RECOVERY OF TRANSITIONAL PROBABILITIES FROM ABSORPTION PROBABILITIES

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ABSTRACT. This paper will examine a probablistic analog of specific resistor networks. Transitional probabilities, probabilities which characterize movement from one node to the next, will replace the conductivities calculated in traditional network problems. While less rigorous mathematically, the probablistic interpretations present complexity in requiring that we consider two probabilities per edge, rather than one conductivity.

1. Introduction

Studying resistor networks probablistically requires different representations of data. Frequently, we will find that the data does not line up symmetrically as it does in regular resistor networks, which tends to limit the amount of information that we can recover from the boundary. It is important to note that in these representations particles on the boundary cannot reenter the system. Boundary nodes are considered *absorbing nodes*.

Moving from one interior node to the next, moving from an interior node to a boundary node, and staying at an interior node (i.e. no movement) are the only allowed movements. The probabilities of these movements are what we will attempt to recover based on the information from the absorbing nodes.

2. Data Abstractions

Assume that our network contains u total nodes, where $u = m+n$ and m and n are boundary and interior nodes respectively. We can then construct a $u \times u$ *transitional matrix*, which we call P. The entries in P, p_{ij}, represent the probability that a particle will move *directly* from node i to node j. Ordering P so that the rows and columns representing boundary information preceed those containing interior information, we construct a matrix in the form:

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}.$$

I represents the $m \times m$ identity matrix, 0 is the $m \times n$ matrix whose entries are all zeros, R is the $n \times m$ matrix representing the probabilities of moving from an interior node to an exterior node, and Q is the $n \times n$ matrix containing the probabilities of moving from one interior node to the next.

¿From the transition matrix P, we would like to derive a matrix B. The entries in B, b_{ij}, represent the probability that a particle which enters the system at node i will be absorbed by node j. We begin by defining a new matrix, N, whose entries n_{ij} represent the expectation that a particle that starts at i will be absorbed by a node adjacent to j. The matrix N, known as the *fundamental matrix*, is an infinite sum defined by $(I-Q)^{-1}$, where I is the $n \times n$ identity matrix. The absorption probabilites contained in B are obtained from N by the following

$$B = (I-Q)^{-1}R.$$

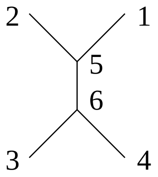
The absorbing probabilites take into calculation the probabilities of moving from an interior node to an external node weighted by the expectation for the non-absorbing states. [**?**]

3. Inverse Problem

The inverse problem requires that we use the information contained in B to recover the entries in P. Given B, we must show that a unique solution for P exists. We do this by showing that certain systems of equations are solvable in terms of the unknown probabilities p_{ij}.

4. Example One

Once a particle reaches the boundary, it is absorbed and ceases to move, therefore, networks with boundary to boundary connections are not very interesting. The first network that I considered was the following



This network has four boundary nodes and two interior nodes. We can easily construct the transition matrix from this information.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ p_{51} & p_{52} & 0 & 0 & p_{55} & p_{56} \\ 0 & 0 & p_{63} & p_{64} & p_{65} & p_{66} \end{bmatrix}.$$

We can now proceed to define the following matrices:

$$R = \begin{bmatrix} p_{51} & p_{52} & 0 & 0 \\ 0 & 0 & p_{63} & p_{64} \end{bmatrix}$$

$$(I-Q) = \begin{bmatrix} 1-p_{55} & -p_{56} \\ -p_{65} & 1-p_{66} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{51} & b_{52} & b_{53} & b_{54} \\ b_{61} & b_{62} & b_{63} & b_{64} \end{bmatrix}.$$

As stated earlier, $B = NR$ and $N = (I - Q)^{-1}$, so $R = (I - Q)B$. Using this equation for R and the matrix above, we can define the following system of equations

$$p_{51} = b_{51} - b_{51}p_{55} - b_{61}p_{56} \tag{4.1}$$
$$p_{52} = b_{52} - b_{52}p_{55} - b_{62}p_{56} \tag{4.2}$$
$$0 = b_{53} - b_{53}p_{55} - b_{63}p_{56} \tag{4.3}$$
$$0 = b_{54} - b_{54}p_{55} - b_{64}p_{56} \tag{4.4}$$

$$0 = b_{61} - b_{51}p_{65} - b_{61}p_{66} \tag{4.5}$$
$$0 = b_{62} - b_{52}p_{65} - b_{62}p_{66} \tag{4.6}$$
$$p_{63} = b_{63} - b_{53}p_{65} - b_{63}p_{66} \tag{4.7}$$
$$p_{64} = b_{64} - b_{54}p_{65} - b_{64}p_{66} \tag{4.8}$$

Writing these equations in the form $Ax = b$, we find

$$\begin{bmatrix} 1 & 0 & 0 & 0 & b_{51} & b_{61} & 0 & 0 \\ 0 & 1 & 0 & 0 & b_{52} & b_{62} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{53} & b_{63} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{54} & b_{64} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{51} & b_{61} \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{52} & b_{62} \\ 0 & 0 & 1 & 0 & 0 & 0 & b_{53} & b_{63} \\ 0 & 0 & 0 & 1 & 0 & 0 & b_{54} & b_{64} \end{bmatrix} \begin{bmatrix} p_{51} \\ p_{52} \\ p_{63} \\ p_{64} \\ p_{55} \\ p_{56} \\ p_{65} \\ p_{66} \end{bmatrix} = \begin{bmatrix} b_{51} \\ b_{52} \\ b_{53} \\ b_{54} \\ b_{61} \\ b_{62} \\ b_{63} \\ b_{64} \end{bmatrix} \tag{4.9}$$

Using our probabilistic representation, however, we must consider two additional equations. The sum of the transition probabilities for each node must add up to one, therefore, we have the following

$$p_{51} + p_{52} + p_{55} + p_{56} = 1$$
$$p_{63} + p_{64} + P_{65} + p_{66} = 1$$

Using these systems of equations, we can begin to characterize the behavior of this network.

If we assume that we are given certain conditions, delineated by the B matrix, we will find certain restrictions on the transitional probabilities.

Let the absorbing probability for any one node be zero, say $b_{51} = 0$, and all other absorbing probabilities lie strictly between zero and one. From the equations, we find

$$p_{66}b_{61} = b_{61} \Rightarrow b_{61} = 0,$$

since if b_{61} were not zero, then p_{66} would have to equal 1. We know that this is not so, because none of the other absorbing probabilities are zero, which would be a necessary result. Therefore, saying that $b_{5i} = 0$, we can say that $b_{6i} = 0$ as well.

Using this information and the equation $p_{51}+p_{56}b_{61}=0$, we can conclude that $p_{51}=0$. Essentially, if the absorption probability for a node is zero, then the transition probability to that node must also be zero.

Using similar logic, we can make the following statements:

if $b_{51}=b_{52}=0$, then $p_{51}=p_{52}=0$ and $p_{55}+p_{56}=1$
if $b_{51}=b_{53}=0$, then $p_{51}=p_{63}=0$
if all $b_{ij}=0$, then $p_{55}+p_{66}=1$, $p_{65}+p_{66}=1$, and all other $p_{ij}=0$.

The general case, or the case in which the absorptions probabilities, $0 < b_{ij} < 1$ requires that we look at subdeterminants to determine whether or not we have a unique solution. Exchanging a few of the rows in our system, we get the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & b_{51} & b_{61} & 0 & 0 \\ 0 & 1 & 0 & 0 & b_{52} & b_{62} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & b_{53} & b_{63} \\ 0 & 0 & 0 & 1 & 0 & 0 & b_{54} & b_{64} \\ 0 & 0 & 0 & 0 & b_{53} & b_{63} & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{54} & b_{64} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{51} & b_{61} \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{52} & b_{62} \end{bmatrix} \begin{bmatrix} p_{51} \\ p_{52} \\ p_{63} \\ p_{64} \\ p_{55} \\ p_{56} \\ p_{65} \\ p_{66} \end{bmatrix} = \begin{bmatrix} b_{51} \\ b_{52} \\ b_{63} \\ b_{64} \\ b_{53} \\ b_{54} \\ b_{61} \\ b_{62} \end{bmatrix}$$

We want to look more closely at the subdeterminants

$$\begin{vmatrix} b_{53} & b_{63} \\ b_{54} & b_{64} \end{vmatrix} \text{ and } \begin{vmatrix} b_{51} & b_{61} \\ b_{52} & b_{62} \end{vmatrix}$$

If we write each of the terms from the B matrix in terms of transition and fundamental probabilities, we find

$$\begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ p_{63} & p_{64} \end{bmatrix} = \begin{bmatrix} b_{53} & b_{54} \\ b_{63} & b_{64} \end{bmatrix} \tag{4.10}$$

$$= \begin{bmatrix} p_{63}n_{12} & p_{64}n_{12} \\ p_{63}n_{22} & p_{64}n_{22} \end{bmatrix}. \tag{4.11}$$

We find quickly, that our equations are not linearly independent and that our determinants will always be zero. We can, however, define a parameterization for our solutions. We have from (4.4) and (4.5) that

$$b_{53}p_{55}+b_{63}p_{56}=b_{53}$$

$$b_{51}p_{65}+b_{61}p_{66}=b_{61}.$$

Say that p_{55} and p_{66} equal x and y respectively. We can then say that

$$p_{56}=\frac{b_{53}-b_{53}x}{b_{63}} \tag{4.12}$$

$$p_{65}=\frac{b_{61}-b_{61}y}{b_{61}}. \tag{4.13}$$

Further, we can define the parameterization that for $0 < x < 1$ and $0 < y < 1$

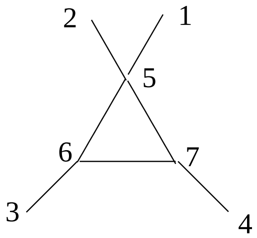
$$p_{56} = \frac{b_{53}(1-x)}{b_{63}} \tag{4.14}$$

$$p_{65} = \frac{b_{61}(1-y)}{b_{51}}. \tag{4.15}$$

We now have a parameterization in terms of the static probabilities of staying at a node. This parameterization characterizes the least amount of information necessary for recovery.

5. Example Two

The next example that I considered was the following structure containing three interior nodes and four boundary nodes.



The transition matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ p_{51} & p_{52} & 0 & 0 & p_{55} & p_{56} & p_{57} \\ 0 & 0 & p_{63} & 0 & p_{65} & p_{66} & p_{67} \\ 0 & 0 & 0 & p_{74} & p_{75} & p_{76} & p_{77} \end{bmatrix}$$

We can also define the following matrices:

$$R = \begin{bmatrix} p_{51} & p_{52} & 0 & 0 \\ 0 & 0 & p_{63} & 0 \\ 0 & 0 & 0 & p_{74} \end{bmatrix}$$

$$(I-Q) = \begin{bmatrix} 1-p_{55} & -p_{56} & -p_{57} \\ -p_{65} & 1-p_{66} & -p_{67} \\ -p_{75} & -p_{76} & 1-p_{77} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{51} & b_{52} & b_{53} & b_{54} \\ b_{61} & b_{62} & b_{63} & b_{64} \\ b_{71} & b_{72} & b_{73} & b_{74} \end{bmatrix}$$

Using these matrices and the equation $R = (I - Q)B$, we can obtain the following useful equations

$$\begin{aligned}
b_{51} - b_{51}p_{55} - b_{61}p_{56} - b_{71}p_{57} &= p_{51} && (5.1)\\
b_{52} - b_{52}p_{55} - b_{62}p_{56} - b_{72}p_{57} &= p_{52} && (5.2)\\
b_{53} - b_{53}p_{55} - b_{63}p_{56} - b_{73}p_{57} &= 0 && (5.3)\\
b_{54} - b_{54}p_{55} - b_{64}p_{56} - b_{74}p_{57} &= 0 && (5.4)\\
-b_{51}p_{65} + b_{61} - b_{61}p_{66} - b_{71}p_{67} &= 0 && (5.5)\\
-b_{52}p_{65} + b_{62} - b_{62}p_{66} - b_{72}p_{67} &= 0 && (5.6)\\
-b_{53}p_{65} + b_{63} - b_{63}p_{66} - b_{73}p_{67} &= p_{63} && (5.7)\\
-b_{54}p_{65} + b_{64} - b_{64}p_{66} - b_{74}p_{67} &= 0 && (5.8)\\
-b_{51}p_{75} - b_{61}p_{76} + b_{71} - b_{71}p_{77} &= 0 && (5.9)\\
-b_{52}p_{75} - b_{62}p_{76} + b_{72} - b_{72}p_{77} &= 0 && (5.10)\\
-b_{53}p_{75} - b_{63}p_{76} + b_{73} - b_{73}p_{77} &= 0 && (5.11)\\
-b_{54}p_{75} - b_{64}p_{76} + b_{74} - b_{74}p_{77} &= p_{74} && (5.12)
\end{aligned}$$

Also, we have the following equations

$$\begin{aligned}
p_{51} + p_{52} + p_{55} + p_{56} + p_{57} &= 1 && (5.13)\\
p_{63} + p_{65} + p_{66} + p_{67} &= 1 && (5.14)\\
p_{74} + p_{75} + p_{76} + p_{77} &= 1 && (5.15)
\end{aligned}$$

Again, writing the equations in matrix form, we obtain the following
(5.16)

$$\begin{bmatrix}
1 & 0 & 0 & 0 & b_{51} & b_{61} & b_{71} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & b_{52} & b_{62} & b_{72} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b_{53} & b_{63} & b_{73} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b_{54} & b_{64} & b_{74} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{51} & b_{61} & b_{71} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{52} & b_{62} & b_{72} & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & b_{53} & b_{63} & b_{73} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{54} & b_{64} & b_{74} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{51} & b_{61} & b_{71}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{52} & b_{62} & b_{72}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{53} & b_{63} & b_{73}\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & b_{54} & b_{64} & b_{74}\\
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
p_{51}\\ p_{51}\\ p_{63}\\ p_{74}\\ p_{55}\\ p_{56}\\ p_{57}\\ p_{65}\\ p_{66}\\ p_{67}\\ p_{75}\\ p_{76}\\ p_{77}
\end{bmatrix}
=
\begin{bmatrix}
b_{51}\\ b_{52}\\ b_{53}\\ b_{54}\\ b_{61}\\ b_{62}\\ b_{63}\\ b_{64}\\ b_{71}\\ b_{72}\\ b_{73}\\ b_{74}\\ 1\\ 1\\ 1
\end{bmatrix}$$

Again, we can examine specific cases, but we will not find much useful information from doing so. Using row reduction, we find that the last three rows are actually absorbed in our boundary considerations. We obtain the following,

(5.17)
$$\begin{bmatrix}
1 & 0 & 0 & 0 & a_{1,5} & a_{1,6} & a_{1,7} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & b_{5,3} & b_{6,3} & b_{7,3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_{5,4} & b_{6,4} & b_{7,4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{5,2} & b_{6,2} & b_{7,2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{5,8} & a_{5,9} & a_{5,10} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{5,4} & b_{6,4} & b_{7,4} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{5,2} & b_{6,2} & b_{7,2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{5,3} & b_{6,3} & b_{7,3} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & a_{9,11} & a_{9,12} & a_{9,13}
\end{bmatrix}$$

Where $a_{i,j}$ are the following,

$$a_{1,5} = b_{5,1} + b_{5,2} - 1, \quad a_{1,6} = b_{6,1} + b_{6,2} - 1, \quad a_{1,7} = b_{7,1} + b_{7,2} - 1$$
$$a_{5,8} = b_{5,3} - 1, \quad a_{5,9} = b_{6,3} - 1, \quad a_{5,10} = b_{7,3} - 1,$$
$$a_{9,11} = b_{5,4} - 1, \quad a_{9,12} = b_{6,4} - 1, \quad a_{9,13} = b_{7,4} - 1.$$

Our new matrix is not invertible; we have four free variables. Once again, we can obtain a parameterization for our solutions, but we cannot actually obtain solutions. It is clear, that we need more than just the boundary information to determine the transitional probabilities for a network with this geometry.