ON A CHARACTERIZATION OF THE LAYERED CASE

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Abstract. We study the properties of the discrete layered case of resistor networks. We will describe the relationship between rotationally invariant systems and circulant matrices, provide a bounding of the eigenvalues of the response matrix, and characterize the components of the layered Kirchhoff matrix.

1. Introduction

A circular network Ω is comprised of m circles and n rays. The $mn+1$ nodes of Ω are points in the plane: a center node $p(0,0)$ and radial nodes $p(i, \frac{2\pi j}{n})$, for $i \in (0 \ldots m-1)$ and $j \in (0 \ldots n)$. Boundary nodes $\partial\Omega$ are defined $p(m, \frac{2\pi j}{n})$ for $j \in (0 \ldots n)$; interior nodes are $int\ \Omega = \Omega - \partial\Omega$.The function γ defined on the edges of Ω is called its *conductivity*. A network is considered γ-harmonic if for each node $p \in \Omega$, the sum of currents out of that node is 0.

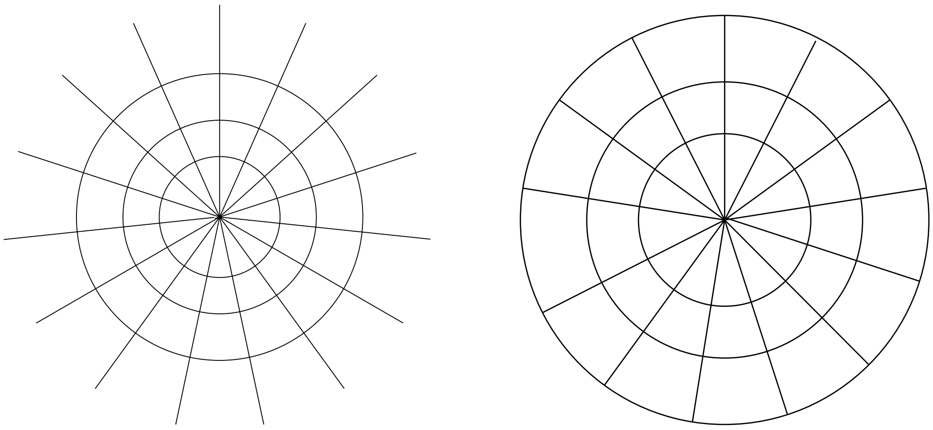


Figure 1. $D(7,15)$ Spikes case (left) and $D(6,13)$ Edges case (right)

In the layered case we have a circular network where the conductivities are constant on layers. In the continuous case, a layer x_0 is defined as the set of points $\{(r,\theta) : r = x_0\}$. In the discrete case we can use a very similar definition based on the way the nodes are arranged in the plane. A single layer consists either of the set of edges between nodes with the same radii or the set of edges between the two sets of nodes with the two distinct radii

r_1 and r_2. We denote a layered network $D(l, n)$, where l is the number of layers and n is the number of rays in the network.

We denote conductivities on layers by $[\sigma_1, \sigma_2, \ldots, \sigma_l]$, where σ_1 is the innermost layer. Layered networks can be divided into two types: those with an odd number of layers (spikes case), and those with an even number of layers (edges case). Since any rotation of a layered network by $\theta = \frac{2\pi i}{n}$ gives us the same network, we can say that such networks are *rotationally invariant.*

The $(mn + 1) \times (mn + 1)$ Kirchhoff Matrix $\mathbf{K}$ takes potentials on the network to currents on the network, and is defined by:

$$\mathbf{K}(i,j) = \begin{cases} -\gamma_{(i,j)} & i \neq j \\ \sum\limits_{i \neq j} (\gamma_{(i,j)}) & i = j \end{cases}$$

The Kirchhoff matrix can be block partitioned in the following manner:

$$\begin{array}{cc|c|c|l} & \partial\Omega & int\ \Omega & & \\ \cline{2-3} \mathbf{K} = & \mathbf{A} & \mathbf{B} & \partial\Omega \\ \cline{2-3} & \mathbf{B}^T & \mathbf{C} & int\ \Omega \\ \cline{2-3} \end{array}$$

FIGURE 2. Block structure of Kirchhoff Matrix

For each conductivity, γ on Ω, the linear map Λ is defined $(\Lambda(\phi))_p = I_u(p)$. The map Λ which takes potentials at the boundary of Λ to currents through the boundary nodes of Ω at a point p is called the Dirichlet-to-Neumann map. From the block structure of the Kirchhoff matrix we can use the Schur Complement to derive the Dirichlet-to-Neumann response matrix, Λ.

$$\Lambda = \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$$

It should be noted that the Kirchhoff matrix is symmetric and since the resulting Λ matrix is itself a Kirchhoff matrix, the response matrix is also symmetric.

In [1], David Ingerman addressed the layered case and studied the relationship between the set of eigenvalues and the set of conductivities. In this paper we will characterize the layered case further and explore its relationship with rotationally invariant systems.

2. Circulant matrices

Definition 2.1. *If a vector $\vec{x}$ has the components $(x_1, x_2, \ldots, x_n)$, the nth* componentwise rotation $rot_n\ \vec{x}$ *consists of those components shifted n places to the left where components that fall off the end are wrapped around to the beginning.*

$$rot_1\ \vec{x} = (x_n, x_1, x_2, \ldots, x_{n-1})$$

Definition 2.2. *A $n \times n$* circulant *matrix $\mathbf{M}$ is a matrix parameterized by its first row j in such a way that any row in the matrix is simply the jth componentwise rotation of the first.*

$$\mathbf{M} = \begin{bmatrix} x_0 & x_{n-1} & . & . & x_1 \\ x_1 & x_0 & x_{n-1} & . & x_2 \\ . & x_1 & x_0 & . & . \\ . & . & . & . & x_{n-1} \\ x_{n-1} & x_{n-2} & . & x_1 & x_0 \end{bmatrix}$$

We can think of a circulant matrix as the matrix equivalent of a rotationally invariant system; clearly any matrix map M from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$M(\vec{x}) = \vec{y} \Rightarrow M(rot_j\ \vec{x}) = rot_j\ \vec{y}$$

is circulant. Furthermore, matrices which have circulant blocks correspond to systems that are rotationally invariant on layers.

Circulant matrices have a special relationship with the discrete Fourier transform, which we can show. We define the matrix $\mathbf{Q}$ as

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 & . & 1 \\ 1 & \omega & \omega^2 & . & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & . & \omega^{2(n-1)} \\ . & . & . & . & . \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & . & \omega^{(n-1)^2} \end{bmatrix}, \ \omega = e^{2\pi i/n} \tag{1}$$

$\mathbf{Q}$ is the the matrix used in the discrete Fourier transform (DFT). The DFT of a vector $\vec{x}$ is defined as

$$\begin{aligned} \mathcal{F}(\vec{x}) &= \frac{1}{n}\bar{\mathbf{Q}}\vec{x} \\ &= \mathbf{Q}^{-1}\vec{x} \end{aligned} \tag{2}$$

In fact, there is an equivalence relationship between the factorization $\mathbf{QEQ}^{-1}$ and circulant matrices, which we can now show.

Theorem 2.3. *The matrix $\mathbf{R}$ is circulant and can be parameterized by a single row if and only if $\mathbf{R}$ can be diagonalized by $\mathbf{Q}$.*

Proof. First, assume

$$\begin{aligned} \mathbf{R} &= \mathbf{QEQ}^{-1} \\ \mathbf{Q}^{-1}\mathbf{R} &= \mathbf{EQ}^{-1} \end{aligned}$$

We multiply by a vector, $\vec{v}$

$$\mathbf{Q}^{-1}\mathbf{R}\vec{v} = \mathbf{E}\mathbf{Q}^{-1}\vec{v}$$

From (2),

$$\begin{aligned}\mathbf{Q}^{-1}(\mathbf{R}\vec{v}) &= \mathcal{F}(\mathbf{R}\vec{v})\\ \mathbf{E}\mathbf{Q}^{-1}\vec{v} &= \mathbf{E}\mathcal{F}(\vec{v})\end{aligned}$$

Define $\vec{u}$ to be the vector formed by the diagonal entries of the matrix $\mathbf{E}$. Let $\vec{x}\bullet\vec{y}$ represent the componentwise product, $(x_1y_1, x_2y_2, \ldots, x_ny_n)$. Then

$$\mathbf{E}\mathcal{F}(\vec{v}) = \vec{u}\bullet\mathcal{F}(\vec{v})$$

Note that $\vec{u}$ is the Discrete Fourier Transform of some vector $\vec{q} = \mathcal{F}^{-1}(\vec{u})$.

$$\vec{u}\bullet\mathcal{F}(\vec{v}) = \mathcal{F}(\vec{q})\bullet\mathcal{F}(\vec{v})$$

We note that the Fourier transform takes convolution to multiplication, hence

$$\mathcal{F}(\vec{q}\circ\vec{v}) = \mathcal{F}(\vec{q})\bullet\mathcal{F}(\vec{v})$$

The convolution $\vec{q}\circ\vec{v}$ is defined as

$$(\vec{q}\circ\vec{v})_k = \sum_{j=1}^{n-1} q_{k-j}v_j$$

where for $(k-j)\leq 0$ we let $q_{k-j} = q_{n-|k-j|}$. This can be expressed in matrix form as $\mathbf{X}\vec{v}$, where

$$\mathbf{X} = \begin{bmatrix} q_0 & q_{n-1} & \cdot & \cdot & q_1\\ q_1 & q_0 & q_{n-1} & \cdot & q_2\\ \cdot & q_1 & q_0 & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & q_{n-1}\\ q_{n-1} & q_{n-2} & \cdot & q_1 & q_0 \end{bmatrix}$$

The form of $\mathbf{X}$ is precisely circulant. Bringing it all together,

$$\begin{aligned}\mathcal{F}(\mathbf{R}\vec{v}) &= \mathcal{F}(\vec{q}\circ\vec{v})\\ \mathbf{R}\vec{v} &= \mathbf{X}\vec{v}\end{aligned}$$

so the response matrix $\mathbf{R}$ is circulant and can be parameterized by the vector $\vec{q}$ which is the first row of $\mathbf{R}$.

We prove the converse by assuming that the we are given a circulant matrix $\mathbf{R}$. We know that circulant matrices are convolution matrices, therefore

$$\mathbf{R}\vec{v} = \vec{q}\circ\vec{v}$$

where $\vec{q}$ is a row of $\mathbf{R}$ and $\vec{v}$ is some vector. We take the Fourier transform of both sides of the equation and once again note that the Fourier transform takes convolution to multiplication.

$$\begin{aligned}\mathcal{F}(\mathbf{R}\vec{v}) &= \mathcal{F}(\vec{q}\circ\vec{v})\\ \mathcal{F}(\vec{q}\circ\vec{v}) &= \mathcal{F}(\vec{q})\bullet\mathcal{F}(\vec{v})\\ \mathcal{F}(\mathbf{R}\vec{v}) &= \mathcal{F}(\vec{q})\bullet\mathcal{F}(\vec{v}) \qquad (3)\end{aligned}$$

Letting $\mathcal{F}(\vec{q}) = \vec{u}$, we express the Fourier transform in matrix form:

$$\mathcal{F}(\vec{q}) \bullet \mathcal{F}(\vec{v}) \quad = \quad \vec{u} \bullet \mathbf{Q}^{-1}\vec{v}$$

Since the componentwise product on the right can be expressed by placing the elements of $\vec{u}$ down the diagonal of a matrix, $\mathbf{E}$,

$$\mathcal{F}(\vec{q}) \bullet \mathcal{F}(\vec{v}) \quad = \quad \mathbf{E}\mathbf{Q}^{-1}\vec{v}$$

Substituting back into (3),

$$\begin{aligned} \mathbf{Q}^{-1}\mathbf{R}\vec{v} &= \mathbf{E}\mathbf{Q}^{-1}\vec{v} \\ \mathbf{R}\vec{v} &= \mathbf{Q}\mathbf{E}\mathbf{Q}^{-1} \end{aligned}$$

Since this is true for all $\vec{v}$, we are done. □

Since $\mathbf{Q}$ diagonalizes all $m \times m$ circulant matrices, $\mathbf{Q}$ must be the matrix of eigenvectors. It follows that all $m \times m$ circulant matrices have the same eigenvectors.

Corollary 2.4. *The vector of eigenvalues of a circulant matrix* $\mathbf{R}$ *is the Discrete Fourier Transform of its parameterization* $\vec{q}$.

Proof. Since $\mathbf{Q}$ is the matrix of eigenvectors. The matrix $\mathbf{E}$ of the diagonaliztion $\mathbf{Q}\mathbf{E}\mathbf{Q}^{-1}$ must be the diagonal matrix of eigenvalues. We know that vector of representation of $\mathbf{E}$ is $\vec{u}$ which in turn is $\mathcal{F}(\vec{q})$. It follows then that the eigenvalues of any circulant matrix are the DFT of its parameterization. □

Because the response matrix Λ is rotationally invariant, we expect it to be a circulant matrix. In [1], Ingerman showed that the eigenvectors of the response matrix Λ in the layered case are of the form

$$e^{ik\theta}|_{\partial_n},\ k \in \mathbb{Z} \tag{4}$$

Λ is known to be symmetric; symmetric real matrices have real eigenvalues. The eigenvectors can be chosen orthonormal, hence by the Spectral Theorem we can diagonalize Λ by the factorization $\Lambda = \mathbf{Q}\mathbf{E}\mathbf{Q}^{-1}$, where the columns of $\mathbf{Q}$ are the eigenvectors, $\mathbf{E}$ is the matrix whose diagonal entries comprise the eigenvalues, and $\mathbf{Q}^{-1} = \mathbf{Q}^T$. Hence,

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 & . & 1 \\ 1 & e^{2\pi i/n} & e^{4\pi i/n} & . & e^{(n-1)\pi i/n} \\ 1 & e^{4\pi i/n} & e^{6\pi i/n} & . & e^{2(n-1)\pi i/n} \\ . & . & . & . & . \\ 1 & e^{(n-1)\pi i/n} & e^{2(n-1)\pi i/n} & . & e^{(n-1)^2\pi i/n} \end{bmatrix} \tag{5}$$

Since this is precisely the matrix $\mathbf{Q}$ defined above, we have our proof.

In the special case of the response matrix Λ, not only is it circulant, but symmetric as well. In the circulant matrix described above, this forces

$x_1 = x_{n-1}$, $x_2 = x_{n-2}$, and so forth. Hence,

$$\Lambda = \begin{bmatrix} x_0 & x_1 & x_2 & . & . & x_2 & x_1 \\ x_1 & x_0 & x_1 & x_2 & . & . & x_2 \\ x_2 & x_1 & x_0 & x_1 & x_2 & . & . \\ . & . & . & . & . & . & x_1 \\ x_1 & x_2 & . & . & . & x_1 & x_0 \end{bmatrix}$$

We call a matrix of this form *circulant symmetric* and note that it can be completely parameterized by the first $\left[\frac{n}{2}\right] + 1$ entries of the first row.

From observation we can generalize the stucture of the components of the Kirchhoff Matrix, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

2.1. A (boundary-boundary edges).

In the spikes case, there are no boundary-boundary edges and the only non-zero entries of $\mathbf{A}$ are the diagonal entries which are equal to the sum of the other entries of that row in the Kirchhoff matrix. The structure of $\mathbf{B}$ shown below implies that these entries must be σ_l. In the edges case, each boundary node is connected to exactly two other boundary nodes. Numbering the nodes in an appropriate fashion, the resulting matrix is circulant:

$$\mathbf{A} = \begin{bmatrix} (\Sigma) & -\sigma_l & 0 & \dots & -\sigma_l \\ -\sigma_l & (\Sigma) & -\sigma_l & \dots & 0 \\ 0 & -\sigma_l & (\Sigma) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_l & 0 & 0 & \dots & (\Sigma) \end{bmatrix}, \text{ where } \Sigma = 2\sigma_l + \sigma_{l-1}$$

2.2. B (boundary-interior connections).

In both the spikes and the edges case, $\mathbf{B}$ represents the single connection from a given boundary node to the nearest interior node. Let β represent the conductivity on this connection; in the spikes case $\beta = \sigma_l$ and in the edges case $\beta = \sigma_{l-1}$. Assuming the nodes are numbered in the appropriate order, $\mathbf{B}$ has following form:

$$\mathbf{B} = \left[\begin{array}{c|c} -\beta\mathbf{I} & \mathbf{0} \end{array}\right]$$

2.3. C (interior-interior connections).

$\mathbf{C}$ represents data for $l-1$ layers of a given network. We can partition $\mathbf{C}$ into layer blocks. On a given layer, each node is connected to two other nodes and forms a circulant pattern similar to $\mathbf{A}$ in the edges case. The radial connections between layers form a diagonal pattern much like $\mathbf{B}$. Thus, the resulting matrix takes on a block form, except for the final row and column which represent the connections to the innermost layer from a single center node. The structure of $\mathbf{C}$ can be described as block circulant:

$$
\mathbf{C} = \quad \begin{array}{c|c|c|c|c|l}
 & & & & & \vdots \\ \cline{2-5}
 & \mathbf{C}_3 & -\sigma_5\mathbf{I} & 0 & 0 & 4 \\ \cline{2-5}
 & -\sigma_5\mathbf{I} & \mathbf{C}_2 & -\sigma_3\mathbf{I} & 0 & 3 \\ \cline{2-5}
 & 0 & -\sigma_3\mathbf{I} & \mathbf{C}_1 & \overline{\sigma_1} & 2 \\ \cline{2-5}
 & 0 & 0 & -\sigma_1 & \Sigma & 1 \\ \cline{2-5}
\cdots & 4 & 3 & 2 & 1 & \text{Layer}
\end{array}
$$

FIGURE 3. Block structure of $\mathbf{C}$

where each $n \times n$ circulant block $\mathbf{C}_r$ has the circulant form:

$$
\mathbf{C}_r(i,j) = \begin{cases} -\sigma_{2r} & |i-j| = 1 \\ \Sigma & i = j \\ 0 & \textit{otherwise} \end{cases}
$$

and Σ is equal to the sum of entries of a given row.

3. Circulant matrices and the Schur Complement

3.1. Generalizing the Schur Complement. Since Λ is circulant and $\mathbf{A}$ is circulant, $\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$ is circulant as well. Denoting the $n \times n$ principle submatrix of $\mathbf{C}^{-1}$ by $(\mathbf{C}^{-1})_0$, we have

$$
\begin{aligned}
\mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T &= \left[\begin{array}{c|c} -\beta\mathbf{I} & \mathbf{0} \end{array}\right] \left[\begin{array}{cc|c} \multicolumn{2}{c|}{(\mathbf{C}^{-1})_0} & \cdots \\ & & \cdots \\ \hline \vdots & \vdots & \ddots \end{array}\right] \left[\begin{array}{c} -\beta\mathbf{I} \\ \hline \mathbf{0} \end{array}\right] \\
&= \beta^2(\mathbf{C}^{-1})_0 \qquad (6)
\end{aligned}
$$

Hence, $(\mathbf{C}^{-1})_0$ is circulant.

3.2. C Inverse. The layer block pattern of $\mathbf{C}$ is preserved in $\mathbf{C}^{-1}$. In fact each of the $n \times n$ layer blocks is circulant with the exception of those blocks which are single rows or columns. These blocks are constant. It will be proven that $\mathbf{C}^{-1}$ takes the block form shown.

where $\mathcal{C}_i$ is some circulant matrix and k is some constant vector.

Theorem 3.1. $\mathbf{C}^{-1}$ *of a network Ω is the Green's function that maps a current source at one interior node to the resulting potentials on int Ω, holding potentials on the boundary to be 0.*

$$\mathbf{C}^{-1} = \begin{array}{|c|c|c|c|c}
\hline
 & & & & \vdots \\
\mathcal{C}_1 & \mathcal{C}_2 & \mathcal{C}_3 & k_4 & 4 \\
\hline
\mathcal{C}_4 & \mathcal{C}_5 & \mathcal{C}_6 & k_3 & 3 \\
\hline
\mathcal{C}_7 & \mathcal{C}_8 & \mathcal{C}_9 & k_2 & 2 \\
\hline
k_4 & k_3 & k_2 & k_1 & 1 \\
\hline
\multicolumn{1}{c}{\cdots\ 4} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{1} & \text{Layer}
\end{array}$$

Proof. We begin with the block form of the Kirchhoff matrix and hold the potential on the boundary constant at 0.

$$\begin{aligned}
\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} 0 \\ \phi \end{bmatrix} &= \begin{bmatrix} I_\partial \\ I_{int} \end{bmatrix} \\
\mathbf{C}\phi &= I_{int} \\
\phi &= \mathbf{C}^{-1} I_{int}
\end{aligned}$$

□

Theorem 3.2. $\mathbf{C}^{-1}$ *is block circulant.*

Proof. For a given unit current source at a interior node p, the resulting potential on the set of interior nodes is the p^{th} column of $\mathbf{C}^{-1}$. The p^{th} column can be subdivided in a block layer fashion as shown. This is similar to the block layer partioning of $\mathbf{C}$.

$$(C^{-1})_p = \begin{bmatrix} \phi_n \\ \phi_{n-1} \\ \vdots \\ \phi_2 \\ \phi_1 \end{bmatrix}$$

Take the potential on a layer l as a result of a unit source at a node p to be ϕ_l as given. Given the rotational symmetry of the network an unit source at the next node ϕ_l' would shift ($\phi_l' = rot_1\ \phi_l$) the resulting potentials on the layer:

$$\phi_l = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \phi_l' = \begin{bmatrix} x_n \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

As we move through each node of a particular layer's subset of the interior nodes, that layer block completes a full rotational sequence forming a circulant matrix. This occurs in each layer block of $\mathbf{C}^{-1}$, therefore $\mathbf{C}^{-1}$ is block circulant.

□

4. Bounding of the eigenvalues

We denote the eigenvalues of a matrix $[e_1, e_2, \ldots, e_n]$. The following will enable us to present a bounding argument for the eigenvalues of Λ:

Theorem 4.1. *An eigenvalue e_k of Λ is equal to the the difference of the corresponding eigenvalues of $\mathbf{A}$ and $k^2(\mathbf{C}^{-1})_0$.*

$$(e_k)_\Lambda = (e_k)_\mathbf{A} - k^2(e_k)_{(\mathbf{C}^{-1})_0} \tag{7}$$

Proof. Because Λ, $\mathbf{A}$, and $k^2(\mathbf{C}^{-1})_0$ are all $n \times n$ circulant matrices, they share the same matrix of eigenvectors, $\mathbf{F}$. The result follows. □

Now, we can derive some useful results about the pieces of the Schur complement which will help us reconstruct properties of the response matrix. Letting β be defined as above, $\beta = \sigma_l$ in the spikes case and $\beta = \sigma_{l-1}$ in the edges case, we have the following:

Theorem 4.2. *The eigenvalues e_k of $(\mathbf{C}^{-1})_0$ are bounded by $0 < e_k \leq \frac{1}{\beta}$*

Proof. To prove that $e_k > 0$, we need the fact $\mathbf{C}$ is positive definite. From [3] we know that all Kirchhoff matrices are positive definite. Lemma 3.1 of [3] shows that any submatrix $K(P; P)$ such that P is a proper subset of the vertices of a given graph Γ is positive definite. Since the set of interior nodes of the graph Γ which form $\mathbf{C}$ is a proper subset, it follows that $\mathbf{C}$ is positive definite. This implies that $\mathbf{C}^{-1}$ is positive definite. One of the conditions that $\mathbf{C}^{-1}$ is positive definite is that each of the principle submatrices is positive definite, so $(\mathbf{C}^{-1})_0$ is positive definite, and hence $e_k > 0$.

In the spikes case, the structure of $\mathbf{A}$ given above shows that its eigenvalues are identically $\beta = \sigma_l$. Since Λ is positive semidefinite, we have

$$0 \leq \beta - \beta^2 e_k = \beta(1 - \beta e_k)$$

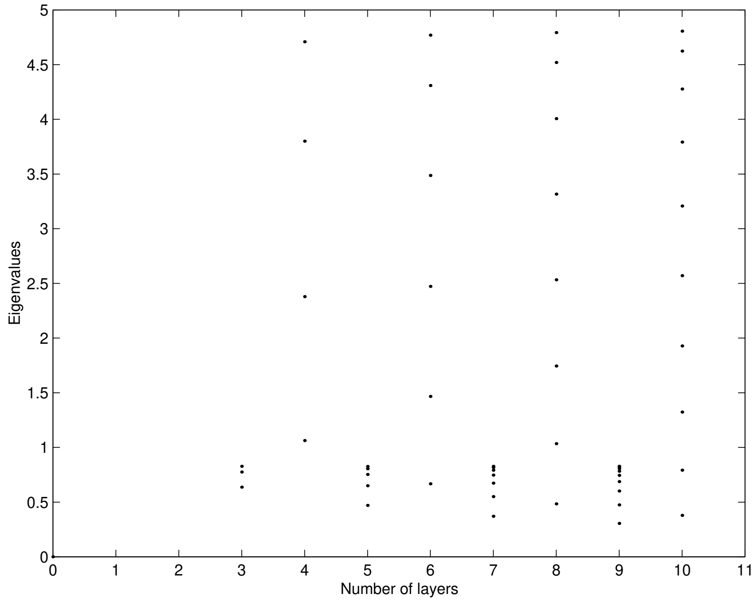
since $\beta > 0$,

$$e_k \leq \frac{1}{\beta}$$

For the edges case, we begin by removing the outside layer. Because this layer consists entirely of boundary-boundary connections, the only component of the Kirchhoff matrix that is affected is $\mathbf{A}$; $\mathbf{B}$ and $\mathbf{C}$ are unchanged. We still have a valid Kirchhoff matrix, so the Schur complement $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$ still produces a valid positive semidefinite response matrix Λ'. Thus, by the argument above the eigenvalues e_k of $(\mathbf{C}^{-1})_0$ are bounded above by $(\beta = \sigma_{l-1})^{-1}$.

□

FIGURE 4. Eigenvalue spread with $\sigma_l \equiv 1$



The eigenvalue spread shown in figure 4 suggests the following, which we can prove.

Theorem 4.3. *The eigenvalues e_k of the response matrix Λ in the layered case are bounded by*

$$0 \le e_k \le \sigma_l$$

in the spikes case and by

$$0 \le e_k \le 4\sigma_l + \sigma_{l-1}$$

in the edges case.

Proof. $e_k \ge 0$ because Λ is a Kirchhoff matrix, and all Kirchhoff matrices are positive semidefinite [3].

In the spikes case, the eigenvalues of $\mathbf{A}$ are exactly equal to σ_l. Since the eigenvalues of $\mathbf{BC}^{-1}\mathbf{B}^T$ are between 0 and σ_l, by the difference formula for eigenvalues e_k is bounded by $0 \le e_k \le \sigma_l$.

In the edges case, the eigenvalues of $\mathbf{A}$ can be found by taking the Fourier transform of its first row, $\vec{q}$. Since $\vec{q}$ is of the form

$$\vec{q} = \begin{bmatrix} (2\sigma_l + \sigma_{l-1}) & -\sigma_l & 0 & 0 & \dots & 0 & -\sigma_l \end{bmatrix}$$

the kth component of the Discrete Fourier Transform of $\vec{q}$ will be

$$e_k = [\mathcal{F}(\vec{q})]_k = (2\sigma_l + \sigma_{l-1}) - e^{\frac{2\pi(k-1)i}{n}}\sigma_l - e^{\frac{2\pi(k-1)i(n-1)}{n}}\sigma_l, \quad k \in [1, 2, \dots, n]$$

By the triangle inequality,

$$\begin{aligned} e_k &\leq |(2\sigma_l + \sigma_{l-1})| + |e^{2\pi(k-1)i/n}\sigma_l| + |e^{2\pi(k-1)i(n-1)/n}\sigma_l| \\ e_k &\leq 4\sigma_l + \sigma_{l-1} \end{aligned}$$

□

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