The Axl Theorem

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Abstract

We define an annular graph to be a graph with n concentric circles and m line segments that bisect each of the circles but no two of which intersect. An annular network is a network on a graph that is topologically equivalent to an annular graph. In this paper, we give sufficient conditions for both partial and full recoveries of annular networks, and we characerize the response matrix for such networks.

1 Introduction

Definition 1 *Let* Γ *be isomorphic to a graph with* n *concentric circles and* m *line segments that bisect each circle and which have endpoints of degree 1. Then we say that* Γ *is an* annular graph*. An* annular network *is a network on an annular graph.*

Figure 1: Annular Network with $n = 2$ and $m = 4$

Let Ω denote the set of all nodes in a network. We designate some nodes as boundary nodes, and we say that $\partial\Omega$ is the set of all boundary nodes. We let $\text{int}\Omega$ be the set of all interior nodes, and we get the identity $\text{int}\Omega \cup \partial\Omega = \Omega$. In the case of an annular network, we say that every node of degree 1 is a boundary node.

Definition 2 *We say that a network is* partially recoverable *if we can solve for the conductiviy γ of at least one edge of the network. A network is* completely recoverable *if we can solve the γ-function for every edge in the network.*

The main purpose of sections 2–4 is the recovery, both partial and complete, of annular networks. In particular, we will give an algorithm that leads to a theorem on partial recoverability. This algorithm will repeatedly utilize Kirchoff's law to force voltages at interior nodes based on given boundary voltages.

2 The "2–4" Case

To whet our appetites for the later theorems, we first consider the simple annular network where $n = 2$ and $m = 4$.

This network is recoverable. We show this by taking the following steps:

1. We first want to find the conductivities on every boundary spike. Because of the symmetry of the network, this is actually easier than it may at first seem. Suppose that we set boundary voltages and currents at nodes 2–4 to be zero, and we set the boundary voltage and current at node 1 to be 1. Then we've used up our eight degrees of freedom on the network (we have eight boundary nodes), and so the boundary voltages are determined at nodes 5–8. By Ohm's law, nodes 10–12 have potential zero, since no current is flowing along those respective boundary spikes. Furthermore, by Kirchoff's law, the potential at node 15 is zero. Thus we wish to solve for the four unknown boundary voltages at nodes 5–8. To do this, we take advantage of the Dirichlet-Neumann response matrix.

   Recall that the Dirichlet-Neumann map sends boundary voltages to boundary currents. By setting up our network as we did, we have four given currents and thus four known equations via the Dirichlet-Neumann map. Thus, we may be able to solve for all four boundary voltages. However, we must first check that the system of equations has a unique solution.

We recall from [1] that given a circular pair P, Q that has a unique set of connections, $\det \Lambda(P;Q) \neq 0$. Since the circular pair in which we are interested consists of all outer boundary nodes in P and all inner boundary nodes in Q, it is geometrically clear that there is only one way to connect the circular pair. Thus, $\det \Lambda(P;Q) \neq 0$. Since the submatrix $\Lambda(P;Q)$ is also the matrix of coefficients in our system of four equations, then its non-zero determinent implies that the system has a unique solution. Thus, we now know the boundary voltages at every boundary node. In particular, we know the voltage α at boundary node 7. We find the current at seven from the now completely determined vector $\Lambda\phi$ where ϕ is the vector of boundary voltages. Thus, we have the current I along edge 7,15 and we know the voltage drop over the edge since $u(15) = 0$. Thus, by Ohm's law:

$$\gamma = (\alpha - 0)I = \alpha I \tag{1}$$

Note that there is nothing special about the nodes we used to recover γ. That is, we can use this algorithm to recover any of the boundary spikes because of the symmetry of the network, either by rotation or deformation. Thus, by repeating this algorithm for each boundary node, we can determine the conductivity of every boundary spike in the network.

2. The next step is to find the conductivities of the interior edges. It turns out that our recovery of the boundary spikes is particularly useful here, because in a certain sense, we can treat every node in the interior of Ω_0 as a boundary node. To do this, we use a formula from [2] that modifies the Λ-matrix for a network that has an edge of known conductivity added to the network. In this case, we simply work backwards...

   Since every node in Ω_0' is a boundary node, then the conductivities fall directly out of the Λ-matrix.

We have now completely found the γ-function for the network Γ. Therefore, in the case where $n = 2$ and $m = 4$, the network Γ is recoverable.

3 Partial Recoverability

Given an annular network, we would like to know if the network is recoverable. In this section, we use an algorithm to demonstrate a set of conditions that guarantee recoverability of boundary spikes. In particular, we will give a theorem regarding partial recoverability in annular networks.

Lemma 1 (Duff's Lemma) *Let* Γ *be an annular network with* n *rings and* m *rays, and define the circular pair* P, Q *such that all* m *outer boundary nodes are in* P *and all* m *inner boundary nodes are in* Q*. Then*

$$\det \Lambda(P; Q) \neq 0 \tag{2}$$

To see that this is true, we note geometrically that the only connection for a node $p_i \in P$ is with its corresponding inner boundary node $q_i \in Q$ on ray i. *At this point, I would like to give something more than an intuitive geometric argument as to why we get a contradiction by connecting, for example* p_i *with* q_{i+1}*. Perhaps something with network topology ...*

Since the connection is unique, then the determinent of the submatrix $\Lambda(P; Q)$ is non-zero by [2]. □

Theorem 1 (Axl's Theorem) *Let* Γ *be an annular network such that* $m \geq 2n$*, then the* γ*-function for the network is partially recoverable. In particular, we can recover each and every boundary spike in* Γ*.*

Comment: *This proof will rely on a generalization of the algortihm that we used in the 2-4 case.*

Recall from the "Five Node Property" that the voltage of an internal node in a γ-harmonic network is the weighted average of the neighboring voltages. The easiest way to take advantage of this property is to make all of the nodes have voltage zero (thus eliminating the question of conductivity for the moment). Keeping this in mind, we set a voltage and current of one at node 1 for the graph and a voltage and current of zero for nodes 2–m. Note that now every neighbor of nodes 2–m has voltage of zero, or alternatively we have $m-1$ nodes on ring 1 with voltage zero. By an extension of the Five Node Property, there are $m-3$ internal nodes on ring 2 with voltage zero. Continuing the patern, the number of nodes, z with voltage zero on ring i is given by

$$z = m - 2i + 1 \tag{3}$$

Thus, when $i = n$, we get

$$z = m - 2n + 1 \tag{4}$$

and since $m \geq 2n$,

$$z \geq 2n - 2n + 1 \Rightarrow z \geq 1 \tag{5}$$

We are therefore guaranteed a voltage of zero for at least one neighbor of an inner boundary node. We've also used up our $2m$ degrees of freedom in settin the boundary voltages and currents, so we have m unknown boundary voltages, all of which are on inner boundary nodes. We also have m known currents, so we end up with m equations and unknowns. Placing all outer boundary nodes in P and all inner boundary nodes in Q, the previous lemma tells us that $\det \Lambda(P; Q) \neq 0$, and this submatrix also corresponds to the system of equations that we wish to solve. Therefore, we can now find every boundary voltage, and in particular, we know the boundary voltage α at an inner boundary node that neighbors a node with voltage zero. Call this node a. Since we now completely know the voltage vector, we can also determine the current I_a out of node a. We can now recover the boundary spike by Ohm's Law:

$$\gamma = I_a(\alpha - 0) = I_a\alpha \tag{6}$$

Using the same symmetry argument as we used in the 2–4 case, we have an algorithm for determining the conductivities for every boundary spike. □

4 Complete Recoveries

We currently have algorithms in place for the complete recovery of annular networks where $n = 2, 3, 4$, and as above $m \geq 2n$.

4.1 Case: $n = 2$

We already know that we can recover the boundary spikes. Furthermore, we can use the same transformation as in the 2–4 case to make every internal node a boundary node, thus leading to the recovery of every edge in the network.

4.2 Case: $n = 3$

Once again, we use the fact that the boundary spikes have been recovered to alter the Λ-matrix such that all nodes on rings 1 and 3 are now boundary nodes. We set a voltage of 1 at node 1, a voltage of zero at nodes 2–m, a voltage and current of zero at node $2m$, and a voltage of zero at $m - 2$ other inner boundary nodes (which include nodes $m+1, 2m-1$), thus leaving one inner boundary node to be determined. By Kirchoff's law, the interior node that neighbors nodes m and $2m$ has a voltage of zero. Finally, we know that the determined boundary node has a unique solution since we are solving only one equation with one unknown, so all boundary voltages are now known.

A current will flow from node 1 to node m. Furthermore, node m has a zero voltage, as does its neighbors. As a result, no current flows from node m to its neighbors, which means that all of the current that flows from node 1 to node m flows out of node m. We can measure the current out of node m from the Dirichlet-Neumann map, and so we get the value of γ for the edge $\{1,m\}$ from Ohm's law. By symmetry, then, we can find the values of the γ-function for all edges on rings 1 and 3. Using the ϵ-transformation again to essentially remove rings 1 and 3, we end up with a single ring and $2m$ boundary nodes. By Axl's Theorem, all of the boundary spikes are recoverable, and a final ϵ-transformation of the Λ-matrix recovers the last ring. (It is worth noting that the only number of nodes that would prevent us from recovering a single ring is 2, because this would now be a pair of nodes in series and hence unrecoverable. This is not an issue here, though, since $m \geq 6$.) Thus we now have the complete γ-function for the network.

4.3 Case: $n = 4$

Once again, we know that we can move to a network with no boundary spikes by Axl's Theorem. Thus, we set the boundary nodes on ring 1 as follows:

NODE	Voltage (Current)
1	1
2	0
3	0(0)
4	0(0)
5	0
$\vdots$	$\vdots$
$m-2$	α
$m-1$	β
m	δ

We set the boundary nodes on ring 4 as:

NODE	Voltage (Current)
$m+1$	0
$m+2$	0
$m+3$	0(0)
$m+4$	0
$\vdots$	$\vdots$
$2m$	0

By setting 3 currents, we leave α, β, and δ determined by the network. There is only one way to connect the nodes with determined voltage to the nodes with a given current (this is not obvious, but a simple geometric argument proves it), so the system is solvable. Thus, we now know the voltage at every boundary node. Using the Dirichlet-Neumann map, we find the current out of node 2, which is the same amount of current as we sent on edge {1,2}. This tells us the conductivity of edge {1,2}, and by extension, we have recovered all of rings 1 and 4. Using another ϵ-transformation, we end up with an annular network of 2 rings, which we have already shown to be completely recoverable. Therefore, our 4 ring annular network is completely recoverable. □

5 On the Λ-Matrix

We turn our attention now to properties of the response matrix for an annular network. In particular, we are interested in characteristics that distinguish an annular network from other networks.

Theorem 2 (Slash's Theorem) *Let* Γ *be an annular graph with* m *rays,* n *rings and the usual ordering of boundary nodes, where* (Γ, γ) *is a particular annular network. Define* $P = \{1, 2, \ldots, m\}$ *and* $Q = \{m+1, m+2, \ldots, 2m\}$*. If* m *is odd, then* $\det \Lambda(P;Q) < 0$*. Alternatively, if* m *is even, then* $\det \Lambda(P;Q) > 0$.

Proof: Consider a network (Γ, γ) defined such that every edge on every ray has conductivity $\gamma = 1$ and every edge on a ring has conductivity $\gamma = \epsilon$:

Put drawing of situation here

To make this proof easier, we reorder the nodes of the matrix. This is a typical ray in the network:

Show network with reordered nodes

Obviously, altering P and Q to reflect this reordering will not affect the value of the submatrix $\Lambda(P;Q)$.

Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(\epsilon) = \det \Lambda_\epsilon(P;Q) \tag{7}$$

where Λ_ϵ is the response matrix for a network with the above pattern and a given value of ϵ. We claim that f is continuous.

We first note that we get Λ from the Schur Complement of the Kirchoff matrix. Thus,

$$\det \Lambda = \det(A - BC^{-1}B^T) \tag{8}$$

where

$$\kappa = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

is the Kirchoff matrix. We also note that the only submatrix of the Kirchoff matrix that is affected by the value of ϵ is C, since

$$A = I_n \tag{9}$$

$$B = B^T = -I_n \tag{10}$$

On the other hand,

$$C_\epsilon = \begin{pmatrix} 2+2\epsilon & -\epsilon & 0 & \dots & -\epsilon & -1 & 0 & \dots \\ -\epsilon & 2+2\epsilon & -\epsilon & \dots & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ -\epsilon & 0 & 0 & \dots & 2+2\epsilon & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a $(mn) \times (mn)$ submatrix of κ.

We define $g : \mathbf{R} \to \mathbf{R}$ such that

$$g(\epsilon) = \det C_\epsilon \tag{11}$$

. Since the determinant function in this case is simply a polynomial function of one variable, ϵ, then g is a continuous function on $\mathbf{R}$. In particular, g is continuous at $\epsilon = 0$.

In the case where $\epsilon = 0$, the matrix C_ϵ becomes:

$$C_0 = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 2 & 0 & 0 & \dots \\ 0 & 0 & 2 & -1 & \dots \\ 0 & 0 & -1 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus,

$$\det C_0 = (2 \times 2 - 1)^m = 3^m \neq 0 \tag{12}$$

We now take the case where $\det C_0 \neq 0$ for $n-1$ rings. Given the internal nodes on a ray i where $1 \leq i \leq m$, let $C_{0,i}$ be the $n \times n$ submatrix of C_0 that corresponds to the edges along ray i.

$$C_0 = \begin{pmatrix} C_{0,1} & 0 & \dots & 0 \\ 0 & C_{0,2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{0,m} \end{pmatrix}$$

$$C_{0,i} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & 2 & -1 & \dots & 0 \\ 0 & 0 & 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}$$

The rows of each submatrix $C_{0,i}$ are linearly independent. Therefore, for every $1 \leq i \leq m$,

$$\det C_{0,i} \neq 0 \tag{13}$$

and so

$$\det C_0 = (\det C_{0,1})^m \neq 0 \tag{14}$$

We now know that the matrix C_0 has an inverse C_0^{-1}. Furthermore, since g is continuous and non-zero for values of ϵ close to 0, then the function $h : \mathbf{R} \to \mathbf{R}$ defined by

$$h(\epsilon) = \frac{1}{g(\epsilon)} = \det C_\epsilon^{-1} \tag{15}$$

is also continuous near zero. As mentioned earlier, A, B and B^T are all constants with respect to ϵ. Thus the determinant fuction $\det \Lambda_\epsilon$ is continuous, and so the function f is continuous.

We set aside the continuity of Λ_ϵ for the moment, and instead focus on the case where $\epsilon = 0$. Recall that every entry $\lambda_{i,j}$ of the Λ-matrix represents the current into boundary node j when boundary node i has voltage 1 and every other boundary node has voltage 0. We get

$$\Lambda_0 = \begin{pmatrix} -\frac{1}{n+1} & 0 & 0 & \dots \\ 0 & -\frac{1}{n+1} & 0 & \dots \\ 0 & 0 & -\frac{1}{n+1} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus,

$$\det \Lambda_0(P;Q) = \left(-\frac{1}{n+1}\right)^m \tag{16}$$

which is clearly non-zero for all m, n. Furthermore, this determinant is clearly negative when m is odd and positive when m is even. The continuity of Λ_ϵ allows us to find a neighborhood W around $\epsilon = 0$ such that for any $\epsilon \in W$, Λ_ϵ has the same sign as Λ_0.

We must now show that for any γ-function on the network, the sign of $\det \Lambda_\gamma(P;Q)$ is the same as $\det \Lambda_\epsilon(P;Q)$. We use contradiction. Suppose WLOG that m is even so that $\det \Lambda_\epsilon(P;Q) > 0$, and suppose that there exists a γ-function such that $\det \Lambda_\gamma(P;Q) < 0$. We create a function $p : [0,1] \to \mathbf{R}$ such that

$$p(t) = \det \Lambda_{t\epsilon+(1-t)\gamma}(P;Q) \tag{17}$$

The determinant function is continuous, so by the Intermediate Value Theorem there exists a value of t on the interval $[0,1]$ such that $p(t) = 0$. This means that the conductivity function $t\epsilon + (1-t)\gamma$ yields a network with response matrix Λ' such that $\det \Lambda'(P;Q) = 0$. This contradicts Duff's Lemma. Therefore, for any γ-function on the network, $\det \Lambda_\gamma(P;Q)$ has the same sign as $\det \Lambda_0(P;Q)$. $\square$