueness fails if we allow the sources to take on different values. To show this, we use a slight modification of the arrangement in remark 1. Assuming* $N \geq 4$, *setting* $I(c_2) = I(c_{N-1}) = 2$, $I(c_3) = \cdots = I(c_{N-2}) = 1$, $I(c_1) = I(c_N) = I(b_1) = \cdots = I(b_{N-2}) = -1$ *and all other net currents zero (so that the voltages are 1 wherever there is a source and zero elsewhere), we see that the boundary measurements are zero. Thus if the sources are allowed to take values in a 2-to-1 ratio, they are not uniquely determined by boundary measurements. For* $N = 3$, *if the sources are in a 2-to-1 ratio they are uniquely determined, but if a source of value 3 is located at the center node* c_2 *and sinks of value -1 are at each of its neighbors, the boundary measurements are again zero, so even in this rather small network, if the sources are allowed to vary signifcantly from each other, they are not determined by boundary measurements.*

Remark 4. *Again, as in the spiked case, once the network becomes more complicated than the networks covered by the uniqueness theorem, uniqueness begins to break down. In the network in figure 6, if sources of value 1 are placed at nodes* $G, H, J, K, L,$ *and* M *and sinks of value -1 are placed at* $A, B, C, D, E,$ *and* F, *the boundary measurements are zero, so uniqueness does not hold in this network. This network is contained as a subnetwork in any unspiked hexagonal network of width everywhere at least two which contains a column of three hexagons, each of which has hexagons both to its left and to its right. Lemma 2 may hence be used to show that uniqueness does not hold in any such network.*

4. Uniqueness of multiple unit sources in triangular and towers of Hanoi networks

4.1. **Triangular networks.** We consider triangular networks Σ_N in which the nodes are arranged in N concentric hexagons around a center node p_0, such as the network in figure 7, and in which the conductance on each edge is one. The hexagons are numbered $1, \ldots, N$ going out from the center, and hexagon k will have nodes (k, r) where $0 \leq r \leq 6k-1$. We adopt the cyclic labeling convention $(k, r) = (k, r + 6k)$, and in general any node labelled $(0, r)$ will refer to node p_0. For $1 \leq k \leq N-1$ and $0 \leq l \leq 5$, the neighbors of node (k, lk) are $(k, lk-1)$,$(k, lk+1)$,$(k-1, l(k-1))$,$(k+1, l(k+1))$,$(k+1, l(k+1)-1)$, and $(k+1, l(k+1)+1)$, and for $1 \leq m \leq k-1$ the neighbors of node $(k, lk+m)$ are $(k, lk+m-1)$,$(k, lk+m+1)$,$(k-1, l(k-1)+m-1)$, $(k-1, l(k-1)+m+1)$, $(k+1, l(k+1)+m)$, and $(k+1, l(k+1)+m+1)$. Note in particular that if $k \leq N-1$, each node in hexagon k has at least two neighbors in hexagon $k+1$.

Lemma 3. *Suppose the voltages on the boundary hexagon* ($k = N$) *of* Σ_N *are all integers, and suppose it is known that every source in* Σ_N *has integer value. Then the voltage at any node inside* Σ_N *is an integer.*

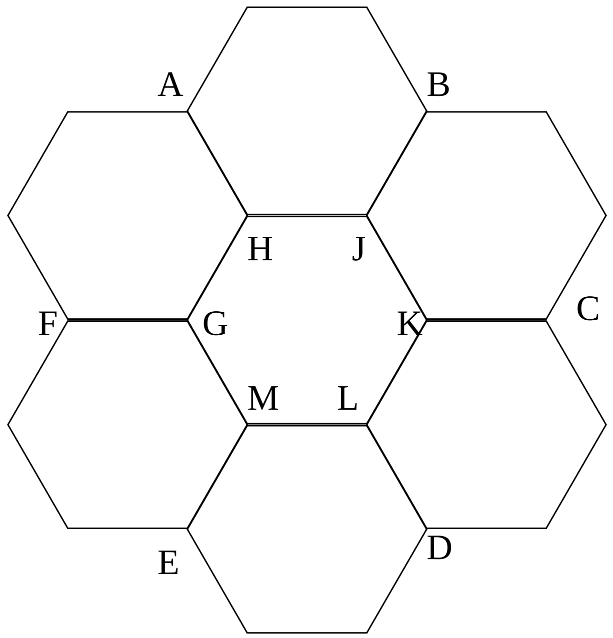

FIGURE 6. A relatively simple unspiked network in which uniqueness fails

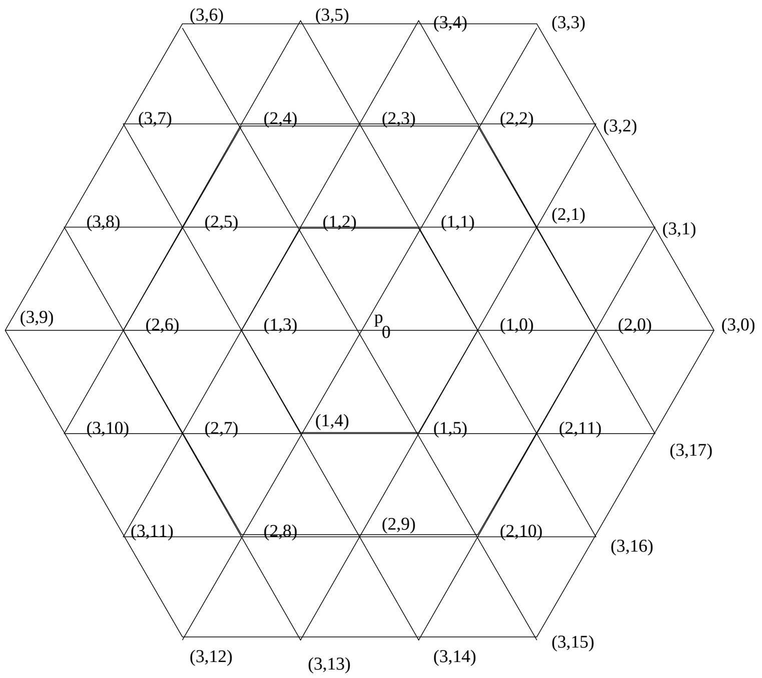

FIGURE 7. A network of three concentric hexagons

Proof. The proof will proceed by induction on $j = N - k$, where k is the label of the hexagon on which the node lies. The case $j = 0$ is contained in the hypothesis of the lemma. Suppose that, where $1 \leq k \leq N - 1$, the voltages on hexagons $k + 1, \ldots, N$ are all integers (i.e., suppose the lemma holds for all natural numbers smaller than j) . Now for $0 \leq l \leq 5$, all neighbors of the node $(k + 1, l(k + 1))$ except node (k, lk) lie on hexagon $k + 1$ or $k + 2$, and hence have integer voltages, as does node $(k+1, l(k+1))$ itself. By assumption, the net current at $(k+1, l(k+1))$ is an integer, so since the conductances are all one, Kirchhoff's Law at node $(k+1, l(k+1))$ requires that $v(k, lk)$ be an integer. Now let $1 \leq m \leq k - 1$ and suppose $v(k, lk + m - 1)$ is an integer. Then $(k+1, l(k+1)+m)$ and all of its neighbors except possibly $(k, kl+m)$ have integer voltage, so since $I(k + 1, l(k + 1) + m)$ is an integer, by Kirchhoff's Law $v(k, kl + m)$ is an integer. So by induction on m, every node on hexagon k has integer voltage. Thus by induction on $j = N - k$, the voltage at every point in Σ_N except possibly p_0 has integer voltage. Applying Kirchhoff's Law at $(1, 0)$ shows that p_0 must also have integer voltage, proving the lemma. □

We now show that if it is known that all current sources in Σ_N must have value 1, the location of the sources in uniquely determined by the voltages and net currents on the boundary (hexagon N). To do this, we consider the difference network Δ of two such networks with identical boundary data, so that the voltages and currents on the boundary of Δ are zero, and the net current at any node is 1, 0, or -1. Uniqueness will be established if we can prove that every voltage in Δ is zero. The method of this proof, and also the proof of Theorem 8, was motivated by the technique used by Chaffee and Whitley in [1] to prove the uniqueness of multiple sources in rectangular networks.

Uniqueness theorem 7 (Triangular networks)**.** *The voltage at every node in Δ is zero. Thus the locations of unit current sources in Σ_N are uniquely determined by boundary data.*

Proof. Since there are only finitely many nodes in Σ_N, the maximum value V of the voltage on Δ is attained at some node in the network; suppose it is attained at node (k, r), and suppose $k < N$. We will show that the maximum voltage is also attained on hexagon $k + 1$. Since the voltage at (k, r) is a maximum, no current flows into (k, r), so (k, r) must not be a sink. If the net current at (k, r) is zero, since none of its neighbors have voltage higher than V, Kirchhoff's Law shows that the voltage at each neighbor of (k, r), in particular at the two (or more) neighbors on hexagon $k + 1$, must be V. Hence the maximum is attained on hexagon $k + 1$. If the net current at (k, r) is one, by Kirchhoff's Law the sum of the voltages at the six neighbors of (k, r) must be $6V - 1$. Now by Lemma 3, V is an integer, as is the voltage at each neighbor of (k, r). Hence 5 of the neighbors of (k, r) have voltage V, and the other one has voltage $V - 1$. Since (k, r) has at least two neighbors on hexagon $k + 1$, at least one of these must have voltage V, so the maximum is attained on hexagon $k + 1$. Induction on k shows that the maximum must then be attained on hexagon N. But hexagon N is the boundary of Δ, and so all of its nodes have voltage zero. Hence $V = 0$. An identical argument shows that the minimum voltage in the network is also zero. Thus every node in Δ has voltage zero. □

Remark 5. *In lemma 2 it was shown that if a small network Γ is embedded in a larger network Σ in such a way that* $\mathrm{int}\Gamma \subset \mathrm{int}\Sigma$*,* $\Sigma \setminus \mathrm{int}\Gamma$ *is connected, and no node in $\Sigma \setminus \Gamma$ neighbors a node in* $\mathrm{int}\Gamma$*, and if the boundary data of Γ do not determine the sources in Γ, then the boundary data of Σ do not determine the sources in Σ. Equivalently, if the boundary data of Σ* do *uniquely determine the sources in Σ (as in the case $\Sigma = \Sigma_N$), then the boundary data of Γ uniquely determine the sources in Γ. Using this fact, since Theorem 7 shows uniqueness of sources for arbitrarily large N, we also have uniqueness for a whole range of triangular networks which may be embedded in networks Σ_N for which a direct uniqueness proof would be difficult because they lack the symmetry properties of Σ_N, such as the network shown in figure 8.*

4.2. **Towers of Hanoi networks.** We will now show a similar result for Towers of Hanoi networks, using the same basic approach. We divide the network into layers, as follows: the innermost layer (layer 1) consists only of the center point if there are an odd number of boundary nodes and of the two center points if there are an even number of boundary nodes, and for $k > 1$ layer k consists of the neighbors of points in layer $k - 1$ which do not lie in layer $k - 2$, as illustrated in Figure 9. The outermost layer (layer N if there are either $2N - 1$ or $2N$ boundary nodes) will then consist of the boundary nodes of the network. We observe that, for $1 \leq k < n$, each node in layer k has at least two neighbors in layer $k + 1$. This and the fact that, as proven in the following lemma, each voltage in the difference network will be an integer, are the basic properties

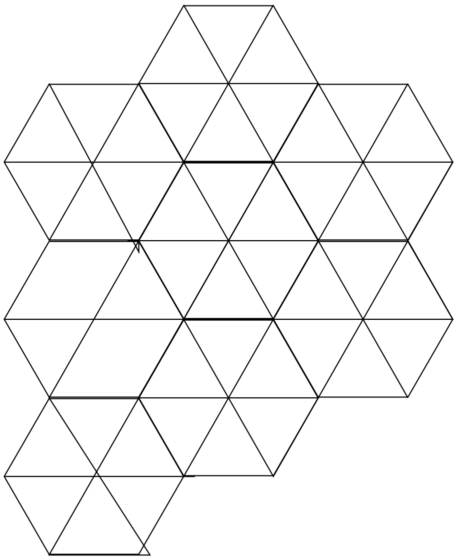

FIGURE 8. Theorem 7 also establishes uniqueness for this network

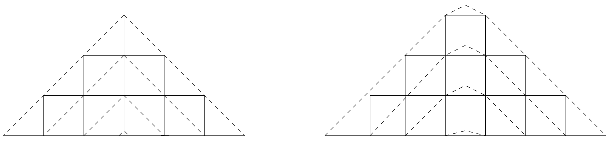

FIGURE 9. Towers of Hanoi networks, with layers indicated by dotted lines

which allow uniqueness of multiple sources to be proven in these networks, and uniqueness in other networks with these properties may be proven just as easily.

Lemma 4. *Let Γ be a towers of Hanoi network with unit resitances on each edge, and suppose each of the voltages on $\partial\Gamma$ and each of the net currents in* $\text{int}\Gamma$ *are known to be integers. Then the voltage at any node in Γ is an integer.*

Proof. We will prove the result for a network with an odd number, say $2N-1$, of boundary nodes; the analagous result for networks with an even number of boundary nodes may be proven similarly. The *kth* layer of the network will have $2k-1$ nodes; we will index these nodes as $(k,-k+1),\ldots,(k,k-1)$ moving from left to right in the network. The neighbors of the center point $(1,0)$ are then $(2,-1),(2,0)$, and $(2,1)$. For $2 \le k \le N-1$, the neighbors of $(k,0)$ are $(k-1,0),(k+1,-1),(k+1,0)$, and $(k+1,1)$, and for $1 \le j \le k-2$, the node $(k,\pm j)$ has neighbors $(k-1,\pm(j-1)),(k-1,\pm j),(k+1,\pm j)$, and $(k+1,\pm(j+1))$, while the node $(k,\pm(k-1))$ has neighbors $(k-1,\pm(k-2)),(k+1,\pm(k-1))$, and $(k+1,\pm k)$. Finally, node $(N,0)$ neighbors only node $(N-1,0)$; for $1 \le j \le N-2$ node $(N,\pm j)$ neighbors nodes $(N-1,\pm(j-1))$ and $(N-1,\pm j)$, and node $(N,\pm(N-1))$ neighbors only node $(N-1,\pm(N-2))$. We now prove the lemma, proceeding, as in the proof of Lemma 3, by induction on $N-k$ where k is the index of the layer. By assumption, each voltage on layer N is an integer. Suppose all voltages of nodes on layers $k,\ldots,N$ are integers. Then $v(k,0)$ and the voltages of each neighbor of $(k,0)$ except possibly $(k-1,0)$ are known to be integers, so Kirchhoff's Law at $(k,0)$ implies $v(k-1,0) \in \mathbb{Z}$. Now suppose that, where $j \le k-2$, we have $v(k-1,\pm j) \in \mathbb{Z}$. Then $(k,\pm(j+1))$ and each of its neighbors except possibly $(k-1,\pm(j+1))$ have integer voltage, so Kircchoff's Law at $(k,\pm(j+1))$ implies $v(k-1,\pm(j+1)) \in \mathbb{Z}$. Hence by induction on j, each node in layer $k-1$ has integer voltage, so by induction on $N-k$, each node in the entire network has integer voltage. $\square$

We now show that, if all sources in a towers of Hanoi network are known to be of unit size, their locations are uniquely determined by boundary measurements. As usual, we assume two such networks Γ and Σ have the same boundary measurements, so that $\Delta = \Gamma - \Sigma$ has boundary voltages and currents zero, and the net current at any node in Δ is 1, 0, or -1.

Uniqueness theorem 8. *The voltage at every node in Δ is zero. Thus the locations of unit sources in Γ are uniquely determined by boundary data.*

Proof. Suppose that the maximum value, V, of the voltage is assumed at some point p on layer k, where $k < N$, of Δ. Then no current flows into p, so p must not be a sink. If p is a harmonic node, the average value of the voltages of the neighbors of p is then V. Since V is the maximum value of the voltage on the entire network, none of the neighbors of p has a voltage exceeding V, so that also none of the neighbors of p may have a voltage less than V. In particular, each neighbor of p which lies on layer $k+1$ has voltage V. Thus the maximum value of the voltage is also assumed at layer $k+1$. The only remaining case is that in which there is a unit source at p. If p has n_p neighbors (n_p will be either three or four), the sum of the voltages of the neighbors of p will then be, by Kirchhoff's Law at p, $n_pV - 1$. Since each of these voltages are integers, the only way this is possible is for one of the neighbors of p to have voltage $V-1$ and the others have voltage V. Since p has two neighbors on layer $k+1$, at least one of these must have voltage V, so that the maximum is attained on layer $k+1$. Thus is any case, if the maximum value of the voltage is attained on layer $k < N$, it is also attained on layer $k+1$. Hence by induction on k, the maximum voltage is attained on layer N, *i.e.*, on the boundary. An identical argument shows that the minimum voltage is also attained on the bounday. But every voltage on $\partial\Delta$ is zero, implying that every voltage in Δ is zero, proving the theorem. □

5. Uniqueness of individual sources in various networks

5.1. **Introduction.** We now examine the question of what boundary data are required to uniquely determine the location of a single source in various resistor networks, including the towers of Hanoi network, circular networks, rectangular networks, and triangular networks. In each case, we find that only a partial set of boundary currents is required to uniquely determine the node at which the source lies. Unlike our results for multiple sources, these results will not require each conductivity to be one. In general we will assume we have two networks, each known to have a single unit source in their interiors, with a certain amount of identical boundary data. The difference network Δ of these two networks will then satisfy the hypotheses of the following lemma, which plays an important role in all of the uniqueness results seen below.

Lemma 5. *Let Θ be a network with at most one unit sink and one unit source, such that the voltage is zero at each node of $\partial\Theta$ and the sum of the boundary currents from Θ is zero. Suppose that for any $p, q \in \Theta$ with q distinct from p there exists a path from q to some boundary node r which does not pass through p. Let p be an interior node in Θ with voltage zero. Then p is a harmonic node.*

Proof. Suppose p is not harmonic. Since if Θ satisfies the hypotheses of the lemma then so does $-\Theta$, we will assume p is a unit source. Since $v(p) = 0$, p is the only source in the circuit, and each boundary node has voltage zero, the maximum principle implies that 0 is the highest voltage attained anywhere in Θ. There is then a unit sink somewhere in intΘ, say at the point q. Due to the assumption on the geometry of the network, we may construct a disjoint path $q = q_0, q_1, \ldots, q_n$, where $q_0, \ldots, q_{n-1} \in$intΘ, $q_n \in \partial\Theta$, and p does not appear in the path. Thus each of the nodes $q_1, \ldots, q_{n-1}$ is harmonic. Since no points in Θ have voltage higher than zero and net current flows in to q, we must have $v(q) < 0$. Let q_i be the first node in the path to have voltage zero; then $i \geq 1$, and since all boundary voltages are zero, $i \leq n$. Suppose $i < n$. Then q_i is harmonic, and by assumption $v(q_{i-1}) < 0$, so since q_{i-1} is a neighbor of q_i, q_i must have a neighbor with positive voltage. But this is impossible since no node in the network has positive voltage. So we must have $i = n$, and in particular $v(q_{n-1}) < 0$. q_{n-1} is a neighbor to the boundary node q_n, which has zero voltage, so some net current flows into the circuit from q_n to q_{n-1}. Since each boundary node has voltage zero and no node in all of Θ has positive voltage, this current may not be compensated for by any current flowing toward the boundary in some other boundary-interior connection. But the sum of the net currents in the boundary nodes was assumed to be zero, so this is a contradiction. Hence p must be harmonic. □

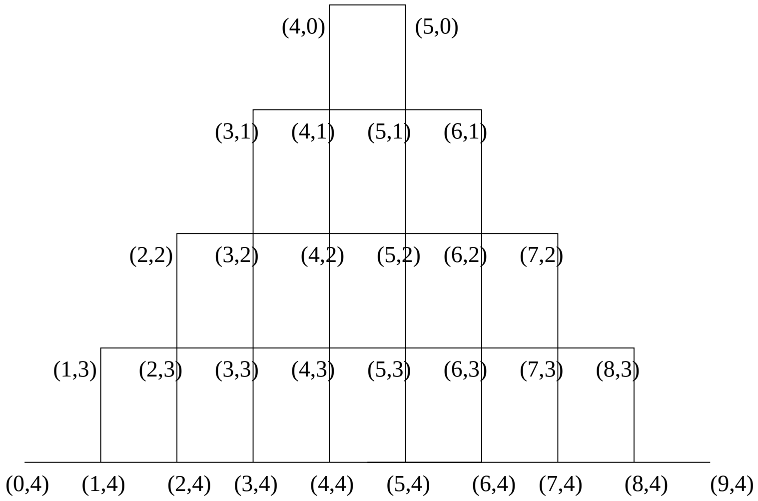

FIGURE 10. Labelling of nodes in a 10-boundary node Towers of Hanoi network

We now turn to the specific networks under examination, beginning with the Towers of Hanoi network.

5.2. **Towers of Hanoi networks.**

Uniqueness theorem 9 (Towers of Hanoi)**.** *Let Γ and Σ be two $2n$-boundary node towers of Hanoi networks, each with one unit source. Suppose Γ and Σ have the same voltage at every boundary node and identical currents at some set of $n-1$ contiguous boundary nodes. Then Γ and Σ have their source at the same node.*

Proof. Consider the difference network $\Delta = \Gamma - \Sigma$. The hypotheses of the theorem show that Δ will have voltage zero at each point on the boundary and current zero at $n-1$ contiguous boundary nodes, that the sum of the boundary currents on Δ is zero, and that Δ has at most one unit source and one unit sink. We wish to show that the voltage at every point in Δ is zero. Noting that Δ satisfies the hypotheses of Lemma 5, let us number the nodes of Δ, as in Figure 10, so that the boundary nodes are $(0, n-1), (1, n-2), \dots, (n-1, 0)$, $(n, 0), (n+1, 1), \dots, (2n-1, n-1)$. By symmetry we need only consider cases where at least half of the nodes on which the current is given to be zero lie on the left side of the graph. We will first consider the case where the $n-1$ boundary nodes known to have current zero are $(0, n-1), \dots, (n-2, 1)$. Kirchhoff's Law at $(0, n-1)$ then gives $v(1, n-1) = 0$, and if, for $1 \le i \le n-3$, $v(i, n-i) = 0$, applying Kirchhoff's Law to $(i, n-i-1)$ gives $v(i+1, n-(i+1)) = 0$. Hence by induction for $1 \le i \le n-2$, $v(i, n-i) = 0$. Now suppose that for all $j \le k$, where $k \ge 0$, we have, for all i such that the node $(i+j, n-i)$ exists, $v(i+j, n-i) = 0$. Then $(1+k, n-1)$ has voltage zero and has neighbors $(k, n-1), (1+k, n-2)$, and $(2+k, n-1)$, of which all but the latter are known to have voltage zero. So by Kirchhoff's Law (which holds at $(1+k, n-1)$ by Lemma 5), $v(2+k, n-1) = 0$. Then if $v(i+1+k, n-i) = 0$, Kirchhoff's Law at $(i+1+k, n-i-1)$ (which exists as long as $(i+2+k, n-i-1)$ exists and is not a boundary node; if $(i+2+k, n-i-1)$ is a boundary node, we already know it has voltage zero), we have $v(i+2+k, n-i) = 0$. So by induction on i, for each i such that the node $(i+1+k, n-i)$ exists, $v(i+1+k, n-i) = 0$. By induction on k, then, each voltage in Δ is zero, as desired.

Note that, in general, if we can establish that the voltage at each point $(i, n-i)$ is zero $(1 \le i \le n-1)$, this will imply that the current at each of the boundary nodes $(n, 0), \dots, (n-2, 1)$ is zero, which is all we need to show since we've already established the result for that case.

Consider the case in which the $n-1$ boundary nodes known to have current zero are $(1, n-2), \dots, (n-1, 0)$. Kirchhoff's Law at $(n-1, 0)$ shows that $v(n-1, 1) = 0$, and if $v(n-j, j) = 0$, then Kirchhoff's Law at $(n-j-1, j)$ implies $v(n-(j+1), j+1) = 0$, so by induction, $v(n-j, j) = 0$

for $1 \leq j \leq n-1$, or equivalently $v(i, n-i) = 0$ for $1 \leq i \leq n-1$, from which the result, by the above remark, follows.

In the remaining cases, the current is given to be zero at the nodes $(j, n-j-1), (j+1, n-j-2), \ldots (n-1, 0), (n, 0), \ldots, (n+j-2, j-2)$ for some $j \geq 2$. Since we restrict our attention to cases where a majority of the given nodes are on the left side of the graph, we will have $2j \leq n-1$. Kirchhoff's Law at nodes $(n-1, 0)$ and $(n, 0)$ gives $v(n-1, 1) = v(n, 1) = 0$. Moving down inductively, similarly to the case discussed in the preceding paragraph, we obtain

$$\begin{aligned} v(j, n-j) = v(j+1, n-j-1) = \cdots = v(n-1, 1) = \\ = v(n, 1) = \cdots = v(n+j-2, j-1) = 0 \end{aligned} \tag{39}$$

Now generally if, for some k, we have $I(n-k-1, k) = I(n+k, k) = 0$ (note that $(n-k-1, k)$ and $(n+k, k)$ are the two boundary nodes with y-coordinate k), and $v(x, k) = 0$ for all x such that the node (x, k) exists (i.e., for $n-k-1 \leq x \leq n+k$), Kirchhoff's Law at (x, k) gives $v(x, k+1) = 0$ for $n-k-1 \leq x \leq n+k$. Since the voltage is zero on the boundary, we have $v(n-(k+1)-1, k+1) = v(n+(k+1), k+1) = 0$. So since for $k \leq j-2$ we have $I(n-k-1, k) = I(n+k, k) = 0$, and since for all x such that the node $(x, 1)$ exists we have $v(x, 1) = 0$, it follows by induction that for all (x, k) with $k \leq j-1$ we have $v(x, k) = 0$. Applying Kirchhoff's Law to the nodes $(n-j+1, j-1), \ldots, (n+j-2, j-1)$ then gives

$$v(n-j+1, j) = \ldots = v(n+j-2, j) = 0 \tag{40}$$

Now $v(n-j-1, j) = 0$ since $(n-j-1, j)$ is a boundary node, and $v(n-j, j) = 0$ by 39, so applying Kirchhoff's Law to nodes $(n-j, j), \ldots, (n+j-3, j)$ gives

$$v(n-j, j+1) = \cdots = v(n+j-3, j+1) = 0 \tag{41}$$

Continuing in this fashion, we eventually obtain

$$v(j-1, n-j) = \cdots = v(3j-2, n-j) = 0 \tag{42}$$

Now generally if $l < m-1$, $r < n-1$, and $v(l, r) = \cdots = v(m, r) = 0$, Kirchhoff's Law at $(l+1, r), \ldots, (m-1, r)$ gives

$$v(l+1, r+1) = \cdots = v(m-1, r+1) = 0 \tag{43}$$

Applying this fact repetitively, starting with equation 42, gives, for all $k \leq j-1$,

$$v(j-1+k, n-j+k) = \cdots = v(3j-2-k, n-j+k) = 0 \tag{44}$$

In particular, for $k = j-1$, we have $v(2j-2, n-1) = v(2j-1, n-1) = 0$. Kirchhoff's Law at node $(2j-2, n-1)$ then yields $v(2j-3, n-1) = 0$. If, where $l \geq 0$, $v(2j-3-l, n-1-l) = 0$, Kirchhoff's Law at node $(2j-3-l, n-2-l)$ (which has voltage zero by equation 44) gives $v(2j-3-(l+1), n-1-(l+1)) = 0$. So by induction on l, we have $v(2j-3-l, n-1-l) = 0$ for $0 \leq l \leq j-2$; equivalently, $v(j-2+k, n-j+k) = 0$ for all k. Suppose that for some $m \geq 2$, whenever $1 \leq r \leq m$ we have $v(j-r+k, n-j+k) = 0$ for all k such that the node $(j-r+k, n-j+k)$ exists. Kirchhoff's Law at $(2j-m-1, n-1)$ (corresponding to $r = m$, $k = j-1$) then gives $v(2j-m-2, n-1) = 0$. If, where $l \geq 0$, $v(2j-m-2-l, n-1-l) = 0$, Kirchhoff's Law at node $(2j-m-2-l, n-2-l)$ (which can be seen to have voltage zero by the hypothesis of the present induction on m), gives $v(2j-m-3-l, n-(l+1)) = 0$. So by induction on l, for all l we have $v(2j-(m+1)-1-l, n-l) = 0$; equivalently, $v(j-(m+1)+k, n-j+k) = 0$ for all k. By induction on m, it then follows that for all $m \geq 2$ and all k such that the node $(j-m+k, n-j+k)$ exists, $v(j-m+k, n-j+k) = 0$. Taking $m = 2r$ and $k = r$, this gives $v(j-r, n-(j-r)) = 0$ for each $r \geq 1$. Since by equation 39 we have $v(j, n-j) = \cdots = v(n-1, 1) = 0$, this shows that each point of form $(i, n-i)$ has voltage zero. By the remark after the first case, this finishes the proof. $\square$

A similar result holds for towers of Hanoi networks with an odd number of boundary nodes; if such a network has $2n+1$ boundary nodes, then all of the boundary voltages, along with the currents at n contiguous boundary points determine the location of the source.

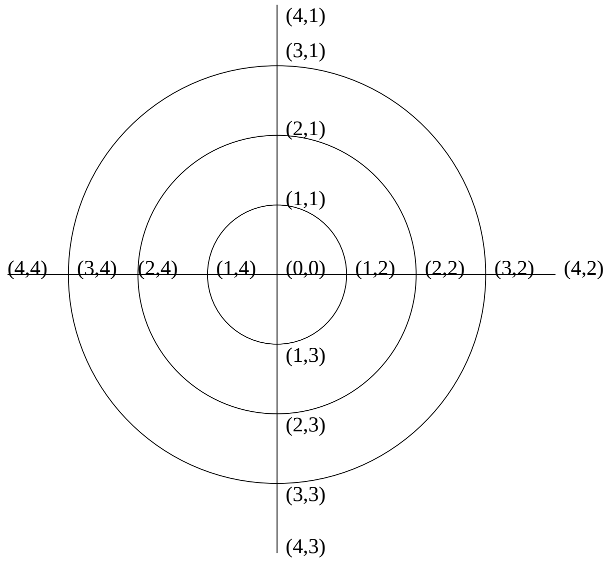

FIGURE 11. Labelling of nodes in a circular network

5.3. **Circular networks.** We show here that, for a spiked circular network with p layers, all of the boundary voltages and any p contiguous boundary currents are sufficient to determine the location of a single source. To prove this, we suppose Γ and Σ are two circular networks, each having exactly one source of unit size, with identical voltages at each boundary node and p identical contiguous boundary currents. The difference network $\Delta = \Gamma - \Sigma$ will then have boundary voltages zero and currents zero at each boundary node for which the currents of Γ and Σ are known to be identical, and the sum of all the boundary currents in Δ will be zero. Δ hence satisfies the hypotheses of Lemma 5.

We consider both networks with a center node and ones without. We assume the network has m circles and n rays, and number the node on the jth circle from the center and the kth ray clockwise from the vertical as (j,k). The boundary nodes are $(m+1,1),\dots,(m+1,n)$, and the center node, if present, is $(0,0)$, as shown in Figure 11. If a center node is present, there are $2m+1$ layers; otherwise there are $2m$ layers.

Uniqueness theorem 10 (Spiked circular networks with center node)**.** *Let Γ and Σ be two spiked circular networks with a center node with m circles and n rays, each known to have exactly one unit source. Suppose the voltages at each boundary node of Γ and Σ are identical, and that Γ and Σ have identical boundary currents at $2m+1$ contiguous boundary nodes. Then Γ and Σ have their source at the same node.*

Proof. Without loss of generality we may assume that the boundary nodes on which Γ and Σ have identical currents are $(m+1,1),\dots,(m+1,2m+1)$. Then in $\Delta = \Gamma - \Sigma$, each boundary voltage is zero and $I(m+1,1) = \cdots = I(m+1,2m+1) = 0$. By Ohm's Law, then, $v(m,1) = \cdots = v(m,2m+1) = 0$ Suppose that, where $0 \le k < m$, we have $v(m-j,j+1) = \cdots v(m-j,2m+1-j) = 0$ for all $j \le k$. Then nodes $(m-k,k+1),\dots,(m-k,2m+1-k)$ are harmonic, and for $k+2 \le r \le 2m-k$, each neighbor of $(m-k,r)$ except $(m-k-1,r)$ is known to have voltage zero. Hence Kirchhoff's Law at $(m-k,r)$ gives $v(m-k-1,r) = 0$ for $k+2 \le r \le 2m-k$, *i.e.*, $v(m-(k+1),(k+1)+1) = \cdots = v(m-(k+1),2m+1-(k+1)) = 0$. Hence by induction on k we have, for all k,

$$v(m-k,k+1) = \cdots = v(m-k,2m+1-k) = 0 \tag{45}$$

In particular, for all $k \le m-1$, the three middle terms in equation 45 give $v(m-k,m) = v(m-k,m+1) = v(m-k,m+2) = 0$. So, where $j = m-k$, for all $j \ge 1$, $v(j,m) = v(j,m+1) = v(j,m+2) = 0$. Kirchhoff's Law at $v(1,m+1)$ then gives $v(0,0) = 0$. Now suppose that, where

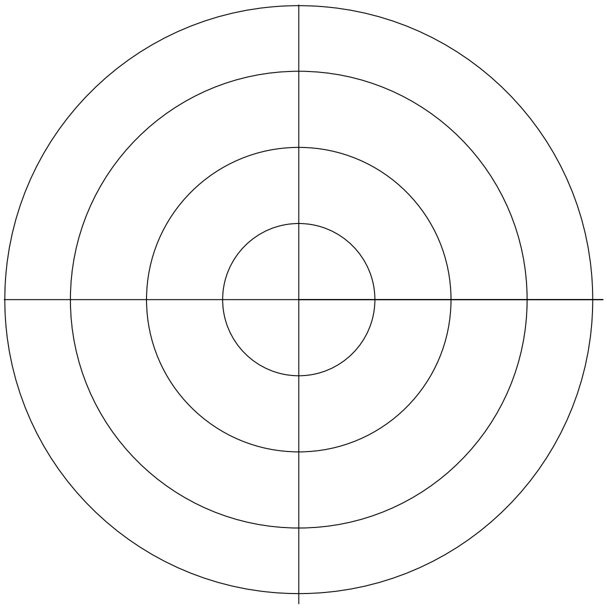

Figure 12. An unspiked circular network

$k \geq 0$, we have, for all j, $v(j, m-k) = \cdots = v(j, m+k+2) = 0$. Since also $v(0,0) = 0$, for all j Kirchhoff's Law at $(j, m-k)$ and $(j, m+k+2)$ gives $v(j, m-k-1) = v(m+k+3) = 0$. So $v(j, m-(k+1)) = \cdots = v(j, m+(k+1)+2) = 0$. By induction on k we then have $v(j,k) = 0$ for all k. Since this holds for all j, every voltage in Δ is zero, so Δ must have no sources, whence the sources in Γ and Σ are located at the same node. $\square$

Uniqueness theorem 11 (Spiked circular networks without center node)**.** *Let Γ and Σ be two spiked circular networks without center node with m circles and n rays, each known to have exactly one unit source. Suppose the voltages at each boundary node of Γ and Σ are identical, and that Γ and Σ have identical boundary currents at $2m$ contiguous boundary nodes. Then Γ and Σ have their source at the same node.*

Proof. Let $\Delta = \Gamma - \Sigma$, so the voltages on $\partial\Delta$ are all zero. Assuming, without loss of generality, that the currents which are known to be identical on Γ and Σ are those at $(m+1, 1), \ldots, (m+1, 2m)$, we then have, in Δ, $I(m+1, 1) = \cdots = I(m+1, 2m) = 0$, so $v(m, 1) = \cdots = v(m, 2m) = 0$. Repeated application of Kirchhoff's Law, similar to that in the proof of Theorem 10, gives, for all k, $v(m-k, k+1) = \cdots = v(m-k, 2m-k) = 0$. So in particular, for all j, $v(j, m) = v(j, m+1) = 0$. If, for some $k \geq 0$, we have that for all j, $v(j, m-k) = \cdots = v(j, m+k+1) = 0$, Kirchhoff's Law at $(j, m-k)$ and $(j, m+k+1)$ then gives $v(j, m-k-1) = v(j, m+k+2) = 0$. So $v(j, m-(k+1)) = \cdots = v(j, m+(k+1)+1) = 0$. Hence by induction on k every voltage in Δ is zero. Thus the sources in Γ and Σ are located in the same node. $\square$

Remark 6. *For unspiked layered circular networks, such as those in Figure 12, the boundary data required for uniqueness of individual sources are the same as that for the spiked network which is obtained by removing the boundary-boundary edges. This is the case because, since all boundary voltages in the difference network are zero, no current flows through the boundary-boundary edges, so that the current flow properties in the difference network are unaffected by the removal of these edges. This and theorems 10 and 11 lead to the general statement that the number of boundary currents required for uniqueness of individual sources in layered circular networks is at most equal to the number of layers if the network is spiked, or one less than the number of layers if it is unspiked.*

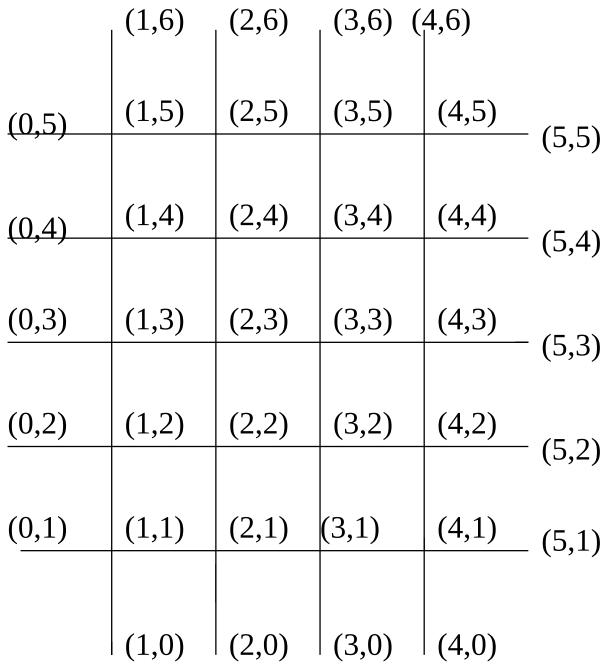

FIGURE 13. Labelling of nodes in a 5x4 rectangular network

Remark 7. *Examination of individual cases shows that considerably less data than that dictated by theorems 10 and 11 are often required for uniqueness of individual sources. I have yet to come across a network with unit conductivities on each edge in which two contiguous boundary currents do not suffice to locate a single source, and even if the boundary currents are not contiguous, only one example has been found (namely the spiked network with center node and four circles and five rays) in which two non-antipodal boundary currents were insufficient.*

5.4. **Rectangular networks.** Let us consider mxn rectangular networks, taking the boundary to consist of only the west, south, and east faces, that is, of nodes $(0,1),\ldots,(0,m)$, $(1,0),\ldots,(1,n)$, and $(n+1,1),\ldots,(n+1,m)$, where the nodes are designated as in figure 13. We will show that, if there is assumed to be exactly one unit source in the network, this source is uniquely determined by the all of the boundary voltages and the boundary currents on the south face. If Γ and Σ are two networks, each known to have a single unit source, with identical voltages on the boundary and identical currents on the south face, the difference network $\Delta = \Gamma - \Sigma$ will then have voltage zero on the boundary and currents zero on the south face, and the sum of the boundary currents in Δ will be zero. Δ hence satisfies the hypotheses of the Lemma 5.

Uniqueness theorem 12 (Rectangular networks)**.** *Let Γ and Σ be two* mxn *rectangular networks, each known to have exactly one unit source. Suppose the voltages at each boundary node (that is, each node on the west, south, and east faces) of Γ and Σ are identical, and that the boundary currents of Γ and Σ are identical on the south face. Then Γ and Σ have their source at the same node.*

Proof. Let $\Delta = \Gamma - \Sigma$. So, in Δ, $v(0,1) = \cdots = v(0,m) = 0$, $v(1,0) = \cdots = v(n,0) = 0$, $v(n+1,1) = \cdots = v(n+1,m) = 0$, and $I(1,0) = \cdots = I(n,0) = 0$. Kirchhoff's Law at nodes $(1,0),\ldots,(n,0)$ then gives $v(1,1) = \cdots = v(n,1) = 0$. Suppose that, for some $j \leq m$, we have $v(1,i) = \cdots = v(n,i) = 0$ for all $i \leq j$ (note this holds for $j = 1$). Each of nodes $(1,j),\ldots,(n,j)$ is then harmonic by Lemma 5, and so, using the fact that $v(0,j) = v(n+1,j) = 0$, Kirchhoff's Law at nodes $(1,j),\ldots,(n,j)$ gives $v(1,j+1) = \cdots = v(n,j+1) = 0$. By induction on j it hence follows that $v(1,j) = \cdots = v(n,j) = 0$ whenever $1 \leq j \leq m+1$. Since all boundary voltages are

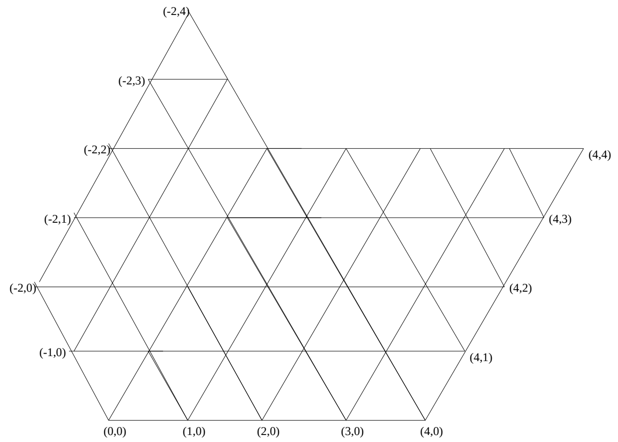

FIGURE 14. A chevron network with m=2, n=4, and p=4, with boundary nodes labelled

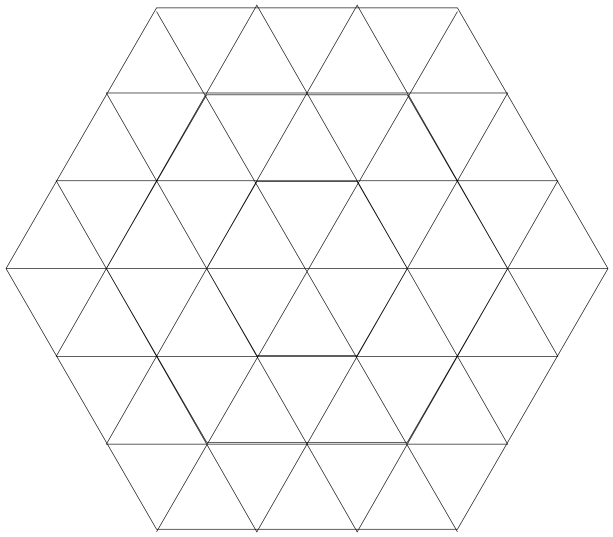

FIGURE 15. The proof of Theorem 13 may easily be modified to apply to this network.

zero, this means every voltage in Δ is zero, so that no current flows anywhere in Δ. Thus there are no sources in Δ, so that Γ and Σ must have their source at the same node. $\square$

Note that if we instead take Γ and Σ to be semi-infinite networks of width n bounded on the south by $(1,0),\ldots,(n,0)$ but unbounded on the north, the above proof remains valid.

5.5. **Triangular networks.** Let us now turn to triangular networks. To keep the details of the arguments simple, we will consider "chevron" networks such as those in figure 14, with boundary consisting of the nodes $(-m,0),\ldots,(n,0)$, $(-m,1),\ldots,(-m,p)$, and $(n,1),\ldots,(n,p)$. It is easily seen that the uniqueness result below applies equally well to certain subnetworks of these chevron networks, such as that in figure 15, and also to the semi-infinite network obtained by taking $p\to\infty$.

Uniqueness theorem 13 (Triangular networks)**.** *Let Γ and Σ be two chevron networks of the same dimensions, each known to have exactly one unit source. Suppose that the voltages at each* *boundary node of* Γ *and* Σ *are identical, as are the currents at nodes* $(-m,0),\dots,(n,0)$. *Then* Γ *and* Σ *have their source at the same node.*

Proof. As usual, let $\Delta = \Gamma - \Sigma$, so that the boundary voltages of Δ are zero, the currents at nodes $(-m,0),\dots,(n,0)$ are zero, and the sum of the boundary currents in Δ are zero. Δ then satisfies the hypotheses of Lemma 5. The neighbors of $(0,0)$ are $(-1,0)$, $(1,0)$ and $(0,1)$, and we have $I(0,0) = v(0,0) = v(-1,0) = v(1,0) = 0$, so by Kirchhoff's Law at $(0,0)$, $v(0,1) = 0$. Where $0 \le k < m$, suppose $v(-k,1) = 0$. Now the neighbors of $(-k-1,0)$ are $(-k,0),(-k-1,1)$, and, if it exists, $(-k-2,0)$. If $(-k-2,0)$ exists, $v(-k-2,0) = 0$, and $I(-k-1,0) = v(-k-1,0) = v(k,0) = 0$, so by Kirchhoff's Law at $(-k-1,0)$, $v(-k-1,1) = 0$. Hence by induction on k we have $v(-m,1) = \cdots = v(0,1) = 0$. A similar argument shows $v(0,1) = \cdots = v(n,1) = 0$. Now suppose that for some j with $1 \le j < p$, we have, for all $i \le j$, $v(-m,i) = \cdots = v(n,i) = 0$. The node $(0,j)$ and all its neighbors except $(0,j+1)$ (namely the nodes $(-1,j),(1,j),(-1,j-1),(0,j-1),(1,j-1)$) are each known to have voltage zero, and by Lemma 5 $I(0,j) = 0$, so by Kirchhoff's Law at $(0,j)$ we have $v(0,j+1) = 0$. Where $0 \le k < m$, suppose $v(-r,j+1) = 0$ for each nonnegative $r \le k$. Now $v(-k,j) = 0$, and by Lemma 5 $I(-k,j) = 0$. Of the neighbors of $(-k,j)$, only $(-k-1,j+1)$ is not known to have voltage zero. Hence by Kirchhoff's Law at $(-k,j)$, $v(-k-1,j+1) = 0$. So by induction on k, $v(-m,j+1) = \cdots = v(0,j+1) = 0$. Since an identical argument shows that $v(0,j+1) = \cdots = v(n,j+1) = 0$, induction on j shows that every voltage in Δ must be zero, so that the sources in Γ and Σ must be at the same node. ☐

6. Recovery of sources and the Green's function

6.1. **Recovery of Sources.** Until now, we have focused on proving that, subject to certain restrictions, the distribution of sources in various networks is uniquely determined by certain sets of boundary data, but we have yet to consider the problem of actually finding the sources from the given boundary data.

Where u_∂ and I_∂ are the vectors representing the voltages and currents on the boundary and u_{int} and I_{int} represent the voltages and currents on the interior, Kirchhoff's equations take the block form

$$\begin{bmatrix} A & B \\ B^t & C \end{bmatrix} \begin{bmatrix} u_\partial \\ u_{int} \end{bmatrix} = \begin{bmatrix} I_\partial \\ I_{int} \end{bmatrix} \tag{46}$$

Thus

$$I_\partial = Au_\partial + Bu_{int} \qquad u_{int} = C^{-1}(I_{int} - B^t u_\partial) \tag{47}$$

Substitution then gives

$$I_\partial = Au_\partial + BC^{-1}(I_{int} - B^t u_\partial) \tag{48}$$

Thus where $\Lambda = A - BC^{-1}B^t$ is the response matrix, we have

$$I_\partial = \Lambda u_\partial + BC^{-1} I_{int} \tag{49}$$

Since Λ and u_∂ are independent of the arrangement of sources, we may take $u_\partial = 0$, so that $I_\partial = BC^{-1}I_{int}$. So if the boundary voltages are held at zero, a unit source at interior node j gives rise to a current of $(BC^{-1})_{rj}$ at boundary node r. Numbering the interior nodes $i = 1,\dots,n$, then, if each interior node has a source of value α_i, the current out of boundary node r will be $\sum_{i=1}^n \alpha_i (BC^{-1})_{rj}$. This gives rise to a very general but also rather inefficient way of locating the sources in a network: for each combination of α_i which satisfy some given restriction (such as that each α_i be either 0 or 1 or that only one of the α_i be nonzero) and for which $\sum_{i=1}^n \alpha_i$ is equal to the total current flowing out of the network, the numbers $\sum_{i=1}^n \alpha_i (BC^{-1})_{rj}$ are compared to the measured boundary currents $(I_\partial)_r$, and the arrangement of sources is the collection of values α_i for which these numbers match. Because there will be a large number of possible choices of α_i for which equality will need to be checked, this algorithm is undesirably complex. However, we have been unable to come up with a more efficient algorithm, partly due to the difficulty of proving results about the Green's function, which is discussed in the next section.

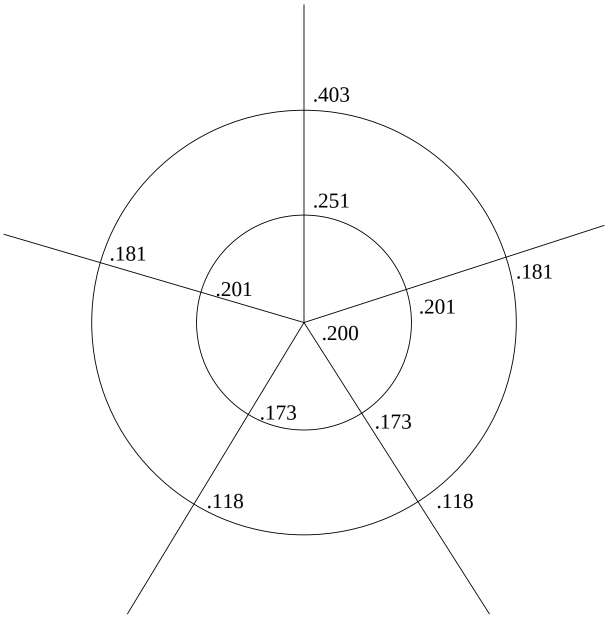

FIGURE 16. Green's function with source at (2,1)

6.2. **The Green's function.** The Green's function g_j centered at the interior node j is defined as the function which assigns to each interior node p the voltage at p when a unit source is placed at node j and all boundary voltages are zero. From equation 47, when the boundary voltages are zero we have $u_{int} = C^{-1}I_{int}$, so, for all interior nodes j and p, $g_j(p) = (C^{-1})_{pj}$. A result of Duffin in [2] states that in infinite lattice networks, the value of the Green's function decreases as one travels further from the source node. One might wonder if a similar result holds for finite networks. If the result were true for rectangular networks, as Chaffee and Whitley point out in [1], this would provide a very simple algorithm for locating unit sources in rectangular networks, since the x coordinate of the source could be determined by finding the largest boundary current on the north and south faces, and the y coordinate could be determined by finding the largest boundary current on the west and east faces. However, this result does not, in general, hold, as is indicated by figures 16 and 17. We see that in a spiked circular network with center node, 2 circles, 5 rays, and unit resistances on each edge, the Green's function centered at node (2,1) takes a larger value at node (1,2) than at (2,2). Likewise, in a 5x5 rectangular network with unit resistances, the Green's function centered at (3,1) takes a larger value at (1,2) than at (1,1). This latter example illustrates that the algorithm proposed in [1] will not work, since the largest boundary current on the west and east faces occurs in row 2, despite the fact that the source is located in row 1. Calculation of individual examples demonstrates that similar anomalous behavior persists in larger rectangular networks and also in circular networks with relatively few rays.

The fact that the Green's function does not always decrease with distance if the source node is not at the center of the graph should not be completely surprising. Indeed, in the continuous case, the level sets for the Green's function on the unit disk with source at some point $z \neq 0$ are circles centered at a point distinct from z (specifically, the level sets are the images of circles centered at the origin under the Mobius transformation which takes 0 to z and maps the disk onto itself). Consequently, there will exist points w_1 and w_2 such that $|z - w_1| > |z - w_2|$ but $g_z(w_1) > g_z(w_2)$. Certain propositions about the behavior of the Green's function which would be expected in analogy with the continuous case do appear to hold true based on examination of certain representative cases, although we have been unable to prove any of them. We list some of them among the following conjectures.

.0079	.0143	.0170	.0143	.0079
.0172	.0322	.0396	.0322	.0172
.0288	.0577	.0769	.0577	.0288
.0404	.0928	.1527	.0928	.0404
.0402	.1203	.3483	.1203	.0402

FIGURE 17. Green's function with source at (3,1)

Conjecture 1. *In a rectangular network, the restriction of the Green's function with source at p to either of the two lines which pass through p decreases as distance from p increases. Likewise, in a circular network, the restriction of the Green's function with source at p to either the circle passing through p or the ray passing through p decreases as distance from p increases.*

Conjecture 2. *In a rectangular network, the Green's function g_p with source at p has the property that the difference between the values of g_p at successive nodes on either of the lines passing through p decreases as distance from p increases.*

Conjecture 3. *In a rectangular network with $2n-1$ columns, if p is a node on the nth column, the maximum of the restriction of g_p to any row of the network occurs on the nth column (i.e., g_p has a "ridge" on the nth column).*

Conjecture 4. *In a (2n-1)x(2n-1) square network, with center node $p = (n, n)$, the restriction of g_p to any square with center p takes its minima on the corners of the square and its maxima on the midpoints of the sides of the square, and has no local extrema elsewhere. In a triangular network with concentric hexagons, such as that in figure 15, with center node p, the restriction of g_p to one of the hexagons centered at p takes its minima at the corners of the hexagons and its maxima at the nodes nearest the midpoints of each of the sides, and has no local extrema elsewhere.*

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[2] R.J. Duffin. "Discrete Potential Theory," *Duke Math. J.* Vol 20 (1953), pp. 233-251.