A NATURAL REPRESENTATION FOR A CLASS OF CIRCULAR PLANAR GRAPHS

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1 Introduction

This paper stems from an observation that all of the methods of Curtis and Morrow in [1] rely on the representation of a circular planar graph in the disc. Traditional graph theory, on the other hand, has used a set-theoretic construction which doesn't say anything about how to represent the graph. There is not an immediately obvious way, given an abstract circular planar graph, to represent it in the disc, and so the question becomes to what degree the prior results come from graph-theoretic properties. This paper shows that there is, in some sense, a natural way of representing a circular planar graph.

The idea is that we regard the graph as an electrical network, assigning positive conductivities to each edge. We then apply a potential to each of the boundary vertices, and we will thus induce a potential at each of the interior vertices. If we regard the potential as a coordinate, applying this procedure once for the x-coordinate and once for the y-coordinate will give, for each node, a point in the plane. Since the conductivities are real-valued, we can use complex potentials, with the properties developed in section 3 of this paper, and solve only one linear algebra problem.

This idea is similar to one developed by Tutte ([2], [3]) for what he calls "nodally 3-connected non-separable graphs," and my results could be, to a certain degree, derived from his, but his terminology is often obscure and we present here an independent and, we hope, clearer derivation.

The major theorem of this paper is that, if the boundary potentials are coordinates of points that lie on the boundary of a strictly convex planar region, and are in the right order along that region, then the interior potentials will give you a natural representation for a large class of graphs. The critical graphs defined by Curtis and Morrow [1] form a proper subset of this collection.

We then go on to prove a necessary (but not sufficient) condition on the order of boundary nodes to give a representation in the disc. This condition is satisfied by all graphs which can be represented in the disc, and by all representations in any strictly convex planar region.

2 Definitions

Definition 2.1 *a graph with boundary is a triple $(V, E, \partial V)$ where V denotes the set of vertices, E denotes the set of edges, and ∂V is a nonempty subset of V which we call the set of boundary vertices. In this paper we shall refer to graphs with boundaries as simply graphs for simplicity.*

Definition 2.2 *a subgraph $G' = (V', E', \partial V')$ of a graph $G = (V, E, \partial V)$ satisfies the following properties*

- *(1) $V' \subset V$*
- *(2) $E' \subset E$, e is an edge of E' if and only if the ends p and q of e are both in V'.*
- *(3) $p \in \partial V'$ iff $p \in \partial V$ or p has a neighbor vertex q that is not in V'.*

With these definitions, we consider different types of graphs.

Definition 2.3 *a path $p \to q$, for $p, q \in V$, is a finite sequence $r_0, r_1, \ldots r_{k-1}, r_k$, where for all i, $r_i \in V$, r_i and r_{i+1} are joined by an edge in E, $p = r_0$, and $q = r_k$.*

Definition 2.4 *a connected graph $G = (V, E, \partial V)$ is a graph such that, for all $p, q \in V$, there is a path $p \to q$.*

Definition 2.5 *a linear graph is a connected graph with two boundary vertices of valence 1, and every interior vertex has valence 2.*

Definition 2.6 *a unipolar graph is a graph $G = (V, E, \partial V)$ where $|\partial V| = 1$. A bipolar graph is a graph where $|\partial V| = 2$.*

We must also specify what we mean by representing a graph.

Definition 2.7 *a vertex map F of $G = (V, E, \partial V)$ is a 1-1 function $F : V \to \mathbb{C}$*

Definition 2.8 *an edge map H of $G = (V, E, \partial V)$ is a function $H : E \to \mathcal{P}(\mathbb{C})$ such that $\forall e \in E : H(e) \neq \emptyset$.*

Definition 2.9 *an edge map H of $G = (V, E, \partial V)$ is in accord with (a vertex map) F if*

- *(i) If e and f are distinct edges incident to a vertex p, $H(e) \cap H(f) = \emptyset$*
- *(ii) If e and f are non-parallel edges incident to a vertex p, then $\overline{H(e)} \cap \overline{H(f)} = \{F(p)\}$*
- *(iii) for all $e \in E$, $\overline{H(e)}$ is homeomorphic to $[0, 1]$*

Definition 2.10 *an induced edge map H of (a vertex map) F is defined by $H(e) = \{z \in \mathbb{C} : z = \lambda F(p) + (1 - \lambda)F(q) \text{ for } \lambda \in (0, 1)\}$ if p and q are the ends of e.*

Note that the induced edge map of a vertex map F may not be an edge map in accord with F. Namely, property (i) might be false if, for example, there are vertices p, q, and r such that p and q are ends of one edge and p and r are ends of another, and the vertex map F is such that $F(q)$ is on the line segment joining $F(p)$ and $F(r)$ (see figure 1).

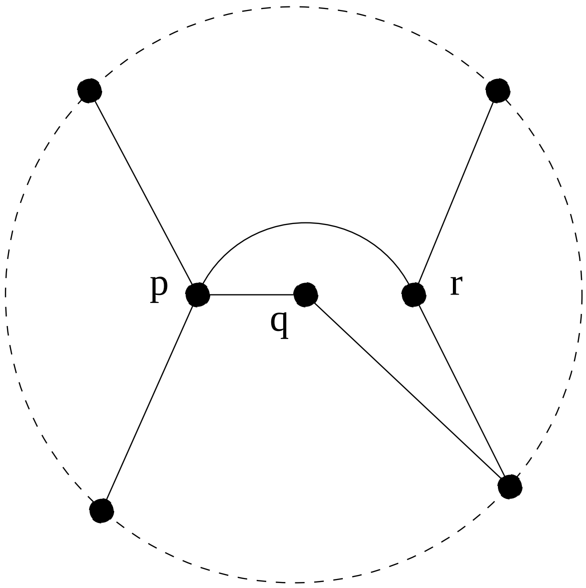

Figure 1: The induced edge map of F need not be in accord with F

Definition 2.11 *a representation function $R = (F, H)$ of a graph G into a bounded connected planar region $S \subset \mathbb{C}$ is an ordered pair such that*

- *(i) F is a vertex map of G*
- *(ii) H is an edge map of G in accord with F*
- *(iii) F maps ∂V into ∂S and maps int V to int S.*
- *(iv) for all e, $f \in E$, if $x \in \overline{H(e)} \cap \overline{H(f)}$, then there is a $v \in V$ with $x = F(v)$, and v is an end of e and f*
- *(v) $\bigcup_{e \in E} H(e) \subset S$*

Definition 2.12 *a representation $R(G)$ of a graph G with a representation function $R = (F, H)$ is defined to be $\bigcup_{e \in E} \overline{H(e)}$.*

Note that by these definitions a representation function (F, H) where H is the induced edge map of F is an example of what Tutte calls a "straight representation" ([2]), but includes information about the boundary of the graphs.

Definition 2.13 *a circular planar graph G is a graph such that there exists a representation function from G into the unit disc.*

Finally, we need a little terminology to deal with representations due to electrical problems. We use the definitions of conductivity and resistor network as found in [1].

Definition 2.14 *a connected graph G is called representable by γ in a strictly convex bounded planar region S if there exists a potential function $\phi : \partial V \to \partial S$ on the boundary of the electrical network $\Gamma = (G, \gamma)$ such that the function F_γ that takes vertices to their potential is a vertex map, the edge map H_γ induced by F_γ is in accord with F_γ, and $R_\gamma = (H_\gamma, F_\gamma)$ is a representation function of G into S. The existence and uniqueness of F_γ are guaranteed by the proof of Theorem 3.2 of [1] and the discussion at the beginning of section 3 of this paper.*

Definition 2.15 *a graph G is called representable if it is representable by γ in S for any conductivity γ and any strictly convex bounded planar region S. It is not clear that there are any such graphs, at the outset, but in section 6 we shall prove that this definition is actually equivalent to the preceding one.*

3 Complex Potential Theory

¿From the proof of Theorem 3.2 in [1], given a resistor network $\Gamma = (G, \gamma)$ and a function $f : \partial V \to \mathbb{R}$, there is a unique function $u : V \to \mathbb{R}$ such that $Ku = b$, where K is the Kirchhoff matrix for Γ, and b is the vector containing the currents generated by f at the boundary of G and 0 for all other entries. Moreover, this vector u depends linearly on f, so if u is the potential vector induced by f, and v is the potential vector induced by g, then $au + bv$ is the potential vector induced by $af + bg$. From an arithmetical point of view, nothing changes if we allow a and b to be complex, so we define $u + iv$ to be the potential vector induced by $f + ig$. Note that the conductances are still postive real numbers.

Before we proceed, we must develop some terminology about convex planar regions. Given a convex planar region S, we defined a point $x \in S$ to be an *extreme point* of S if the equation $x = cy + (1-c)z$ for $0 < c < 1$ is only satisfied when $x = y = z$. The *convex hull* of a set Z is the set of points $\{x : x = cy + (1-c)z \text{ for some } y, z \in Z, 0 \le c \le 1\}$. Also, if $Z = \{z_1, z_2 \ldots z_n\}$, any point x in the convex hull of Z can be represented as $x = \sum_{i=1}^{n} c_i z_i$, where for each i, $0 \le c_i$ and $\sum_{i=1}^{n} c_i = 1$. We define the *relative interior* of a convex hull of a finite set Z to be the set obtained by requiring $0 < c_i$ for each i in the preceding observation.

¿From the perturbability of the coefficients c_i which define the relative interior, we can get a geometric picture of the concept. We must proceed by cases.

When Z consists of a single point z, then the convex hull of Z and the relative interior of that convex hull both also consist solely of z. When Z consists of two distinct points, call them y and z, or of more than two colinear points on the line segment between y and z, then the convex hull of Z consists of the points $x = cy + (1-c)z$ with $0 \leq c \leq 1$, in other words of the line segment connecting y and z. The relative interior of this convex hull is the line segment without its endpoints. When Z contains at least three non-colinear points, the convex hull of Z is the smallest convex polygon which contains all the points of Z, and its relative interior coincides with its geometric interior when regarded as a region in the plane.

The point of this discussion has been to develop a vocabulary in which we can easily state the following maximum principle for complex potentials:

Theorem 3.1 (Maximum Principle for Complex Potentials) *Let $\Gamma = (G, \gamma)$ be a resistor network, p be an interior vertex of Γ, and u be a complex potential function on Γ. Then $u(p)$ is in the relative interior of the convex hull of the potentials at its neighbor vertices. That is, for every p, denote by $\mathcal{N}(p)$ the set of vertices q for which p and q are ends of an edge in Γ, and by $u(\mathcal{N}(p))$ the potentials obtained by vertices in $\mathcal{N}(p)$. Then for all interior vertices p, $u(p)$ is in the relative interior of the convex hull of $u(\mathcal{N}(p))$.*

Proof: Kirchhoff's law for interior nodes, namely that there is no net current out of any interior node, can be written arithmetically as

$$\sum_{q \in \mathcal{N}(p)} \gamma(pq)(u(q) - u(p)) = 0$$

If we move all the occurences of $u(p)$ to the other side of the equation, and if we let $S = \sum_{q \in \mathcal{N}(p)} \gamma(pq)$, we have

$$u(p) = \frac{1}{S} \sum_{q \in \mathcal{N}(p)} \gamma(pq) u(q)$$

If p has n neighbor vertices, which we number $q_1 \ldots q_n$, and we let $c_i = \frac{\gamma(pq_i)}{S}$, then the positivity of $\gamma(pq)$ for all edges implies that $c_i > 0$, and the definition of S guarantees that $\sum_{i=1}^{n} c_i = 1$, so by the definitions of convex hull and relative interior, we have that for all interior nodes p, $u(p)$ is in the relative interior of the convex hull of $u(\mathcal{N}(p))$.

Corollary 3.1 *Let Γ be an electrical network, p be an interior node of Γ, and u be a potential function on Γ. Then $u(p)$ is in the relative interior of the convex hull of the boundary potentials, $u(\partial V)$.*

Proof: Repeated application of the above theorem.

4 Continuity of the Forward Dirichlet Matrix

In this section we describe the dependence of a certain matrix on the conductivities of a circular planar network. Specifically, as defined in [1], the Kirchhoff matrix of a network $\Gamma = (G, \gamma)$ is a matrix such that, if c represents the currents out of every node and u is the vector of potentials at each node, $Ku = c$. K is symmetric, and if the nodes are ordered such that the boundary nodes precede the interior nodes, K has a block structure:

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

In this block structure, A represents the boundary-boundary edges, B represents the boundary-interior edges, and C represents the interior-interior edges. If we let ϕ denote the potentials at the boundary nodes and ψ denote the potentials at the interior nodes, then the potential vector u has the corresponding block structure $u = \begin{bmatrix} \phi \\ \psi \end{bmatrix}$, and Kirchhoff's law that the current out of the network at any interior node is zero can be summarized by block multiplication: $B^T\phi + C\psi = 0$. Since C is invertible (from Lemma 3.1 of [1]), we can express the dependence of ψ on ϕ as $\psi = -C^{-1}B^T\phi$. If we let $D = -C^{-1}B^T$, then $\psi = D\phi$, and D can be called the forward Dirichlet matrix. Thus ψ depends linearly on ϕ, and therefore continuously.

It is clear from the definitions of K and D that D depends on the conductance of an individual edge in a rational way. Thus, by Lemma 3.1 of [1], C is invertible for any connected circular planar graph, and it is clear that D depends continuously on any individual conductance as long as that conductance is positive. The question then becomes whether or not this continuity is preserved as individual conductances go to zero or to infinity. Setting a conductance to be zero is electrically equivalent to deleting the edge, and if deleting the edge gives a graph that is itself connected, then, again from Lemma 3.1 of [1], the new C matrix is still invertible. If, on the other hand, the new graph is disconnected, then a reordering of the interior nodes will give a block structure

$$C = \begin{bmatrix} C_I & 0 \\ 0 & C_{II} \end{bmatrix}$$

where C_I and C_{II} are the interior-interior portion of the Kirchhoff matrices for the two disjoint subgraphs. Since each of these subgraphs is a connected circular planar graph, then both C_I and C_{II} are invertible, and thus C is still invertible. Thus the matrix $D = -C^{-1}B^T$ is still defined, and since it depends rationally on the conductance, this means that D varies continuously with an individual conductance as it varies over all non-negative numbers (because rational functions don't have jump discontinuities).

When we ask whether D varies continuously with an individual conductance as that conductance goes to infinity, we are really asking if there is a finite limit that is approached. Since D is a rational function, it suffices to show that each conductance σ is present to greater or equal degree in the denominators of each element than in the numerators. We must consider separately whether the edge we are interested in is a boundary-interior or interior-interior edge (if it is a boundary-boundary edge, D doesn't depend on σ at all, and hence varies continuously). First, if the edge e with conductance σ is an interior-interior edge, then the corresponding matrix C looks like

$$C = \begin{bmatrix} \ddots & \vdots & & \vdots & \\ \cdots & \sigma + a & \cdots & -\sigma & \cdots \\ & \vdots & \ddots & \vdots & \\ \cdots & -\sigma & \cdots & \sigma + b & \cdots \\ & \vdots & & \vdots & \ddots \end{bmatrix}$$

where a and b are constants with respect to σ. Clearly, then, $\det C$ has a σ^2 term and no term of higher degree. Thus, C^{-1} approaches a limit as $\sigma \to \infty$. Since σ does not occur in B, clearly $D = -C^{-1}B^T$ approaches a limit as $\sigma \to \infty$.

The case when e is a boundary-interior edge is even simpler. Since σ occurs in only one term in C, then every element of C^{-1} except those in one row and one column depends in a linear fractional manner on σ, and the elements in that row and column do not have a σ in their numerator, but only in their denominator. Since σ occurs once in B, it is not hard to see that it is the column that lacks a σ in the denominator that gets multiplied by a σ in B in order to form the product $D = -C^{-1}B^T$. Thus, D approaches a limit as $\sigma \to \infty$.

Notice that allowing a conductance σ of an edge e to go to infinity is electrically equivalent to contracting the edge e.

What we have thus demonstrated, through this discussion, is the following lemma:

Lemma 4.1 (Continuity Lemma) *Let $\Gamma = (G, \gamma)$ be a circular planar network, fixing γ except for the conductance σ of an edge e. The resulting forward Dirichlet matrix D varies continuously with σ as σ varies over $[0, \infty)$ and approaches a finite limit as $\sigma \to \infty$.*

5 A Geometric Class of Circular Planar Graphs

The major results of this paper, in section 6, do not apply to all circular planar graphs, but rather only to a subset defined by a certain geometric property. The definition of this property is motivated by an observation about which graphs are not representable. From Corollary 3.1 it follows that a unipolar graph or subgraph will always have constant potential, so any attempt to represent it using potentials will always map the entire graph to a single point. Similarly, a bipolar graph will always have potentials lying along the line segment connecting the potentials of the two boundary vertices. It is clear, therefore, that any graph containing a nontrivial unipolar subgraph or a nonlinear bipolar subgraph will not be representable. This section defines the geometric property and proves that it is equivalent to requiring that a graph have neither of these types of subgraphs.

First, we need a set of combinatoric manipulations that will change one graph into another. For one, we can add a *boundary pendant*. That is, given a graph $G = (V, E, \partial V)$, we can add a vertex p to form a new graph G' in the following way: we select a boundary vertex $q \in \partial V$, and let $G' = (V + p, E + pq, \partial V + p)$. We can also add an edge between two distinct vertices which are not already joined by a linear bipolar subgraph (loops are not allowed). This is called *splitting a cell* and is the opposite of deletion. Finally, we can replace one interior vertex p with two interior vertices p_1 and p_2, such that each edge incident to p is now incident to either p_1 or p_2 (but not both), p_1 and p_2 each is a boundary vertex or has a valence 2 or greater. This is called *splitting a vertex* and is the opposite of contraction of an edge that isn't part of a Δ.

We also need a graph that is, in some sense, basic. This is accomplished by the simplest non-trivial graph, with two boundary vertices and a single edge connecting them. We shall refer to this graph as the *base graph*. It is depicted in figure 2, and is clearly circular planar.

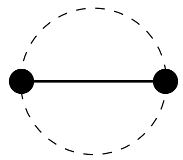

Figure 2: The base graph

With this terminology, we can proceed to the definition of the geometric property we are interested in:

Definition 5.1 *A circular planar graph G is said to have the* <u>*geometric property*</u> *if there is a sequence of graphs $G_0, G_1, \ldots G_n = G$ such that G_0 is the base graph, and for each i, G_{i+1} is obtained from G_i by adding a boundary pendant or splitting a vertex or a cell.*

Note that if G is a circular planar graph, every graph in such a sequence must be circular planar, because the manipulations only add edges; they don't delete them. Clearly, then, every graph G_i in such a sequence must also have the geometric property. Specifically, the base graph has the geometric property, and will be used as the base case in the inductive proofs in the following section. Notice also that no graph that has the geometric property will have either a nontrivial unipolar subgraph (to do so would require adding an interior pendant) or a nonlinear bipolar subgraph (to do so would require adding either an interior pendant or an edge between two nodes joined by a linear bipolar graph). The converse of this statement, that no graph with a nontrivial unipolar subgraph or a nonlinear bipolar subgraph has the geometric property, is proven in the next lemma, and shows that the geometric property is the analog for circular planar graphs of Tutte's conditions for non-separable nodally 3-connected graphs ([3]).

Lemma 5.1 *If a connected graph G has no non-trivial unipolar subgraphs and no non-linear bipolar subgraphs, then it has the geometric property.*

Proof: We proceed inductively. Clearly the base graph satisfies the hypothesis, and has the geometric property. We then assume that all graphs with fewer than N edges and which have no non-trivial unipolar subgraphs or non-linear bipolar subgraphs have the geometric property. Then we need to prove that given any graph G with N edges, there is at least one edge that can be removed to give a graph G' with $N-1$ edges which satisfies the hypothesis.

If G has any pair of edges in series (that is, an interior vertex of valence 2), then you could contract one of these edges to get G'. If G has any boundary-boundary edges, one could be deleted to obtain G' (or contracted if it is a boundary pendant). Otherwise, every interior vertex in G has valence at least 3 and there are no boundary-boundary edges. If G has any edge that is not part of a Δ, you can contract it without generating a non-linear bipolar subgraph, thus obtaining a G' with $N-1$ edges and which, by the inductive hypothesis, has the geometric property. Otherwise, every edge in G is part of a Δ, and any interior-interior edge could be deleted without generating a non-linear bipolar subgraph, thus obtaining G'. But any graph G wherein every edge is part of a Δ and without any boundary-boundary edges has to have an interior-interior edge (otherwise it has no Δs).

6 Dirichlet Representation of Graphs

The previous section started with the observation that any graph which contains a nontrivial unipolar subgraph or a nonlinear bipolar subgraph is not representable. Now that we have developed the basic terminology and preliminary results, we can prove the converse statement, and thus the major result of this paper: that a graph G is representable if and only if it has no nontrivial unipolar subgraphs and no nonlinear bipolar subgraphs (Theorem 6.3). The result will follow from other, related results.

Theorem 6.1 *Given a circular planar graph G with the geometric property and a strictly convex bounded planar region S, there is a conductivity γ such that G is representable by γ in S.*

Proof: We proceed by induction on the number of edges of G. The base graph is trivially representable by any γ in any strictly convex planar region S. Pick the conductance of this edge to be any positive real number. Let N be the number of edges in G. Now we assume, inductively, that any circular planar graph with the geometric property and fewer than N edges is representable by some γ in S. By the definition of the geometric property there is a sequence of graphs $G_0, \ldots G_{N-1}, G_N = G$ such that G_0 is the base graph and G_i is obtained from

G_{i-1} by adding a boundary pendant, or splitting a cell or a vertex. In specific, G_{N-1} has the geometric property, and has $N-1$ edges, so by the inductive hypothesis there is a conductivity γ' such that G_{N-1} is representable by γ' in S. There are three ways we could have obtained G from G_{N-1}; we proceed by cases.

- If G was obtained from G_{N-1} by adding a boundary pendant, the interior potentials do not depend on the conductance of the new boundary-boundary edge. Since G is finite, there is a neighborhood about the vertex q that doesn't contain any other vertices. We assign p a potential that is on ∂S and in that neighborhood, and we choose the conductance of pq to be any positive real number. Thus there is a 1-1 boundary potential function ϕ and a conductivity γ such that G is representable by γ in S.

- If G was obtained from G_{N-1} by splitting a cell, let m be the number of internal vertices (numbered $v_1 \ldots v_m$) of G. The finiteness of G guarantees that there exist positive real numbers $\epsilon_1 \ldots \epsilon_m$ such that, if F denotes the vertex map from G_{N-1} into S, $F(v_i)$ can be perturbed within a neighborhood of radius ϵ_i for each internal vertex v_i, and F will remain a vertex map of G_{N-1} into S, the induced edge map H will be in accord with F, and $R = (F, H)$ will be a representation function of G_{N-1} in S. We add the edge e to G_{N-1}, obtaining G, but declare the conductance of e to be zero. Now, by Lemma 4.1, there is a number $\delta > 0$ such that, if the conductance of e is less than δ, $F(v_i)$ will be perturbed less than ϵ_i for all i. So we let the conductance of e be $\frac{\delta}{2}$, and thus there is a conductivity γ such that G is representable by γ in S.

- If G was obtained from G_{N-1} by splitting a vertex p into p_1 and p_2, let F denote the vertex map of G_{N-1}. By an argument parallel to the one for splitting a cell and by Lemma 4.1, there is a large positive number R such that, if the conductance of the edge joining p_1 and p_2 is larger than R, then $F(v)$ for every interior vertex v will be perturbed by a small amount. The only thing that remains left to be checked is that the vertices p_1 and p_2 don't separate in such a way that any edges meet where they're not supposed to. Since G is circular planar and is derived from G_{N-1} by splitting p, there must be a pair of rays emanating from $F(p)$ with the following properties: the union of the two rays splits S into two disjoint nonempty parts S_1 and S_2; the image $F(q)$ of each neighbor q of p_1, with the exception of p_2, lies in S_1, and likewise $F(q)$ for each neighbor q of p_2, with the exception of p_1, lies in S_2. See figure 3. If the union of the rays forms a line, then the maximum principle guarrantees that if the conductance of the edge joining p_1 and p_2 is high enough, then p_1 will be in S_1 and p_2 will be in S_2, so there is a conductivity γ such that G is representible by γ. If, on the other hand, the two rays do not form a line, then one of the two regions, wlog S_1, will have an acute corner at $F(p)$. If the conductance of the edge joining p_1 and p_2 is high enough, then $u(p_1)$ will not be in the relative interior of $u(\mathcal{N}(p_1) - p_2)$, so it is clear that no edge incident to p_1 intersect any edge incident to p_2. Thus there is a conductivity γ such that G is representible by γ. This completes the proof.

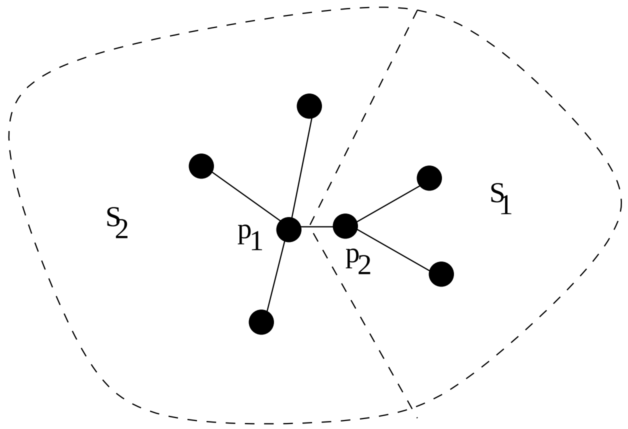

Figure 3: splitting a vertex

Note that the preceding theorem only guarantees the existence of a conductivity γ, and the proof implies the existence of conductivities with very large and very small conductances. We have no bound on how large or small we may need a conductance to be; we are only guaranteed that it is positive and finite. The next theorem, then, establishes that the conductivity itself is unimportant, and thus that we can select any conductivity we like.

Theorem 6.2 *Let G be a graph and S be a bounded strictly convex planar region such that G is representable by γ in S for some γ. Then G is representible by γ in S for any conductivity γ.*

Proof: Let γ_0 be a conductivity such that G is representable by γ_0 in S. We fix $\gamma = \gamma_0$ except for a single conductance σ of an edge e. We denote by σ_0 the conductance of the edge e in the conductivity γ_0. The proof of the Theorem 6.1 guarrantees the perturbability of the interior potentials, and Lemma 4.1 thus guarrantees that the set of σ such that G is representable by γ in S is an open set. We shall show by contradiction that this set is all of $\mathbb{R}^+$.

As a bit of notation, we shall denote the Dirichlet representation of G due to the conductivity γ by $R_\sigma = (F_\sigma, H_\sigma)$ to emphasize that we are only allowing the conductance σ to vary.

If the set $\Sigma = \{\sigma : G \text{ is representable by } \gamma \text{ in } S\}$ is bounded above, let $\mu = \sup \Sigma$. This μ is clearly finite, so the potentials F_μ still satisfy the Maximum Principle (Theorem 3.1). However, since Σ is an open set, then R_μ is not a representation function of G into S. Since $R_{\mu-\epsilon}$ is a representation function of G into S for all small $\epsilon > 0$, and since Lemma 4.1 guarantees that R_σ

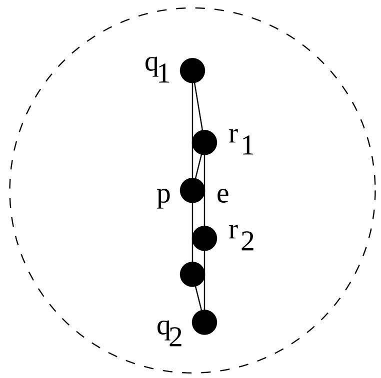

Figure 4: A portion of $R_{\mu-\epsilon}$ if $F(p) \in H(e)$

varies continuously with σ, then either F_μ must not be a vertex map of G, or $F_\mu(p) \in H_\mu(e)$ for some $p \in \text{int}V, e \in E$. We consider the latter possibility first.

Since G is connected, and since $R_{\mu-\epsilon}$ is a representation function for small ϵ, the Maximum Principle can only be satisfied if $F_\mu(q)$ is on the line of which $H_\mu(e)$ is a segment whenever $q \in \mathcal{N}(p)$. See figure 4. Let r_1 and r_2 denote the ends of edge e. If r_1 is not a neighbor of p, and indeed perhaps even if it is, then there is an edge f such that $F_\mu(r_1) \in H_\mu(f)$, and we can repeat this process for r_1 and r_2 and their neighbors. The end is only reached when we find two vertices q_1 and q_2 such that all linear subgraphs of G that are sufficiently large and that go through either the vertex p or the edge e must go through q_1 and q_2. The process is guaranteed to stop, since G is finite, and q_1 and q_2 might be boundary vertices. The repeated application of the Maximum Principle guarantees that there is no vertex q whose potential $F_\mu(q)$ is on the open line segment connecting $F_\mu(q_1)$ to $F_\mu(q_2)$ but which has a neighbor whose potential is not also on the open line segment. In other words, if we take the subgraph of G whose vertices consists precisely of the vertices q_1, q_2, and the vertices q such that $F_\mu(q)$ lies in the line segment between $F_\mu(q_1)$ and $F_\mu(q_2)$, then it is a nonlinear bipolar subgraph of G, which contradicts our hypothesis that G is representable by γ_0 in S, since graphs with nonlinear bipolar subgraphs are not representable by any conductivity in any planar region S. So if F_μ is a vertex map, then R_μ is a representation function of G into S.

The only way F_μ could not be a vertex map is if there exist two vertices p_1 and p_2 such that $F_\mu(p_1) = F_\mu(p_2)$. Let $I = \{p : F_\mu(p) = F_\mu(p_1)\}$. Since the potential of boundary vertices is fixed and there aren't any nontrivial unipolar subgraphs (because G is representable by γ_0), then the Maximum Principle ensures that no member of I is a boundary vertex. Since $R_{\mu-\epsilon}$ is a representation function of G into S for all small ϵ, and since R_σ varies continuously with σ, the Maximum Principle implies that $F_\mu(q)$ lies on a certain line (which passes through $F_\mu(p_1)$) whenever q has a neighbor $p \in I$. Since there are no unipolar

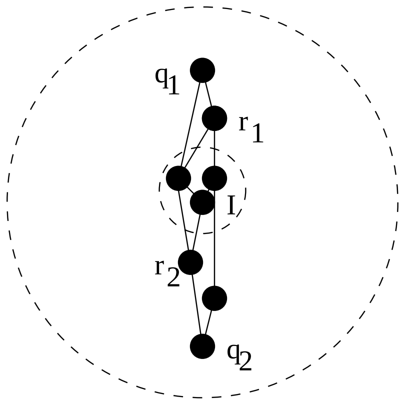

Figure 5: A portion of $R_{\mu-\epsilon}$ if F_μ is not a vertex map

subgraphs in G, there are at least two vertices r_1 and r_2 that are not in I but that have neighbors in I. If these two vertices are the only vertices which are not in I but have neighbors in I, then the subgraph of G with vertices consisting of r_1, r_2, and the vertices in I form a nonlinear bipolar subgraph (a quick arithmetic check ensures that this situation cannot happen with finite conductivities in a linear bipolar subgraph). If r_1 and r_2 are not the only vertices not in I but with neighbors in I, then there is either a vertex p and an edge e such that $F_\mu(p) \in H_\mu(e)$, or there is another pair of vertices with equal potential. See 5. In the former case, the arguments in the preceding situation construct a nonlinear bipolar subgraph of G, and in the latter case, we can repeat this process until we come across two vertices q_1 and q_2 such that every linear subgraph of sufficient size through a vertex in I must pass through q_1 and q_2. This means that there is a nonlinear bipolar subgraph of G with boundary vertices q_1 and q_2. The presence of a nonlinear bipolar subgraph of G contradicts the hypothesis that G is representable by γ_0 in S.

Thus F_μ must be a vertex map of G into S, and R_μ is a represenation function of G into S. But this means that $\mu \in \Sigma$, which contradicts the definition of μ as the supremum of Σ since Σ is open. It is clear, then, that Σ is not bounded above. The same arguments can be made about the infimum of Σ, substituting the word 'positive' for 'finite' and the appropriate inequalities. This shows that we can vary σ over $\mathbb{R}^+$ and G will still be representable by γ in S. The full theorem follows by changing the conductivity γ_0 into γ one conductance at a time, which is possible since G is finite.

Now, the main theorem of this paper can be stated and proven succinctly:

Theorem 6.3 *A connected circular planar graph G is representible if and only if it has no nontrivial unipolar subgraphs and no nonlinear bipolar subgraphs.*

Proof: If G does not have any nontrivial unipolar subgraphs or nonlinear bipolar subgraphs, by Lemma 5.1 it has the geometric property, and thus by Theorem

6.1 for each strictly convex bounded planar region S there is a conductivity γ such that G is representable by γ in S. By the preceding theorem, for each S, G is representable by γ in S for all conductivities γ, and thus G is representable by γ in S for all γ and all strictly convex bounded planar regions S. Thus, G is representible. The converse is obvious because if G has a nonlinear bipolar subgraph or a nontrivial unipolar subgraph, then it is not representible by γ in S for any conductivities γ or any planar region S. Thus it cannot be representible.

7 Boundary Orders

All the preceding discussion has assumed that the potentials of the boundary vertices have been well chosen so that the induced potentials on the interior form a vertex map, etc. From the proof of Theorem 6.2, it is apparent that if a boundary potential function works for one conductivity, it works for all conductivities. Furthermore, it is relatively simple to prove, given the results and methods in the previous section, that given a strictly convex region and a Dirichlet representation of a graph G in that region, that you can vary the boundary potentials of G along the boundary of the strictly convex region, and as long as the boundary vertices occur in the same order as one traverses ∂S, then the Dirichlet representation with the new boundary potential is also a representation function of G in S.

The key question then becomes in what order one should put the boundary vertices along ∂S. Given a graph $G = (V, E, \partial V)$, there is nothing to indicate an order. This section, then, establishes some facts about the order in which the boundary vertices must appear. We begin with some relevant definitions.

Definition 7.1 *For $G = (V, E, \partial V)$ and $p,\ q \in V$, $\underline{d_G(p, q)}$ is the length of the shortest path $p \to q$.*

Since the shortest path is obviously non-self-intersecting, and the set of non-self-intersecting paths $p \to q$ is finite, $d_G(p, q)$ is well-defined if G is connected. Also, since reversing the order of a path $p \to q$ gives a path $q \to p$, $d_G(p, q) = d_G(q, p)$. It is equally clear that the triangle inequality must hold, namely that $\forall p, q, r \in V, d_G(p, q) \leq d_G(p, r) + d_G(r, q)$.

Definition 7.2 *A boundary order P of G is a sequence $\{p_i\}$ of boundary vertices of G such that each boundary vertex occurs precisely once in P. We label the vertices circularly, so if $N = |\partial V|$, $p_{N+k} = p_k$.*

Definition 7.3 *The norm of a boundary order $P = \{p_i\}$ on a connected graph G is $\|P\|_G = \sum_{i=1}^{N} d_G(p_i, p_{i+1})$, where $N = |\partial V|$.*

Definition 7.4 *A perimeter π_P of a boundary order $P = \{p_i\}$ on a connected graph G is a path $p_1 \to p_2 \to \ldots \to p_N \to p_1$, the length of which is $\|P\|_G$.*

Definition 7.5 *Given a representation $R(G)$ of a connected circular planar graph G in the unit disc, the* <u>*induced boundary order*</u> *$P = \{p_i\}$ of $R(G)$ is a boundary order on G such that for each i, the circular arc $p_i p_{i+1}$ contains no other boundary vertex.*

Definition 7.6 *The* <u>*order set*</u> *$\mathcal{P}(G)$ is the set of boundary orders on G. If G is finite, so is $\mathcal{P}(G)$.*

Now that we have the language, we can state and prove a property of induced boundary orders:

Theorem 7.1 *Let $G = (V, E, \partial V)$ be a connected circular planar graph, and let $P_0 = \{p_i\}$ be an induced boundary order of a representation $R(G)$ in the unit disc. Then $\forall P \in \mathcal{P}(G) : \|P\|_G \geq \|P_0\|_G$.*

Proof: Let π_0 be a perimeter of P_0 on G, and let π be a perimeter of P on G. Let G' be the graph consisting solely of the vertices in π and the edges whose endpoints are consecutive in π. The boundary vertices of G' are the same as the boundary vertices of G. Note that G' is not a subgraph of G as it is defined in section 2.

Let $R = (F, H)$ be a representation function from G to the unit disc, such that the induced boundary order of $R(G)$ is P_0, and let $R' = (F', H')$ be the restriction of R to G'. Since $R'(G')$ is compact, the complement of $R'(G')$ is a disjoint union of open subsets of the complex plane, one of which, call it S, is unbounded. Then there is a shortest path $\pi' = (p_1 \to p_2 \to \ldots \to p_N \to p_1)$, such that if e is an edge whose endpoints occur consecutively in π', then $H'(e) \subset \partial S$. Since π' includes the boundary vertices in the same order as π_0, and π_0 is a perimeter of P_0, length$(\pi') \geq$ length(π_0).

For an example, see Figure 6, where P_0 is the induced boundary order by this representation and $P = (1, 4, 3, 5, 2, 7, 6)$. In this example, a possible G' is shown by the darkened edges, and those edges not in G' are dotted lines, and S includes everything but the shaded regions. In this example, there is only one edge in G' that is not in ∂S, but in more elaborate examples arbitrary numbers of edges from G' can be left out.

Every edge e with endpoints consecutive in π' is an edge in G', and hence has endpoints consecutive in π. Also, since π' is a shortest path and the union of the $\overline{H'(e)}$ for e with endpoints consecutive in π' is equal to the boundary of S, no edge is traversed in π' more than twice. If an edge e in π' is traversed twice, the definition of π' implies that points in S occur on both sides of e, so deleting e would create a disconnected graph. Since π is a closed path, then π must traverse e at least twice. Thus, length$(\pi) \geq$ length(π').

Finally, we see that $\|P\|_G =$ length$(\pi) \geq$ length$(\pi') \geq$ length$(\pi_0) = \|P_0\|_G$, and the theorem is proved.

The converse of this statement is not necessarily true. The clearest instance of this is a tree. Because of the lack of cycles, there are many boundary orders which result in a representation. But if you construct a graph from a tree by adding a linear bipolar subgraph of sufficiently large size between two boundary

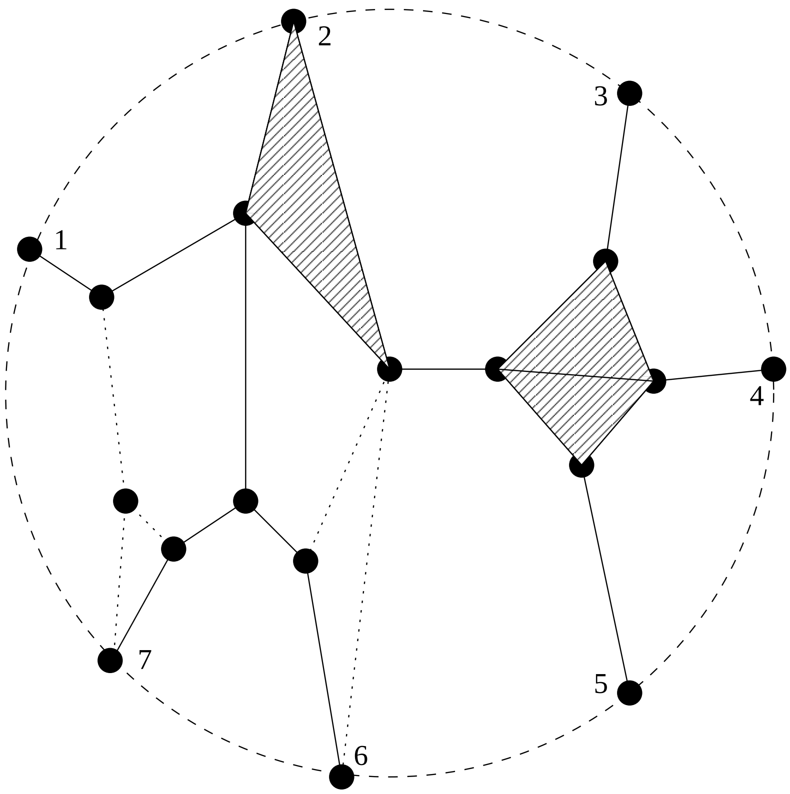

Figure 6: An example of our construction of G' and S

vertices, you will not affect the norm of any boundary order, but you might, as a side effect, prevent certain boundary orders from being induced orders. As an example, any ordering of the boundary vertices in the graph in part (a) of figure 7 is an induced order of a representation, and hence the norms are constant. Adding the linear bipolar subgraph of length 3, resulting in the graph in (b), does not affect any distances, and hence not any boundary order norms, but now the boundary order {1,3,2,4} is no longer the induced boundary order of any representation.

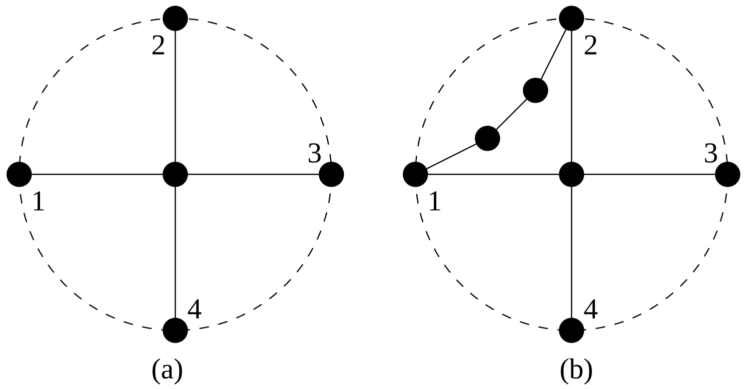

Figure 7: a tree with a linear bipolar subgraph added

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